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Existence of traveling waves in Fermi–Pasta–Ulam type systems on 2D–lattice

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(*Presented by I. I. Skrypnik*)

Abstract. The article deals with the Fermi–Pasta–Ulam type systems that describes an infinite systems of particles on 2D–lattice. The main result concerns the existence of traveling waves solutions with periodic and vanishing profiles. By means of critical point theory, we obtain sufficient conditions for the existence of such solutions.

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1. Introduction

In the present paper we study the Fermi–Pasta–Ulam type systems that describes the dynamics of an infinite systems of nonlinearly coupled particles on a two dimensional lattice. Let $q_{n,m} = q_{n,m}(t)$ be a coordinate of the (n,m)-th particle at time t. It is assumed that each particle interacts nonlinearly with its four nearest neighbors. The equations of motion of the system considered is of the form

$$\ddot{q}_{n,m} = W_1'(q_{n+1,m} - q_{n,m}) - W_1'(q_{n,m} - q_{n-1,m})$$

$$+W_2'(q_{n,m+1} - q_{n,m}) - W_2'(q_{n,m} - q_{n,m-1}), (n,m) \in \mathbb{Z}^2, \qquad (1.1)$$

where W_1 and W_2 are the potentials of interaction. Equations (1.1) form an infinite system of ordinary differential equations.

Systems of such type are of interest in view of numerous applications in physics [1, 16, 17]. Among the solutions of such systems, traveling waves deserve special attention. A comprehensive presentation of existing results on traveling waves for 1D Fermi–Pasta–Ulam lattices is given in [20]. The existence of periodic and solitary traveling waves in

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Fermi–Pasta–Ulam system on 2D–lattice is studied in [5] and [13]. In papers [4, 15, 18, 19] traveling waves for infinite systems of linearly coupled oscillators on 2D–lattice are studied, while [9] and [22] deal with periodic in time solutions for such systems. In [7] it is obtained a result on the existence of subsonic periodic traveling waves for the system of nonlinearly coupled nonlinear oscillators on 2D–lattice, while in [8] supersonic periodic traveling waves for such systems are studied. Paper [6] is devoted to the existence of solitary traveling waves for such systems. Papers [3, 10–12] is devoted to the existence of periodic, homoclinic and heteroclinic traveling waves for the discrete sine–Gordon type equations on 2D–lattice. Note that in [14] the existence of standing waves in discrete nonlinear Shrödinger type equations on 2D–lattice is studied.

In contrast to the previous ones (see [5] and [13]), where traveling waves with periodic and vanishing derivative of the wave profile were studied, in this paper the case of traveling waves solutions with periodic and vanishing profiles is considered.

2. Statement of a problem

A traveling wave solution of Eq. (1.1) is a function of the form

$$q_{n,m}(t) = u(n\cos\varphi + m\sin\varphi - ct),$$

where the profile function $u(s)$ of the wave, or simply profile, satisfies the equation

$$\begin{aligned} c^2u''(s) &= W_1'(u(s+\cos\varphi) - u(s)) - W_1'(u(s) - u(s-\cos\varphi)) \\ &+W_2'(u(s+\sin\varphi) - u(s)) - W_2'(u(s) - u(s-\sin\varphi)). \end{aligned} \tag{2.1}$$

In what follows, a solution of Eq. (2.1) is understood as a function $u(s)$ from the space $C^2(\mathbb{R})$ satisfying Eq. (2.1) for all $s \in \mathbb{R}$.

We consider two types of solutions: periodic and solitary traveling waves. In the first case profile satisfies periodic condition:

$$u(s+2k) = u(s),\ s \in \mathbb{R},\ k > 0, \tag{2.2}$$

and in the second case profile satisfies boundary conditions (vanishing):

$$\lim_{s\to\pm\infty} u(s) = u(\pm\infty) = 0. \tag{2.3}$$

Let X_k be the Hilbert space defined by

$$X_k = \left\{u \in H^1_{loc}(\mathbb{R}) :\ u(s+2k) = u(s),\ u(0) = 0\right\}$$

with the scalar product

$$(u,v)_k = \int_{-k}^{k} u'(s)v'(s)ds$$

and corresponding norm $\|u\|_k = (u,u)^{\frac{1}{2}}$. This space is a closed subspace of the space

$$\tilde{X}_k = \{u \in H^1_{loc}(\mathbb{R}) : u'(s+2k) = u'(s), u(0) = 0\}$$

with the same scalar product.

Let X be the closure of $C_0^\infty(\mathbb{R})$ with respect to the norm

$$\|u\| = \left(\int_{-\infty}^{+\infty} (u'(s))^2 ds \right)^{\frac{1}{2}}.$$

Obviously, X is a closed subspace of the space

$$\tilde{X} := \{u \in H^1_{loc}(\mathbb{R}) : u' \in L^2(\mathbb{R})\}$$

with the scalar product

$$(u,v)_{\tilde{X}} = u(0)v(0) + \int_{-\infty}^{+\infty} u'(s)v'(s)ds,$$

therefore, the functions from the space X satisfy the conditions (2.3).

Assume

(*i*) $W_i(r) = \frac{c_i}{2}r^2 + f_i(r)$, *where* $c_i \in \mathbb{R}$, $f_i \in C^1(\mathbb{R})$, $f_i(0) = f_i'(0) = 0$ *and* $f_i'(r) = o(r)$ *as* $r \to 0$, $i = 1, 2$;

(*ii*) *exist* $r_0 \in \mathbb{R}$ *and* $\mu > 2$ *such that* $f_i(r_0) > 0$ *and*

$$\mu f_i(r) \le r f_i'(r),\ r \in \mathbb{R},\ i = 1, 2.$$

Lemma 2.1. *Under the assumption* (*ii*) *there exist constants* $d > 0$ *and* $d_0 \ge 0$ *independent of* i *such that*

$$f_i(r) \ge d|r|^\mu - d_0,\ i = 1, 2. \tag{2.4}$$

Proof. Let fix $r_0 > 0$. Since

$$f_i'(r) \geq \mu \frac{f_i(r)}{r}$$

then, by standard results for differential inequalities $f_i(r) \geq y(r)$ as $r \geq r_0$, where $y(r)$ is solution of differential equation

$$y'(r) = \frac{\mu}{r} y(r)$$

with initial data $y(r_0) = f_i(r_0)$. Obviously,

$$y(r) = \frac{f_i(r_0)}{r_0^\mu} r^\mu.$$

Hence,

$$f_i(r) \geq \frac{f_i(r_0)}{r_0^\mu} r^\mu, r \geq r_0.$$

Then for all $r \geq 0$

$$f_i(r) \geq f_i(r_0)(\frac{r^\mu}{r_0^\mu} - 1) = \frac{f_i(r_0)}{r_0^\mu} r^\mu - f_i(r_0).$$

Similarly, for $r \leq 0$

$$f_i(r) \geq \frac{f_i(-r_0)}{r_0^\mu} |r^\mu| - f_i(r_0).$$

Thus, we obtain (2.4) with

$$d = \min[\frac{f_i(-r_0)}{r_0^\mu}, \frac{f_i(r_0)}{r_0^\mu}],$$

$$d_0 = \max[f_i(r_0), f_i(-r_0)].$$

□

On the spaces X_k and X, we consider the functionals, respectively

$$J_k(u) = \int_{-k}^{k} \left[\frac{c^2}{2} (u'(s))^2 - W_1(Au(s)) - W_2(Bu(s)) \right] ds,$$

$$J(u) = \int_{-\infty}^{+\infty} \left[\frac{c^2}{2} (u'(s))^2 - W_1(Au(s)) - W_2(Bu(s)) \right] ds,$$

where

$$(Au)(s) := u(s+\cos\varphi) - u(s) = \int\limits_{s}^{s+\cos\varphi} u'(\tau)d\tau,$$

$$(Bu)(s) := u(s+\sin\varphi) - u(s) = \int\limits_{s}^{s+\sin\varphi} u'(\tau)d\tau.$$

It is not so difficult to verify that the critical points of these functionals in the spaces X_k and X are solutions of the equation (2.1) satisfying the conditions (2.2) and (2.3), respectively.

By direct calculation we obtain the following lemma.

Lemma 2.2. *The operators A and B are bounded and we have*

$$\|Au\|_{L^2(-k,k)} \le |\cos\varphi| \cdot \|u'\|_{L^2(-k,k)}, \ \|Bu\|_{L^2(-k,k)} \le |\sin\varphi| \cdot \|u'\|_{L^2(-k,k)},$$

and

$$\|Au\|_{L^\infty(-k,k)} \le l_1(k) \cdot \|u'\|_{L^2(-k,k)}, \ \|Bu\|_{L^\infty(-k,k)} \le l_2(k) \cdot \|u'\|_{L^2(-k,k)},$$

where

$$l_1(k) = \begin{cases} |\cos\varphi|\sqrt{\left[\frac{1}{2k}\right]+1}, & 0 < 2k < 1, \\ |\cos\varphi|, & 2k \ge 1, \end{cases}$$

and

$$l_2(k) = \begin{cases} |\sin\varphi|\sqrt{\left[\frac{1}{2k}\right]+1}, & 0 < 2k < 1, \\ |\sin\varphi|, & 2k \ge 1, \end{cases}$$

where $\left[\frac{1}{2k}\right]$ denotes the integer part of $\frac{1}{2k}$.

3. Periodic waves

The main result of this section is the following theorem, which establishes the existence of periodic waves.

Theorem 3.1. *Assume (i) and (ii). Then for every $k > 0$ and $c^2 > a := \max\{c_1, c_2, 0\}$ Eq. (2.1) has a nonconstant solution u that satisfies condition (2.2).*

To prove this theorem, we need the mountain pass theorem.

Let I – C^1-functional on a Hilbert space H. We say that I satisfies the *Palais–Smale condition*, if the following condition is satisfied:

(PS) Let $\{u_n\} \subset H$ *be a such sequence that* $\{I(u_n)\}$ *is bounded and* $I'(u_n) \to 0$, $n \to \infty$. *Then* $\{u_n\}$ *contains a convergent subsequence.*

Now we formulate the mountain pass theorem (see [20, 21, 23]).

Theorem 3.2. ***(Mountain pass theorem).*** *Let* I — C^1*-functional on a Hilbert space* H *with norm* $\|\cdot\|$, *which satisfies the Palais–Smale condition. Assume that there exist* $e \in H$ *and* $r > 0$ *such that* $\|e\| > r$ *and* $\beta := \inf\limits_{\|u\|=r} I(u) > I(0) \geq I(e)$. *Then there exists a critical point* $u \in H$ *of* I *such that the critical value*

$$I(u) := b = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) \geq \beta,$$

where $\Gamma := \{\gamma \in C([0,1], H) : \gamma(0) = 0, \gamma(1) = e\}$. *Moreover,*

$$I(u) \leq \sup_{\tau\geq 0} I(\tau e).$$

We verify the conditions of the mountain pass theorem for the functional J_k.

Lemma 3.1. *Under the assumptions of theorem 3.1 functional* J_k *satisfies the Palais–Smale condition.*

Proof. Let $\{u_n\} \subset X_k$ be a Palais–Smale sequence of J_k at level b, i.e. $J_k(u_n) \to b$ and $J_k'(u_n) \to 0$ as $n \to \infty$. Then, for n large enough, we have

$$b + 1 + \frac{1}{\mu}\|u_n\|_k \geq J_k(u_n) - \frac{1}{\mu}\langle J_k'(u_n), u_n\rangle$$

$$= \left(\frac{1}{2} - \frac{1}{\mu}\right) \int\limits_{-k}^{k} \left[c^2(u_n'(s))^2 - c_1(Au_n(s))^2 - c_2(Bu_n(s))^2\right] ds$$

$$+ \int\limits_{-k}^{k} \left[\frac{1}{\mu} f_1'(Au_n(s))Au_n(s) - f_1(Au_n(s)\right] ds$$

$$+ \int\limits_{-k}^{k} \left[\frac{1}{\mu} f_2'(Bu_n(s))Bu_n(s) - f_2(Bu_n(s)\right] ds.$$

Due to the assumptions on potentials the second and third integrals are nonnegative and, by Lemma 2.2, we have

$$b + 1 + \frac{1}{\mu}\|u_n\|_k \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)(c^2 - a)\|u_n\|_k^2.$$

And this implies that the sequence $\{u_n\}$ is bounded in X_k.

Then, up to a subsequence (with the same denotation), $u_n \to u$ weakly in X_k, hence, $Au_n \to Au$ and $Bu_n \to Bu$ weakly in X_k, and strongly in $L^2(-k,k)$ and $C([-k,k])$ (by the compactness of Sobolev embedding). A straightforward calculation shows that

$$c^2\|u_n - u\|_k^2 = \int_{-k}^{k} c^2(u_n'(s) - u'(s))^2 ds$$

$$= \langle J_k'(u_n) - J_k'(u), u_n - u\rangle$$

$$+c_1\|Au_n - Au\|^2_{L^2(-k,k)} + c_2\|Bu_n - Bu\|^2_{L^2(-k,k)}$$

$$+\int_{-k}^{k} \left(f_1'(Au_n(s)) - f_1'(Au(s))\right)(Au_n(s) - Au(s))\, ds$$

$$+\int_{-k}^{k} \left(f_2'(Bu_n(s)) - f_2'(Bu(s))\right)(Bu_n(s) - Bu(s))\, ds.$$

Obviously that all the terms on the right part converge to 0 (first, fourth and fifth by weak convergence, second and third terms converge to 0 by strong convergence). Thus, $\|u_n - u\|_k \to 0$ as $n \to \infty$, and proof is complete. □

Lemma 3.2. *Under the assumptions of theorem 3.1 there exist $r_0 > 0$ and $\alpha_0 > 0$ independent of k such that* $\inf\limits_{\|u\|_k = r_0} J_k(u) > \alpha_0$.

Proof. We represent the functional J_k in the form

$$J_k(u) = \frac{1}{2}\Psi_k(u) - S_k(u),$$

where

$$\Psi_k(u) = \int_{-k}^{k} \left[c^2(u'(s))^2 - c_1(Au(s))^2 - c_2(Bu(s))^2\right] ds,$$

$$S_k(u) = \int_{-k}^{k} \left[f_1(Au(s)) + f_2(Bu(s))\right] ds.$$

Then, by lemma 2.2, we have

$$J_k(u) + S_k(u) = \frac{1}{2}\Psi_k(u) \geq \frac{c^2 - a}{2}\|u\|_k^2.$$

We show that $S_k(u) = o(\|u\|_k^2)$. Due to (i), for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\max\{f_1(r), f_2(r)\} \leq \frac{\varepsilon r^2}{2}$$

as $|r| \leq \delta$.

We set

$$r_0 = \frac{\delta}{\max\{l_1(k), l_2(k)\}},$$

where $l_1(k), l_2(k)$ from lemma 2.2. And take $u \in X_k$ with norm $\|u\|_k = r_0$. Then, by lemma 2.2, for almost everyone s we have

$$|Au(s)| \leq \|Au\|_{L^\infty(-k,k)} \leq l_1(k)\|u\|_k \leq \delta,$$

$$|Bu(s)| \leq \|Bu\|_{L^\infty(-k,k)} \leq l_2(k)\|u\|_k \leq \delta.$$

Hence,

$$S_k(u) \leq \frac{\varepsilon}{2}\int_{-k}^{k} \left[(Au(s))^2 + (Bu(s))^2\right] ds \leq \frac{\varepsilon}{2}\|u\|_k^2.$$

By arbitrariness of $\varepsilon > 0$, we have

$$S_k(u) = o(\|u\|_k^2).$$

In particular, if we choose ε such that $0 < \varepsilon < c^2 - a$, then we obtain

$$J_k(u) \geq (c^2 - a - \varepsilon)\frac{r_0^2}{2} > 0$$

and the lemma is proved. □

Lemma 3.3. *Under the assumptions of theorem 3.1 there exists $e \in X_k$ with norm $\|e\|_k > r_0$ such that $J_k(e) \leq 0$.*

Proof. By lemma 2.1, for all r

$$f_i(r) \geq d|r|^\mu - d_0.$$

Let $u \in X_k \setminus \{0\}$ and $r > 0$. Then we have

$$J_k(ru) = \frac{1}{2}\int_{-k}^{k} \left[c^2 r^2 (u'(s))^2 - c_1 r^2 (Au(s))^2 - c_2 r^2 (Bu(s))^2\right] ds$$

$$- \int_{-k}^{k} [f_1(A(ru(s))) + f_2(B(ru(s)))]\, ds$$

$$\leq \frac{r^2}{2} \int_{-k}^{k} \left[c^2 r^2 (u'(s))^2 - c_1 r^2 (Au(s))^2 - c_2 r^2 (Bu(s))^2\right] ds$$

$$-dr^{\mu} \int_{-k}^{k} [|Au(s)|^{\mu} + |Bu(s)|^{\mu}]\, ds + 4kd_0.$$

Since $\mu > 2$, $J_k(ru) \to -\infty$ as $r \to +\infty$, hence, there exists $r_0 = r_0(u) > 0$ such that $J_k(ru) \leq 0$ for all $r > r_0$ and the lemma is proved. □

Proof of theorem 3.1. Lemmas 3.1–3.3 show that J_k satisfies all conditions of mountain pass theorem. Hence, J_k has nontrivial critical point $u \in X_k$, which is a C^2-solution of Eq. (2.1) that satisfy (2.2). Obviously that u is nonconstant. The proof is complete. □

4. Solitary waves

The main result of this section is the following theorem, which establishes the existence of solitary waves.

Theorem 4.1. *Assume (i), (ii) and* $c^2 > a := \max\{c_1, c_2, 0\}$ *Then Eq. (2.1) has a nonconstant solution u that satisfies boundary conditions (2.3).*

Proof. As we have pointed out in section 3, the functional J satisfies the mountain pass geometry in $\tilde{X}$. Since there exists $e \in X$ such that $J(e) < 0$, the functional J also satisfies the mountain pass geometry in X. Then, by the version of mountain pass theorem without Palais–Smale (see [23], Theorem 1.15), there exists a Palais–Smale sequence $\{u_n\} \subset X$ at level b, i.e. $J(u_n) \to b$ and $J'(u_n) \to 0$ in X^*.

As usual, the sequence $\{u_n\}$ is bounded in X. Furthermore, $\|u_n\|$ is bounded below by a positive constant, hence, $\|u_n\| \not\to 0$. Then we can assume that $u_n \to u$ weakly in X. In addition, for every $r > 0$ there exist $\theta > 0$, subsequence of $\{u_n\}$ (with the same denotation) and $\{\eta_n\} \subset \mathbb{R}$ such that

$$\int_{\eta_n - r}^{\eta_n + r} \left[(Au_n(s))^2 + (Bu_n(s))^2\right] ds \geq \theta.$$

Replacing $u_n(s)$ by $u_n(s-\eta_n)$, we obtain

$$\int_{-r}^{r} \left[(Au_n(s))^2 + (Bu_n(s))^2\right] ds \geq \theta$$

and the new sequence $\{u_n\}$ still form a Palais–Smale sequence. Due to the compactness of Sobolev embedding, $Au_n \to Au$ and $Bu_n \to Bu$ in the space $L^\infty_{loc}(\mathbb{R})$, i.e. uniformly on segments, and, hence,

$$\int_{-r}^{r} \left[(Au(s))^2 + (Bu(s))^2\right] ds \geq \theta > 0.$$

And this implies that $u \neq 0$.

Let $g \in C_0^\infty(\mathbb{R})$. Then for k large enough: $[-k,k] \supset \operatorname{supp} Ag \cup \operatorname{supp} Bg =: S$. For such k we denote by $g_k \in X_k$ the $2k$-periodic extension of $g_{[-k,k]}$. Thus, we have

$$\langle J'(u), g\rangle$$

$$= \int_{-\infty}^{+\infty} \left[c^2 u'(s)g'(s) - W_1'(Au(s))Ag(s) - W_2'(Bu(s))Bg(s)\right] ds$$

$$= \int_S \left[c^2 u'(s)g'(s) - W_1'(Au(s))Ag(s) - W_2'(Bu(s))Bg(s)\right] ds$$

$$= \lim_{k\to\infty} \int_S \left[c^2 u_k'(s)g'(s) - W_1'(Au_k(s))Ag(s) - W_2'(Bu_k(s))Bg(s)\right] ds$$

$$= \lim_{k\to\infty} \int_{-k}^{k} \left[c^2 u_k'(s)g_k'(s)\right.$$

$$\left. - W_1'(Au_k(s))Ag_k(s) - W_2'(Bu_k(s))Bg_k(s)\right] ds$$

$$= \lim_{k\to\infty} \langle J'(u_k), g_k\rangle = 0.$$

Hence, u is nontrivial solution of Eq. (2.1) that satisfies boundary conditions (2.3). Obviously that u is nonconstant. The proof is complete. □

Conclusion

Thus, in the present paper we obtain some results on the existence of periodic and solitary traveling waves in Fermi–Pasta–Ulam type systems on a two-dimensional lattice.

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