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Fredholm eigenvalues and quasiconformal geometry of polygons

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Abstract. An important open problem in geometric complex analysis is to establish algorithms for explicit determination of the basic curvelinear and analytic functionals intrinsically connected with conformal and quasiconformal maps, such as their Teichmüller and Grunsky norms, Fredholm eigenvalues and the quasireflection coefficient. This is important also for the potential theory but has not been solved even for convex polygons. This case has intrinsic interest in view of the connection of polygons with the geometry of the universal Teichmüller space and approximation theory.

This survey extends our previous survey of 2005 and presents the new approaches and recent essential progress in this field of geometric complex analysis and potential theory, having various important applications. Another new topic concerns quasireflections across finite collections of quasiintervals (to which the notion of Fredholm eigenvalues also can be extended).

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1. Fredholm eigenvalues and quasiconformal reflections: general theory

1.1. Quasireflections and quasicurves

The classical Brouwer–Kerekjarto theorem ([13, 31], see also [90]) says that every periodic homeomorphism of the sphere S^2 is topologically equivalent to a rotation, or to a product of a rotation and a reflection across a diametral plane. The first case corresponds to orientation preserving homeomorphisms (and then E consists of two points), the second

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one is orientation reversing, and then either the fixed point set E is empty (which is excluded in our situation) or it is a topological circle.

We are concerned with homeomorphisms reversing orientation. Such homeomorphisms of order 2 are topological involutions of S^2 with $f \circ f = \mathrm{id}$ and are called topological reflections.

We shall consider here **quasiconformal reflections** or quasireflections on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = S^2$, i.e., the orientation reversing quasiconformal automorphisms of order 2 (involutions) of the sphere with $f \circ f = \mathrm{id}$. The topological circles admitting such reflections are called **quasicircles**. Such circles are locally *quasi-intervals*, i.e., the images of straight line segments under quasiconformal maps of the sphere S^2. Any quasireflection preserves pointwise fixed a quasicircle $L \subset \widehat{\mathbb{C}}$ interchanging its inner and outer domains.

Under **quasiconformal map** $w(z)$ of a domain $D \subset \widehat{\mathbb{C}}$, we understand an orientation preserving generalized solution of the Beltrami equation (uniformly elliptic system of the first order)

$$\frac{\partial w}{\partial \overline{z}} = \mu(z) \frac{\partial w}{\partial z}, \quad z \in D,$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)$$

are the distributional partial derivatives, μ is a given function from $L_\infty(D)$ with $\|\mu\|_\infty < 1$, called the **Beltrami coefficient** (or complex dilatation) of the map w, and the quantity $k(w) = \|\mu\|_\infty$ is the (quasiconformal) **dilatation** of this map. There are some equivalent analytic and geometric definitions of such maps.

Quasiconformality preserves (up to bounded perturbations) the main intrinsic properties of conformal maps (see, e.g., [5, 34, 66]).

Qualitatively, any quasicircle L is characterized, due to [4], by uniform boundedness of the cross-ratios for all ordered quadruples (z_1, z_2, z_3, z_4) of the distinct points on L; namely,

$$\frac{\overline{z_1 z_2}}{\overline{z_1 z_3}} \, \frac{\overline{z_3 z_4}}{\overline{z_2 z_4}} \le C < \infty$$

for any quadruple of points z_1, z_2, z_3, z_4 on L following this order. Using a fractional linear transformation, one can send one of the points, for example, z_4, to infinity; then the above inequality assumes the form

$$\Big|\frac{z_2 - z_1}{z_3 - z_1}\Big| \le C.$$

This is shown in [4] by applying the properties of quasisymmetric maps. Ahlfors has established also that if a topological circle L admits quasireflections (i.e., is a quasicircle), then there exists a differentiable quasireflection across L which is (euclidian) biLipschitz-continuous. This property is very useful in various applications. On its extension to hyperbolic M-bilipschitz reflections see [26].

Geometrically, a quasicircle is characterized by the property that, for any two points z_1, z_2 on L, the ratio of the chordal distance $|z_1 - z_2|$ to the diameters of the corresponding subarcs with these endpoints is uniformly bounded. Note also that every quasicircle has zero two-dimensional Lebesgue measure.

Other characterizations of quasicircles are given, for example, in [25, 68, 79]. We will not touch here the extension of this theory to higher dimensions.

Quasireflections across more general sets $E \subset \widehat{\mathbb{C}}$ also appear in certain questions and are of independent interest. Those sets admitting quasireflections are called **quasiconformal mirrors**.

One defines for each mirror E its **reflection coefficient**

$$q_E = \inf k(f) = \inf \ \|\partial_z f / \partial_{\bar{z}} f\|_\infty \tag{1}$$

and **quasiconformal dilatation**

$$Q_E = (1 + q_E)/(1 - q_E) \geq 1;$$

the infimum in (1) is taken over all quasireflections across E, provided those exist, and is attained by some quasireflection f_0.

When $E = L$ is a quasicircle, the corresponding quantity

$$k_E = \inf\{k(f_*) : f_*(S^1) = E\} \tag{2}$$

and the reflection coefficient q_E can be estimated in terms of one another; moreover, due to [5], [57], we have

$$Q_E = K_E := \Big(\frac{1 + k_E}{1 - k_E}\Big)^2. \tag{3}$$

The infimum in (2) is taken over all orientation preserving quasiconformal automorhisms f_* carrying the unit circle onto L, and $k(f_*) = \|\partial_{\bar{z}} f_* / \partial_z f_*\|_\infty$.

Theorem 1. *For any set $E \subset \widehat{\mathbb{C}}$ which admits quasireflections, there is a quasicircle $L \supset E$ with the same reflection coefficient; therefore,*

$$Q_E = \min\{Q_L : \ L \supset E \ \text{ quasicircle}\}. \tag{4}$$

The proof of this important theorem was given for finite sets $E = \{z_1, \dots, z_n\}$ by Kühnau in [60], using Teichmüller's theorem on extremal quasiconformal maps applied to the homotopy classes of homeomorphisms of the punctured spheres, and extended to arbitrary sets $E \subset \widehat{\mathbb{C}}$ by the author in [39].

Theorem 1 yields, in particular, that similar to (3) for any set $E \subset \widehat{\mathbb{C}}$, its quasiconformal dilatation satsfies

$$Q_E = (1 + k_E)^2/(1 - k_E)^2,$$

where $k_E = \inf \|\partial_{\overline{z}} f/\partial_z f\|_\infty$ over all quasicircles $L \supset E$ and all orientation preserving quasiconformal homeomorphisms $f: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $f(\widehat{\mathbb{R}}) = L$.

This theorem implies various quantitative consequences. A new its application will be given in the last section.

We point out that the conformal symmetry on the extended complex plane is strictly rigid and reduces to reflection $z \mapsto \bar{z}$ within conjugation by transformations $g \in PSL(2, \mathbb{C})$. The quasiconformal symmetry avoids such rigidity and is possible over quasicircles. Theorem 1 shows that, in fact, this case is the most general one, since for any set $E \subset \widehat{\mathbb{C}}$ we have $Q_E = \infty$, unless E is a subset of a quasicircle with the same reflection coefficient.

Let us mention also that a somewhat different construction of quasiconformal reflections across Jordan curves has been provided in [21]; it relies on the conformally natural extension of homeomorphisms of the circle introduced by Douady and Earle [17].

The quasireflection coefficients of curves are closely connected with intrinsic functionals of conformal and quasiconformal maps such as their Teichmüller and Grunsky norms and the first Fredholm eigenvalue, which imply a deep quantitative characterization of the features of these maps.

One of the main problem here, important also in applications of geometric complex analysis, is to establish the algorithms for explicit determination of these quantities for individual quasicircles or quasiintervals. This was remains open a long time even for generic quadrilaterals.

1.2. Fredholm eigenvalues

Recall that the **Fredholm eigenvalues** ρ_n of an oriented smooth closed Jordan curve L on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ are the eigenvalues of its double-layer potential, or equivalently, of the integral equation

$$u(z) + \frac{\rho}{\pi} \int\limits_L u(\zeta) \frac{\partial}{\partial n_\zeta} \log \frac{1}{|\zeta - z|} ds_\zeta = h(z),$$

which has has many applications (here n_ζ is the outer normal and ds_ζ is the length element at $\zeta \in L$).

The least positive eigenvalue $\rho_L = \rho_1$ plays a crucial role and is naturally connected with conformal and quasiconformal maps . It can be defined for any oriented closed Jordan curve L by

$$\frac{1}{\rho_L} = \sup \frac{|\mathcal{D}_G(u) - \mathcal{D}_{G^*}(u)|}{\mathcal{D}_G(u) + \mathcal{D}_{G^*}(u)},$$

where G and G^* are, respectively, the interior and exterior of L; $\mathcal{D}$ denotes the Dirichlet integral, and the supremum is taken over all functions u continuous on $\widehat{\mathbb{C}}$ and harmonic on $G \cup G^*$. In particular, $\rho_L = \infty$ only for the circle.

An upper bound for ρ_L is given by Ahlfors' inequality [3]

$$\frac{1}{\rho_L} \leq q_L, \tag{5}$$

where q_L denotes the minimal dilatation of quasireflections across L.

In view of the invariance of all quantities in (5) under the action of the Möbius group $PSL(2, \mathbb{C})/\pm\mathbf{1}$, it suffices to consider the quasiconformal homeomorphisms of the sphere carrying S^1 onto L whose Beltrami coefficients $\mu_f(z) = \partial_{\bar{z}} f/\partial_z f$ have support in the unit disk $\mathbb{D} = \{|z| < 1\}$, and $f|\mathbb{D}^*(z) = z + b_0 + b_1 z^{-1} + ...$, where $\mathbb{D}^* = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ (or in the upper half-plane $U = \{\Im z > 0\}$). Then q_L is equal to the minimum $k_0(f)$ of dilatations $k(f) = \|\mu\|_\infty$ of quasiconformal extensions of the function $f^* = f|\mathbb{D}^*$ into $\mathbb{D}$.

The inequality (5) serves as a background for defining the value ρ_L, being combined with the features of Grunsky inequalities given by the classical Kühnau–Schiffer theorem. The related results can be found, e.g. in surveys [42, 57, 61] and the references cited there.

In the following sections, we provide a new general approach.

1.3. The Grunsky and Milin inequalities

Let

$$\mathbb{D} = \{z : \ |z| < 1\}, \ \mathbb{D}^* = \{z \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} : \ |z| > 1\}.$$

In 1939, Grunsky discovered the necessary and sufficient conditions for univalence of a holomorphic function in a finitely connected domain on the extended complex plane $\widehat{\mathbb{C}}$ in terms of an infinite system of the coefficient inequalities. In particular, his theorem for the canonical disk $\mathbb{D}^*$ yields that a holomorphic function $f(z) = z + \text{const} + O(z^{-1})$ in a neighborhood of $z = \infty$ can be extended to a univalent holomorphic function on the $\mathbb{D}^*$ if and only if its Grunsky coefficients α_{mn} satisfy

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| \le 1, \tag{6}$$

where α_{mn} are defined by

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (\mathbb{D}^*)^2, \tag{7}$$

the sequence $\mathbf{x} = (x_n)$ runs over the unit sphere $S(l^2)$ of the Hilbert space l^2 with norm $\|\mathbf{x}\|^2 = \sum\limits_1^\infty |x_n|^2$, and the principal branch of the logarithmic function is chosen (cf. [30]). The quantity

$$\varkappa(f) = \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| : \ \mathbf{x} = (x_n) \in S(l^2) \Big\} \le 1 \tag{8}$$

is called the **Grunsky norm** of f.

For the functions with k-quasiconformal extensions ($k < 1$), we have instead of (8) a stronger bound

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| \le k \quad \text{for any} \ \ \mathbf{x} = (x_n) \in S(l^2), \tag{9}$$

established first in [53] (see also [42, 46]). Then

$$\varkappa(f) \le k(f), \tag{10}$$

where $k(f)$ denotes the **Teichmüller norm** of f which is equal to the infimum of dilatations $k(w^\mu) = \|\mu\|_\infty$ of quasiconformal extensions of f to $\widehat{\mathbb{C}}$. Here w^μ denotes a homeomorphic solution to the Beltrami equation $\partial_{\overline{z}} w = \mu \partial_z w$ on $\mathbb{C}$ extending f.

Note that the Grunsky (matrix) operator

$$\mathcal{G}(f) = (\sqrt{mn}\ \alpha_{mn}(f))_{m,n=1}^{\infty}$$

acts as a linear operator $l^2 \to l^2$ contracting the norms of elements $\mathbf{x} \in l^2$; the norm of this operator equals $\varkappa(f)$ (cf. [28, 29]).

For most functions f, we have in (10) the strong inequality $\varkappa(f) < k(f)$ (moreover, the functions satisfying this inequality form a dense subset of Σ), while the functions with the equal norms play a crucial role in many applications (see [42, 46, 52, 54–56]).

The method of Grunsky inequalities was generalized in several directions, even to bordered Riemann surfaces X with a finite number of boundary components(cf. [30, 66, 71, 79, 86]; see also [27]). In the general case, the generating function (7) must be replaced by a bilinear differential

$$-\log \frac{f(z) - f(\zeta)}{z - \zeta} - R_X(z, \zeta) = \sum_{m,n=1}^{\infty} \beta_{mn}\ \varphi_m(z)\varphi_n(\zeta) : \ X \times X \to \mathbb{C}, \tag{11}$$

where the surface kernel $R_X(z, \zeta)$ relates to the conformal map $j_\theta(z, \zeta)$ of X onto the sphere $\widehat{\mathbb{C}}$ slit along arcs of logarithmic spirals inclined at the angle $\theta \in [0, \pi)$ to a ray issuing from the origin so that $j_\theta(\zeta, \zeta) = 0$ and

$$j_\theta(z) = (z - z_\theta)^{-1} + \text{const} + O(1/(z - z_\theta)) \quad \text{as} \ \ z \to z_\theta = j_\theta^{-1}(\infty)$$

(in fact, only the maps j_0 and $j_{\pi/2}$ are applied). Here $\{\varphi_n\}_1^\infty$ is a canonical system of holomorphic functions on X such that (in a local parameter)

$$\varphi_n(z) = \frac{a_{n,n}}{z^n} + \frac{a_{n+1,n}}{z^{n+1}} + \dots \quad \text{with} \ \ a_{n,n} > 0, \quad n = 1, 2, \dots,$$

and the derivatives (linear holomorphic differentials) φ_n' form a complete orthonormal system in $H^2(X)$.

We shall deal only with simply connected domains $X = D^* \ni \infty$ with quasiconformal boundaries (quasidisks). For any such domain, the kernel R_D vanishes identically on $D^* \times D^*$, and the expansion (11) assumes the form

$$-\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=1}^{\infty} \frac{\beta_{mn}}{\sqrt{mn}\ \chi(z)^m\ \chi(\zeta)^n}, \tag{12}$$

where χ denotes a conformal map of D^* onto the disk $\mathbb{D}^*$ so that $\chi(\infty) = \infty,\ \chi'(\infty) > 0$.

Each coefficient $\alpha_{mn}(f)$ in (12) is represented as a polynomial of a finite number of the initial coefficients $b_1, b_2, \dots, b_s$ of f; hence it depends holomorphically on Beltramicoefficients of quasiconformal extensions of f as well as on the Schwarzian derivatives

$$S_f(z) = \Big(\frac{f''(z)}{f'(z)}\Big)' - \frac{1}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2, \quad z \in D^*. \tag{13}$$

These derivatives range over a bounded domain in the complex Banach space $\mathbf{B}(D^*)$ of hyperbolically bounded holomorphic functions $\varphi \in \mathbb{D}^*$ with norm

$$\|\varphi\|_{\mathbf{B}} = \sup_{D^*} \lambda_{D^*}^{-2}(z)|\varphi(z)|,$$

where $\lambda_{D^*}(z)|dz|$ denotes the hyperbolic metric of D^* of Gaussian curvature -4. This domain models the **universal Teichmüller space $\mathbf{T}$** with the base point $\chi'(\infty)D^*$ (in holomorphic Bers' embedding of $\mathbf{T}$).

A theorem of Milin [71] extending the Grunsky univalence criterion for the disk $\mathbb{D}^*$ to multiply connected domains D^* states that a holomorphic function $f(z) = z + \text{const} + O(z^{-1})$ in a neighborhood of $z = \infty$ can be continued to a univalent function in the whole domain D^* if and only if the coefficients β_{mn} in (12) satisfy, similar to the classical case of the disk $\mathbb{D}^*$, the inequality

$$\Big| \sum_{m,n=1}^{\infty} \beta_{mn}\, x_m x_n \Big| \le 1 \tag{14}$$

for any point $\mathbf{x} = (x_n) \in S(l^2)$. We call the quantity

$$\varkappa_{D^*}(f) = \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \beta_{mn}\, x_m x_n \Big| : \ \mathbf{x} = (x_n) \in S(l^2) \Big\}, \tag{15}$$

the **generalized Grunsky norm** of f. By (14), $\varkappa_{D^*}(f) \le 1$ for any f from the class $\Sigma(D^*)$ of univalent functions in D^* with hydrodynamical normalization

$$f(z) = z + b_0 + b_1 z^{-1} + \dots \quad \text{near} \ \ z = \infty.$$

The inequality $\varkappa_{D^*}(f) \le 1$ is necessary and sufficient for univalence of f in D^* (see [30, 71, 79]).

The norm (15) also is dominated by the Teichmüller norm $k(f)$ of this map. Similar to (10),

$$\varkappa_{D^*}(f) \le k(f) = \tanh \tau_{\mathbf{T}}(\mathbf{0}, S_F),$$

where $\tau_{\mathbf{T}}$ denotes the Teichmüller distance on the universal Teichmüller space $\mathbf{T}$ with the base point D, and for the most of univalent functions, we also have here the strict inequality.

The quasiconformal theory of generic Grunsky coefficients has been developed in [47]. This technique is a powerful tool in geometric complex analysis having fundamental applications in the Teichmüller space theory and other fields.

Note that in the case $D^* = \mathbb{D}^*$, $\beta_{mn} = \sqrt{mn}\, \alpha_{mn}$; for this disk, we shall use the notations Σ and $\varkappa(f)$. We denote by S the canonical class of univalent functions $F(z) = z + a_2 z^2 + \dots$ in the unit disk $\mathbb{D}$.

The Grunsky norm of univalent functions $F \in S$ is defined similar to (5), (6); so any such $F(z)$ and its inversion $f(z) = 1/F(1/z)$ univalent in D^* have the same Grunsky coefficients α_{mn}. Technically it is more convenient to deal with functions univalent in $\mathbb{D}^*$.

1.4. Extremal quasiconformality

A crucial point here is that the Teichmüller norm on Σ is intrinsically connected with **integrable holomorphic quadratic differentials** $\psi(z)dz^2$ on the complementary domain

$$D = \widehat{\mathbb{C}} \setminus \overline{D^*}$$

(the elements of the subspace $A_1(D)$ of $L_1(D)$ formed by holomorphic functions), while the Grunsky norm naturally relates to the **abelian** structure determined by the set of quadratic differentials

$$A_1^2(D) = \{\psi \in A_1(D) : \ \psi = \omega^2\}$$

having only zeros of even order on D.

We describe the general intrinsic features. Let L be a quasicircle passing through the points $0, 1, \infty$ which is the common boundary of two domains D and D^*. Let L be an oriented quasiconformal Jordan curve (quasicircle) on the Riemann sphere $\widehat{\mathbb{C}}$ with the interior and exterior domains D and D^*. Denote by $\lambda_D(z)|dz|$ the hyperbolic metric of D of Gaussian curvature -4 and by $\delta_D(z) = \operatorname{dist}(z, \partial D)$ the Euclidean distance from the point $z \in D$ to the boundary. Then

$$\frac{1}{4} \le \lambda_D(z)\delta_D(z) \le 1,$$

where the right hand inequality follows from the Schwarz lemma and the left from Koebe's $\frac{1}{4}$ theorem.

Consider the unit ball of Beltrami coefficients supported on D,

$$\operatorname{Belt}(D)_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu|D^* = 0 \quad \|\mu\|_\infty < 1\}$$

and take the corresponding quasiconformal automorphisms $w^\mu(z)$ of the sphere $\widehat{\mathbb{C}}$ satisfying on $\mathbb{C}$ the Beltrami equation $\partial_{\overline{z}} w = \mu \partial_z w$ preserving the points $0, 1, \infty$ fixed. Recall that $k(w) = \|\mu_w\|_\infty$ is the dilatation of the map w.

Take the equivalence classes $[\mu]$ and $[w^\mu]$ letting the coefficients μ_1 and μ_2 from $\operatorname{Belt}(D^*)_1$ be equivalent if the corresponding maps w^{μ_1} and w^{μ_2} coincide on L (and hence on $\overline{D}$). These classes are in one-to-one correspondence with the Schwarzians S_{w^μ} on D^* which fill a bounded domain in the space $\mathbf{B}_2(D^*)$ modelling the universal Teichmüller space $\mathbf{T} = \mathbf{T}(D^*)$ with the base point D^*. The quotient map

$$\phi_{\mathbf{T}} : \ \operatorname{Belt}(D)_1 \to \mathbf{T}, \quad \phi_{\mathbf{T}}(\mu) = S_{w^\mu}$$

is holomorphic (as the map from $L_\infty(D)$ to $\mathbf{B}_2(D)$). Its intrinsic *Teichmüller metric* is defined by

$$\tau_{\mathbf{T}}(\phi_{\mathbf{T}}(\mu), \phi_{\mathbf{T}}(\nu)) = \frac{1}{2} \inf\{\log K\big(w^{\mu_*} \circ \big(w^{\nu_*}\big)^{-1}\big) : \mu_* \in \phi_{\mathbf{T}}(\mu), \nu_* \in \phi_{\mathbf{T}}(\nu)\},$$

It is the integral form of the infinitesimal Finsler metric

$$F_{\mathbf{T}}(\phi_{\mathbf{T}}(\mu), \phi'_{\mathbf{T}}(\mu)\nu) = \inf\{\|\nu_*/(1-|\mu|^2)\|_\infty : \ \phi'_{\mathbf{T}}(\mu)\nu_* = \phi'_{\mathbf{T}}(\mu)\nu\}$$

on the tangent bundle $\mathcal{T}\mathbf{T}$ of $\mathbf{T}$, which is locally Lipschitzian.

The Grunsky coefficients give rise to another Finsler structure $F(\mathbf{x}, v)$ on the bundle $\mathcal{T}\mathbf{T}$. It is dominated by the canonical Finsler structure $F_{\mathbf{T}}(\mathbf{x}, v)$ and allows one to reconstruct the Grunsky norm along the geodesic Teichmüller disks in $\mathbf{T}$ (see [41]).

We call the Beltrami coefficient $\mu \in \mathrm{Belt}(D^*)_1$ **extremal** (in its class) if

$$\|\mu\|_\infty = \inf\{\|\nu\|_\infty : \ \ \phi_{\mathbf{T}}(\nu) = \phi_{\mathbf{T}}(\mu)\}$$

and call μ **infinitesimally extremal** if

$$\|\mu\|_\infty = \inf\{\|\nu\|_\infty : \ \ \nu \in L_\infty(D^*), \ \ \phi'_{\mathbf{T}}(\mathbf{0})\nu = \phi'_{\mathbf{T}}(\mathbf{0})\mu\}.$$

Any infinitesimally extremal Beltrami coefficient μ is globally extremal (and vice versa), and by the basic Hamilton-Krushkal-Reich-Strebel theorem the extremality of μ is equivalent to the equality

$$\|\mu\|_\infty = \inf\{| < \mu, \psi >_{D^*} | : \ \ \psi \in A(D) : \ \|\psi\| = 1\}$$

(where $A(D)$ is the space of the integrable holomorphic quadratic differentials on D (the subspace of $L_1(D)$ formed by holomorphic functions on D) and the pairing

$$\langle \mu, \psi \rangle_D = \iint_D \mu(z)\psi(z)dxdy, \quad \mu \in L_\infty(D), \ \psi \in L_1(D) \ \ (z = x + iy).$$

Let $w_0 := w^{\mu_0}$ be an extremal representative of its class $[w_0]$ with dilatation

$$k(w_0) = \|\mu_0\|_\infty = \inf\{k(w^\mu) : w^\mu|L = w_0|L\},$$

and assume that there exists in this class a quasiconformal map w_1 whose Beltrami coefficient μ_{A_1} satisfies the inequality $\operatorname{ess\,sup}_{A_r} |\mu_{w_1}(z)| < k(w_0)$ in some ring domain $\mathcal{R} = D^* \setminus G$ complement to a domain $G \supset D^*$. Any such w_1 is called the **frame map** for the class $[w_0]$, and the corresponding point in the universal Teichmüller space $\mathbf{T}$ is called the **Strebel point**.

These points have the following important properties.

Theorem 2. *(i) If a class $[f]$ has a frame map, then the extremal map f_0 in this class (minimizing the dilatation $\|\mu\|_\infty$) is unique and either a conformal or a Teichmüller map with Beltrami coefficient $\mu_0 = k|\psi_0|/\psi_0$ on D, defined by an integrable holomorphic quadratic differential ψ_0 on D and a constant $k \in (0, 1)$* [92].

(ii) The set of Strebel points is open and dense in $\mathbf{T}$ [24, 65].

The first assertion holds, for example, for asymptotically conformal curves L. Similar results hold also for arbitrary Riemann surfaces (cf. [19, 24]).

Recall that a Jordan curve $L \subset \mathbb{C}$ is called **asymptotically conformal** if for any pair of points $a, b \in L$,

$$\max_{z \in L} \frac{|a - z| + |z - b|}{|a - b|} \to 1 \quad \text{as} \quad |a - b| \to 0,$$

where z lies between a and b.

Such curves are quasicircles without corners and can be rather pathological (see, e.g., [81, p. 249]. In particular, all C^1-smooth curves are asymptotically conformal.

The polygonal lines are not asymptotically conformal, and the presence of angles causes non-uniqueness of extremal quasireflections.

The boundary dilatation $H(f)$ admits also a local version $H_p(f)$ involving the Beltrami coefficients supported in the neighborhoods of a boundary point $p \in \partial D$. Moreover (see, e.g., [24, Ch. 17]), $H(f) = \sup_{p \in \partial D} H_p(f)$, and the points with $H_p(f) = H(f)$ are called **substantial** for f and for its equivalence class.

On the unique and non-unique extremality see, e.g., [12, 18, 34, 70, 81, 91, 92, 99].

The extremal quasiconformality is naturally connected with extremal quasireflections.

1.5. Complex geometry and basic Finsler metrics on universal Teichmüller space

Recall that the universal Teichmüller space $\mathbf{T}$ is the space of quasisymmetric homeomorphisms h of the unit circle $S^1 = \partial\mathbb{D}$ factorized by Möbius transformations. Its topology and real geometry are determined by the Teichmüller metric which naturally arises from extensions of these homeomorphisms h to the unit disk. This space admits also the complex structure of a complex Banach manifold (and this is valid for all Teichmüller spaces).

One of the fundamental notions of geometric complex analysis is the invariant Kobayashi metric on hyperbolic complex manifolds, even in the infinite dimensional Banach or locally convex complex spaces.

The canonical complex Banach structure on the space $\mathbf{T}$ is defined by factorization of the ball of Beltrami coefficients

$$\mathrm{Belt}(\mathbb{D})_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu|\mathbb{D}^* = 0, \ \|\mu\| < 1\},$$

letting $\mu, \nu \in \mathrm{Belt}(\mathbb{D})_1$ be equivalent if the corresponding maps $w^\mu, w^\nu \in \Sigma^0$ coincide on S^1 (hence, on $\overline{\mathbb{D}^*}$) and passing to Schwarzian derivatives S_{f^μ}. The defining projection $\phi_{\mathbf{T}} : \mu \to S_{w^\mu}$ is a holomorphic map from $L_\infty(\mathbb{D})$ to $\mathbf{B}$. The equivalence class of a map w^μ will be denoted by $[w^\mu]$.

An intrinsic complete metric on the space $\mathbf{T}$ is the Teichmüller metric, defined above in Section 1.4, with its infinitesimal Finsler form (structure) $F_{\mathbf{T}}(\phi_{\mathbf{T}}(\mu), \phi'_{\mathbf{T}}(\mu)\nu), \ \mu \in \mathrm{Belt}(\mathbb{D})_1; \ \nu, \nu_* \in L_\infty(\mathbb{C})$.

The space $\mathbf{T}$ as a complex Banach manifold also has invariant metrics. Two of these (the largest and the smallest metrics) are of special interest. They are called the Kobayashi and the Carathéodory metrics, respectively, and are defined as follows.

The **Kobayashi metric** $d_{\mathbf{T}}$ on $\mathbf{T}$ is the largest pseudometric d on $\mathbf{T}$ does not get increased by holomorphic maps $h : \ \mathbb{D} \to \mathbf{T}$ so that for any two points $\psi_1, \ \psi_2 \in \mathbf{T}$, we have

$$d_{\mathbf{T}}(\psi_1, \psi_2) \le \inf\{d_{\mathbb{D}}(0, t) : \ h(0) = \psi_1, \ h(t) = \psi_2\},$$

where $d_{\mathbb{D}}$ is the **hyperbolic Poincaré metric** on $\mathbb{D}$ of Gaussian curvature -4, with the differential form

$$ds = \lambda_{\mathrm{hyp}}(z)|dz| := |dz|/(1 - |z|^2).$$

The **Carathéodory** distance between ψ_1 and ψ_2 in $\mathbf{T}$ is

$$c_{\mathbf{T}}(\psi_1, \psi_2) = \sup d_{\mathbb{D}}(h(\psi_1), h(\psi_2)),$$

where the supremum is taken over all holomorphic maps $h : \ \mathbb{D} \to \mathbf{T}$.

The corresponding differential (infinitesimal) forms of the Kobayashi and Carathéodory metrics are defined for the points (ψ, v) of the tangent bundle $\mathcal{T}(\mathbf{T})$, respectively, by

$$\mathcal{K}_{\mathbf{T}}(\psi, v) = \inf\{1/r : \ r > 0, \ h \in \mathrm{Hol}(\mathbb{D}_r, \mathbf{T}), h(0) = \psi, dh(0) = v\},$$
$$\mathcal{C}_{\mathbf{T}}(\psi, v) = \sup\{|df(\psi)v| : \ f \in \mathrm{Hol}(\mathbf{T}, \mathbb{D}), f(\psi) = 0\},$$

where $\mathrm{Hol}(X, Y)$ denotes the collection of holomorphic maps of a complex manifold X into Y and $\mathbb{D}_r$ is the disk $\{|z| < r\}$.

The Schwarz lemma implies that the Carathéodory metric is dominated by the Kobayashi metric (and similarly for their infinitesimal forms). We shall use here mostly the Kobayashi metric.

Due to the fundamental Gardiner-Royden theorem, the Kobayashi metric on any Teichmüller spaces is equal to its Teichmüller metric (see [18, 20, 24, 84]).

For the the universal Teichmüller space $\mathbf{T}$, we have the following strengthened version of this theorem for universal Teichmüller space given in [37].

Theorem 3. *The Teichmüller metric $\tau_{\mathbf{T}}(\psi_1, \psi_2)$ of either of the spaces $\mathbf{T}$ or $\mathbf{T}(\mathbb{D}^*)$ is plurisubharmonic separately in each of its arguments; hence, the pluricomplex Green function of $\mathbf{T}$ equals*

$$g_{\mathbf{T}}(\psi_1, \psi_2) = \log \tanh \tau_{\mathbf{T}}(\psi_1, \psi_2) = \log k(\psi_1, \psi_2),$$

where k is the norm of extremal Beltrami coefficient defining the distance between the points ψ_1, ψ_2 in $\mathbf{T}$ (and similar for the space $\mathbf{T}(\mathbb{D}^*)$).

The differential (infinitesimal) Kobayashi metric $\mathcal{K}_{\mathbf{T}}(\psi, v)$ on the tangent bundle $\mathcal{T}(\mathbf{T})$ of $\mathbf{T}$ is logarithmically plurisubharmonic in $\psi \in \mathbf{T}$, equals the infinitesimal Finsler form $F_{\mathbf{T}}(\psi, v)$ of metric $\tau_{\mathbf{T}}$ and has constant holomorphic sectional curvature $\kappa_{\mathcal{K}}(\psi, v) = -4$ on the tangent bundle $\mathcal{T}(\mathbf{T})$.

In other words, the Teichmüller–Kobayashi metric is the largest invariant plurisubharmonic metric on $\mathbf{T}$.

The **generalized Gaussian curvature** κ_λ of an upper semicontinuous Finsler metric $ds = \lambda(t)|dt|$ in a domain $\Omega \subset \mathbb{C}$ is defined by

$$\kappa_\lambda(t) = -\frac{\mathbb{D} \log \lambda(t)}{\lambda(t)^2},$$

where $\mathbb{D}$ is the **generalized Laplacian**

$$\mathbb{D}\lambda(t) = 4 \liminf_{r \to 0} \frac{1}{r^2} \Big\{ \frac{1}{2\pi} \int_0^{2\pi} \lambda(t + re^{i\theta}) d\theta - \lambda(t) \Big\}$$

(provided that $-\infty \le \lambda(t) < \infty$). Similar to C^2 functions, for which $\mathbb{D}$ coincides with the usual Laplacian, one obtains that λ is subharmonic on Ω if and only if $\mathbb{D}\lambda(t) \ge 0$; hence, at the points t_0 of local maximuma of λ with $\lambda(t_0) > -\infty$, we have $\mathbb{D}\lambda(t_0) \le 0$.

The **sectional holomorphic curvature** of a Finsler metric on a complex Banach manifold X is defined in a similar way as the supremum of the curvatures over appropriate collections of holomorphic maps from the disk into X for a given tangent direction in the image.

The holomorphic curvature of the Kobayashi metric $\mathcal{K}(x,v)$ of any complete hyperbolic manifold X satisfies $\kappa_{\mathcal{K}_X} \geq -4$ at all points (x,v) of the tangent bundle $\mathcal{T}(X)$ of X, and for the Carathéodory metric $\mathcal{C}_X$ we have $\kappa_{\mathcal{C}}(x,v) \leq -4$.

Finally, the **pluricomplex Green function** of a domain X on a complex Banach space manifold E is defined as $g_X(x,y) = \sup u_y(x)$ $(x, y \in X)$, where supremum is taken over all plurisubharmonic functions $u_y(x) : \ X \to [-\infty, 0)$ satisfying $u_y(x) = \log \|x-y\| + O(1)$ in a neighborhood of the pole y. Here $\|\cdot\|$ is the norm on X and the remainder term $O(1)$ is bounded from above. If X is hyperbolic and its Kobayashi metric d_X is logarithmically plurisubharmonic, then $g_X(x,y) = \log\tanh \ d_X(x,y)$, which yields the representation of $g_{\mathbf{T}}$ in Theorem 3.

For details and general properties of invariant metrics, we refer to [15,32] (see also [1,42]).

Theorem 3 has various applications in geometric function theory and in complex geometry Teichmüller spaces. Its proof involves the technique of the Grunsky coefficient inequalities.

Plurisubharmonicity of a function $u(x)$ on a domain D in a Banach space X means that $u(x)$ is upper continuous in D and its restriction to the intersection of D with any complex line L is subharmonic.

A deep Zhuravlev's theorem implies that the intersection of the universal Teichmüller space $\mathbf{T}$ with every complex line is a union of simply connected planar (moreover, this holds for any Teichmüller space); see, [53, p. 75–82], [104].

1.6. The Grunsky–Milin inequalities revised

Denote by $\Sigma^0(D^*)$ the subclass of $\Sigma(D^*)$ formed by univalent $\widehat{\mathbb{C}}$-holomorphic functions in D^* with expansions $f(z) = z + b_0 + b_1 z^{-1} + \dots$ near $z = \infty$ admitting quasiconformal extensions to $\widehat{\mathbb{C}}$. It is dense in $\Sigma(D^*)$ in the weak topology of locally uniform convergence on D^*

Each Grunsky coefficient $\alpha_{mn}(f)$ is a polynomial of a finite number of the initial coefficients $b_1, b_2, \dots, b_{m+n-1}$ of f; hence it depends holomorphically on Beltrami coefficients of extensions of f as well as on the Schwarzian derivatives $S_f \in \mathbf{B}_2(D^*)$.

Consider the set

$$A_1^2(D) = \{\psi \in A_1(D) : \ \psi = \omega^2\}$$

consisting of the integrable holomorphic quadratic differentials on D having only zeros of even order and put

$$\alpha_D(f) = \sup \, \{|\langle \mu_0, \psi \rangle_D| : \ \psi \in A_1^2, \ \|\psi\|_{A_1(D)} = 1\}.$$

The following theorem from [47] completely describes the relation between the Grunsky and Teichmüller norms (more special results were obtained in [35, 54]).

Theorem 4. *For all $f \in \Sigma^0(D^*)$,*

$$\varkappa_{D^*}(f) \le k \frac{k + \alpha_D(f)}{1 + \alpha_D(f) k}, \quad k = k(f),$$

and $\varkappa_{D^*}(f) < k$ unless

$$\alpha_D(f) = \|\mu_0\|_\infty, \tag{16}$$

where μ_0 is an extremal Beltrami coefficient in the equivalence class $[f]$. The last equality is equivalent to $\varkappa_{D^*}(f) = k(f)$.

If $\varkappa(f) = k(f)$ and the equivalence class of f (the collection of maps equal to f on $S^1 = \partial D^*$) is a Strebel point, then the extremal μ_0 in this class is necessarily of the form

$$\mu_0 = \|\mu_0\|_\infty |\psi_0|/\psi_0 \quad \text{with} \quad \psi_0 \in A_1^2(D). \tag{17}$$

Note that geometrically (16) means the equality of the Carathéodory and Teichmüller distances on the geodesic disk $\{\phi_{\mathbf{T}}(t\mu_0/\|\mu_0\|) : t \in \mathbb{D}\}$ in the universal Teichmüller space $\mathbf{T}$ and that the mentioned above the strict inequality $\varkappa(f) < k(f)$ is valid on the (open) dense subset of Σ^0 in both strong and weak topologies (i.e., in the Teichmüller distance and in locally uniform convergence on D^*).

An important property of the Grunsky coefficients $\alpha_{mn}(f) = \alpha_{mn}(S_F)$ is that these coefficients are holomorphic functions of the Schwarzians $\varphi = S_f$ on the universal Teichmüller space $\mathbf{T}$. Therefore, for every $f \in \Sigma^0$ and each $\mathbf{x} = (x_n) \in S(l^2)$, the series

$$h_{\mathbf{x}}(\varphi) = \sum_{m,n=1}^{\infty} \sqrt{mn}\, \alpha_{mn}(\varphi) x_m x_n \tag{18}$$

defines a holomorphic map of the space $\mathbf{T}$ into the unit disk $\mathbb{D}$, and $\varkappa_{D^*}(F) = \sup_{\mathbf{x}} |h_{\mathbf{x}}(S_F)|$.

The convergence and holomorphy of the series (18) simply follow from the inequalities

$$\Big| \sum_{m=j}^{M} \sum_{n=l}^{N} \sqrt{mn}\, \alpha_{mn} x_m x_n \Big|^2 \le \sum_{m=j}^{M} |x_m|^2 \sum_{n=l}^{N} |x_n|^2$$

(for any finite M, N) which, in turn, are a consequence of the classical area theorem (see, e.g., [80, p. 61], [72, p. 193]).

Using Parseval's equality, one obtains that the elements of the distinguished set $A_1^2(\mathbb{D})$ are represented in the form

$$\psi(z) = \frac{1}{\pi} \sum_{m+n=2}^{\infty} \sqrt{mn}\; x_m x_n z^{m+n-2} \tag{19}$$

with $\mathbf{x} = (x_n) \in l^2$ so that $\|\mathbf{x}\|_{l^2} = \|\psi\|_{A_1}$ (see [35]). This result extends to arbitrary domains D with quasiconformal boundaries but the proof is much more complicated (see [46]).

Let us mention also that a new model of the universal Teichmüller space using the Grunsky coefficients has been constructed in [48].

1.7. The first Fredholm eigenvalue and Grunsky norm

One of the basic tools in quantitative estimating the Freholm eigenvalues ρ_L of quasicircles is given by the classical Kühnau-Schiffer theorem mentioned above. This theorem states that *the value ρ_L is reciprocal to the Grunsky norm $\varkappa(f)$ of the Riemann mapping function of the exterior domain of L* (see. [55, 85]).

Another important tool is the following Kühnau's jump inequality [57]:

If a closed curve $L \subset \widehat{\mathbb{C}}$ contains two analytic arcs with the interior intersection angle $\pi\alpha'$, then

$$\frac{1}{\rho_L} \geq |1 - |\alpha'||. \tag{20}$$

This implies the lower estimate for q_L and $1/\rho_L$. By approximation, this inequality extends to smooth arcs.

One of the standard ways of establishing the reflection coefficients q_L (respectively, the Fredholm eigenvalues ρ_L) consists of verifying wether the equality in (5) or the equality $\varkappa(f^*) = k_0(f^*)$ hold for a given curve L (cf. [35, 56–58, 103]).

This was an open problem a long time even for the rectangles stated by R. Kühnau, after it was established only [57], [103] that the answer is in affirmative for the square and for close rectangles $\mathcal{R}$ whose moduli $m(\mathcal{R})$ vary in the interval $1 \leq m(\mathcal{R}) < 1.037$; moreover, in this case $q_L = 1/\rho_L = 1/2$. The method exploited relied on an explicit construction of an extremal reflection. The complete answer was given in [41].

The relation between the basic curvelinear functionals intrinsically connected with conformal and quasiconformal maps is described in Kühau's paper [64].

1.8. Holomorphic motions

Let E be a subset of $\widehat{\mathbb{C}}$ containing at least three points.

A **holomorphic motion** of E is a function $f : E \times \mathbb{D} \to \widehat{\mathbb{C}}$ such that:

(a) for every fixed $z \in E$, the function $t \mapsto f(z,t) : E \times \mathbb{D} \to \widehat{\mathbb{C}}$ is holomorphic in $\mathbb{D}$;

(b) for every fixed $t \in \mathbb{D}$, the map $f(z,t) = f_t(z) : E \to \widehat{\mathbb{C}}$ is injective;

(c) $f(z,0) = z$ for all $z \in E$.

The remarkable lambda-lemma of Mañé, Sad and Sullivan [69] yields that such holomorphic dependence on the time parameter provides quasiconformality of f in the space parameter z. Moreover: *(i) f extends to a holomorphic motion of the closure $\overline{E}$ of E;*
(ii) each $f_t(z) = f(t,z) : \overline{E} \to \widehat{\mathbb{C}}$ is quasiconformal; (iii) f is jointly continuous in (z,t).

Quasiconformality here means, in the general case, the boundedness of the distortion of the circles centered at the points $z \in E$ or of the cross-ratios of the ordered quadruples of points of E.

The Slodkowski lifting theorem ([89], see also [8, 14, 16]) solves the problem of Sullivan and Thurston on the extension of holomorphic motions from any set to a whole sphere:

Extended lambda-lemma. *Any holomorphic motion $f : E \times \mathbb{D} \to \widehat{\mathbb{C}}$ can be extended to a holomorphic motion $\widetilde{f} : \widehat{\mathbb{C}} \times \mathbb{D} \to \widehat{\mathbb{C}}$, with $\widetilde{f}|E \times \mathbb{D} = f$.*

The corresponding Beltrami differentials $\mu_{\widetilde{f}_t}(z) = \partial_{\bar{z}} \widetilde{f}(z,t)/\partial_z \widetilde{f}(z,t)$ are holomorphic in t via elements of $L_\infty(\mathbb{C})$, and Schwarz's lemma yields

$$\|\mu_{\widetilde{f}_t}\|_\infty \le |t|,$$

or equivalently, the maximal dilatations $K(\widetilde{f}_t) \le (1+|t|)/(1-|t|)$. This bound cannot be improved in the general case.

Holomorphic motions have been important in the study of dynamical systems, Kleinian groups, holomorphic families of conformal maps and of Riemann surfaces as well as in many other fields (see, e.g., [8, 14, 18, 43, 69, 74–76, 88, 94, 95], and the references there.

There is an intrinsic connection between holomorphic motions and Teichmüller spaces, first mentioned by Bers and Royden in [10]. McMullen and Sullivan introduced in [76] the Teichmüller spaces for arbitrary holomorphic dynamical systems, and this approach is now one of the basic in complex dynamics.

2. Unbounded convex polygons

2.1. Main theorem

The inequalities (5), (20) have served a long time as the main tool for establishing the exact or approximate values of the Fredholm value ρ_L and allowed to establish it only for some special collections of curves and arcs.

In this section, we present, following [41, 48], a new method which enables us to solve the indicated problems for large classes of convex domains and of their fractional linear images. This method involves in an essential way the complex geometry of the universal Teichmüller space $\mathbf{T}$ and the Finsler metrics on holomorphic disks in $\mathbf{T}$ as well as the properties of holomorphic motions on such disks.

It is based on the following general theorem for unbounded convex domains giving an explicit representation of the main associated curvelinear and analytic functionals invariants by geometric characteristics of these domains solving the problem for unbounded convex domains completely.

Theorem 5. *For every unbounded convex domain $D \subset \mathbb{C}$ with piecewise $C^{1+\delta}$-smooth boundary L ($\delta > 0$) (and all its fractional linear images), we have the equalities*

$$q_L = 1/\rho_L = \varkappa(f) = \varkappa(f^*) = k_0(f) = k_0(f^*) = 1 - |\alpha|, \tag{21}$$

where f and f^* denote the appropriately normalized conformal maps $\mathbb{D} \to D$ and $\mathbb{D}^* \to D^* = \widehat{\mathbb{C}} \setminus \overline{D}$, respectively, $k_0(f)$ and $k_0(f^*)$ are the minimal dilatations of their quasiconformal extensions to $\widehat{\mathbb{C}}$; $\varkappa(f)$ and $\varkappa(f^*)$ stand for their Grunsky norms, and $\pi|\alpha|$ is the opening of the least interior angle between the boundary arcs $L_j \subset L$. Here $0 < \alpha < 1$ if the corresponding vertex is finite and $-1 < \alpha < 0$ for the angle at the vertex at infinity.

The same is true also for the unbounded concave domains (the complements of convex ones) which do not contain ∞; for those one must replace the last term by $|\beta| - 1$, where $\pi|\beta|$ is the opening of the largest interior angle of D.

The proof of Theorem 5 is outlined in [41], [43]. In the next section we provide an extension of this important theorem to nonconvex polygons giving the detailed proof.

The equalities of type (21) were known earlier only for special closed curves (see [54,57,61,103]), for example, for polygons bounded by circular arcs with a common inner tangent circle. The proof of Theorem 5 involves a completely different approach; it relies on the properties of holomorphic motions.

Let us mention also that the geometric assumptions of Theorem 5 are applied in the proof in an essential way. Its assertion extends neither to the arbitrary unbounded nonconvex or nonconcave domains nor to the arbitrary bounded convex domains.

This theorem has various important consequences. It distinguishes a broad class of domains, whose geometric properties provide the explicit values of intrinsic conformal and quasiconformal characteristics of these domains.

2.2. Examples

1. Let L be a closed unbounded curve with the convex interior which is $C^{1+\delta}$ smooth at all finite points and has at infinity the asymptotes approaching the interior angle $\pi\alpha < 0$. For any such curve, Theorem 5 yields the equalities

$$q_L = 1/\rho_L = 1 - |\alpha|. \tag{22}$$

2. More generally, assume that L also has a finite angle point z_0 with the angle opening $\pi\alpha_0$. Then, similar to (22),

$$q_L = 1/\rho_L = \max(1 - |\alpha_0|, 1 - |\alpha_\infty|).$$

Simultaneously this quantity gives the exact value of the reflection coefficient for any convex curvelinear lune bounded by two smooth arcs with the common endpoints a, b, because any such lune is a Moebius image of the exterior domain for the above curve L.

Other quantitative examples illustrating Theorem 5 are presented in [43].

3. Extension to unbounded non-convex polygons

3.1. An open question

An open question is to establish the extent in which Theorem 5 can be prolonged to arbitrary unbounded polygons.

Our goal is to show that this is possible for unbounded rectilinear polygons for which the extent of deviation from convexity is sufficiently small.

This extension essentially increases the collections of individual polygonal curves and arcs with explicitly established Fredholm eigenvalues and reflection coefficients.

3.2. Main theorem

Let P_n be a rectilinear polygon with the finite vertices $A_1, A_2, \dots, A_{n-1}$ and with vertex $A_\infty = \infty$, and let the interior angle at the vertex A_j be equal to $\pi\alpha_j$ and at A_∞ be equal to $\pi\alpha_\infty$, where $\alpha_\infty < 0$ and all $a_j \neq 1$, so that $\alpha_1 + \dots + \alpha_{n-1} + \alpha_\infty = 2$. Let f_n be the conformal map of the upper half-plane $U = \{z : \Im z > 0\}$ onto P_n which without loss of generality, can be normalized by $f_n(z) = z - i + O(z - i)$ as $z \to i$ (assuming that P_n contains the origin $w = 0$).

An important geometric characteristic of polynomials is the quantity

$$|1 - |\alpha|| = \max\{|1 - |\alpha_1||, \dots, |1 - |\alpha_{n-1}||,\ |1 - |\alpha_\infty||\}; \tag{23}$$

it valuates the local boundary quasiconformal dilatation of P_n.

Using this quantity, we first prove that an assertion similar to Theorem 5 fails for the generic rectilinear polygons.

Theorem 6. *There exist rectilinear polygons P_n whose conformal mapping functions f_n satisfy*

$$\varkappa(f_n) = k(f_n) > |1 - |a||, \tag{24}$$

where a is defined via (23).

Proof. We shall use the rectangles P_4; in this case all $\alpha_j = 1/2$. It is known that the mapping function f_4 of any rectangle has equal Grunsky and Teichmüller norms,

$$\varkappa(f_4) = k(f_4)$$

(see [40, 57, 103]).

Using the Moebius map $\sigma :\ z \mapsto 1/z$, we transform the rectangle into a (nonconvex) circular quadrilateral $\sigma(P_4)$ with angles $\pi/2$ and mutually orthogonal edges so that two unbounded edges from these are rectilinear and two bounded are circular, and note that for sufficiently long rectangles must be

$$k(\widehat{f_4}) = \varkappa(\widehat{f_4}) = 1/\rho_{\sigma(\partial P_4)} > 1/2, \tag{25}$$

where $\widehat{f_4}$ denotes the conformal map $\mathbb{D} \to \sigma(P_4)$.

Indeed, as was established by Kühnau [57]), the quadrilaterals with the side ratios (conformal module) greater than 3.31 have the reflection coefficient $q_{\partial P_4} > 1/2$ (the last inequality follows also from the fact that the long rectangles give in the limit a half-strip with two unbounded parallel sides. Such a domain is not a quasidisk, so its reflection coefficient equals 1); this proves (25).

Any circular quadrilateral $\sigma(P_4)$ satisfying (25) can be approximated by appropriate rectilinear polygons P_n. Assuming now that the equalities of type (21) or (24) are valid for all such polygons, one obtains a contradiction with (25), because both dilatation $k(f)$ and $q_{\partial P}$ are lower continuous functionals under locally uniform convergence of quasiconformal maps (i.e., $k(f) \le \liminf k(f_n)$ as $f_n \to f$ in the indicated topology, and similarly for the reflection coefficient). This contradiction proves the theorem. □

For the indicated polygons, we also have the strict inequality $1/\rho_{\partial P_n} < |1 - |a||$ giving only a lower bound for $\rho_{\partial P_n}$.

3.3. The main result of this section

The main result of this section is

Theorem 7. [49] *Let P_n be a unbounded rectilinear polygon, neither convex nor concave, and hence contain the vertices A_j whose inner angles $\pi\alpha_j$ have openings $\pi\alpha_j$ with $1 < \alpha_j < 2$. Assume that all such α_j satisfy*

$$\alpha_j - 1 < |1 - |\alpha||, \tag{26}$$

where α is given by (23) (which means that the maximal value in (23) is attained at some vertex A_j with $0 < |a_j| < 1$).

For any such polygon, taking appropriate Moebius map $\sigma : \mathbb{D} \to U$, we have the equalities

$$\varkappa(f_n \circ \sigma) = k(f_n) = q_{\partial P_n} = 1/\rho_{\partial P_n} = |1 - |\alpha||. \tag{27}$$

Proof. Let P_n be an unbounded rectilinear polygon. Its conformal mapping function $f_n : U \to P_n$ fixing the infinite point and with $f_n(i) = 0$ is represented by the Schwarz–Christoffel integral

$$f_n(\zeta) = d_1 \int\limits_0^z (\xi - a_1)^{\alpha_1 - 1} \dots (\xi - a_{n-1})^{\alpha_{n-1} - 1} d\xi + d_0, \tag{28}$$

where all $a_j = f_*^{-1}(A_j) \in \mathbb{R}$ and d_0, d_1 are the corresponding complex constants. The logarithmic derivative $b_f = (\log f')' = f''/f'$ of this map has the form

$$b_{f_n}(z) = \sum_1^{n-1} (\alpha_j - 1)(z - a_j)^{-1}.$$

Letting $I_\alpha = \{t \in \mathbb{R} : \ -1/|1-|\alpha|| < t < 1/|1-|\alpha||\}$, $\mathbb{D}_\alpha = \{t \in \mathbb{C} : \ |t| < 1/|1-|\alpha||\}$, we construct for f_n an ambient complex isotopy (holomorpic motion)

$$w(z,t) : \ U \times \mathbb{D}_\alpha \to \widehat{\mathbb{C}}, \tag{29}$$

(containing f_n as a fiber map), which is injective in the space coordinate z for any fixed t, holomorphic in t for a fixed z and $w(z,0) \equiv z$.

First observe that for real $r \in I_\alpha$ the solution W_r to the equation $w''(z) = rb_{f_4}(z)w'(z)$ with the initial conditions $w_r(i) = i, \ w_r(\infty) = \infty$ satisfies

$$b_{W_r}(z) = \sum_1^{n-1} r\frac{\alpha_j - 1}{z - a_j} = \sum_1^{n} \frac{\alpha_j(r) - 1}{z - a_j},$$

where

$$\alpha_j(r) = r(\alpha_j - 1) + 1. \tag{30}$$

If the interior angles of the initial polygon P_n satisfy the assumption (26), then all the functions W_r are represented by an integral of type (27) (replacing α_j by $\alpha_j(r)$, and with suitable constants d_{0r}, d_{1r}).

Geometrically this means that the exterior angle $2\pi - \pi\alpha_j(r)$ at any finite vertex $A_j(r)$ decreases with r (but the value $\alpha_j(r) - 1$ increases if $1 < \alpha_j < 2$). Under the assumption (26), the admissible bounds for the possible values of angles ensure the univalence of this integral on U for every indicated t. This yields that every $W_r(U)$ also is a polygon with the interior angles $\pi\alpha_j(r)$ for $r \neq 0$, while $W_0(U) = U$.

Now we pass to the conformal map $g_n(\zeta) = f_n \circ \sigma_0(\zeta)$ of the unit disk $\mathbb{D}$ onto P_n, using the function $\sigma_0(\zeta) = (1+\zeta)/(1-\zeta)$. This map is represented similar to (28) by

$$g_n(\zeta) = d_1 \int_0^\zeta \prod_1^n (\xi - e_j)^{\alpha_j - 1} d\xi + d_0,$$

where the points e_j are the preimages of vertices $e_j = g_n^{-1}(A_j)$ on the unit circle $\{|\zeta| = 1\}$. Pick d_1 to have $g_n'(0) = 1$. For this function, we have a natural complex isotopy

$$\widetilde{w}_t(\zeta) = \frac{1}{t} g_n(t\zeta) : \ \mathbb{D} \times \mathbb{D} \to \mathbb{C}, \tag{31}$$

with

$$b_{\widetilde{w_t}}(\zeta) = \frac{\widetilde{w}_t''(\zeta)}{\widetilde{w}_t'(\zeta)} = t\frac{g_n''(t\zeta)}{g_n'(t\zeta)} = tb_{g_n}(t\zeta). \tag{32}$$

Following (31), we set for $t = re^{i\theta}$,

$$\widetilde{w}_t(\zeta) = e^{-i\theta} W_r \circ \sigma_0(e^{i\theta}\zeta).$$

The relations (32) yield that this function also is univalent in $\mathbb{D}$.

The corresponding Schwarzians $S_{\widetilde{w}_r}(\zeta) = rb'_{\widetilde{w}_r}(\zeta) - r^2 b_{\widetilde{w}_r}(\zeta)^2/2$ fill a real analytic line Γ in the universal Teichmüller space $\mathbf{T}$ (modeled as a bounded domain in the complex Banach space $\mathbf{B}$ of hyperbolically bounded holomorphic functions on $\mathbb{D}$). This line is located in the holomorphic disk $\widetilde{\Omega} = \mathbf{b}(G) \subset \mathbf{T}$, where $\mathbf{b}$ denotes the map $t \mapsto S_{\widetilde{w}_t}$ and $G \supset I_\alpha$ is a simply connected planar domain.

By Zhuravlev's theorem (see [51,104]), this domain contains for each $r \in I_\alpha$ also the points $S_{\widetilde{w}_t}$ with $|t| \le r$ (representing the curvelinear polygons with piecewise analytic boundaries).

This generates the holomorphic motions (complex isotopies) $\widetilde{w}(\zeta, t) : \mathbb{D} \times G \to \widehat{\mathbb{C}}$ and $w(z,t) : U \to \widehat{\mathbb{C}}$ with $w(z,1) = f_n(z)$.

The basic lambda-lemma for holomorphic motions implies that every fiber map $w_t(z)$ is the restriction to U of a quasiconformal automorphism $\widetilde{W}_t(z)$ of the whole sphere $\widehat{\mathbb{C}}$, and the Beltrami coefficients

$$\mu(z,t) = \partial_{\overline{z}} \widehat{W}_t(z) / \partial_z \widehat{W}_t(z), \quad t \in \mathbb{D}_\alpha,$$

in the lower half-plane $U^* = \{z : \ \Im z < 0\}$ depend holomorphically on t as elements of the space $L_\infty(U^*)$.

So we have a holomorphic map $\mu(\cdot, t)$ from the disk $\mathbb{D}_\alpha$ into the unit ball of Beltrami coefficients supported on U^*,

$$\mathrm{Belt}(U^*)_1 = \{\mu \in L_\infty(\mathbb{C}) : \quad \mu(z)|U = 0, \quad \|\mu\| < 1\},$$

and the classical Schwarz lemma implies the estimate

$$k(\widehat{W}_t) = \|\mu_{\widehat{W}_t}\|_\infty \le |1 - |\alpha|||t|.$$

It follows that the extremal dilatation of the initial map $f_n(z) = \widehat{W}_1(z)|U$ satisfies

$$k(f_n) \le |1 - |\alpha||.$$

Hence, also $q_{\partial P_n} \le |1 - |\alpha||$ and by the inequality (10), $\varkappa(f_n) \le |1 - |\alpha||$.

On the other hand, Kühnau's lower bound (20) implies

$$\frac{1}{\rho_{\partial P_n}} \ge |1 - |\alpha||.$$

Together with (5), this yields that the polygon P_n admits all equalities (27) completing the proof of the theorem. □

3.4. Some applications

Theorem 7 widens the collections of curves with explicitly given Fredholm eigenvalues and reflection coefficients.

For example, let L be a saw-tooth quasicircle with a finite number of triangular and trapezoidal teeth joined by rectilinear segments. We assume that the angles of these teeth satisfy the condition (26). Then we have the following consequence of Theorem 7.

Corollary 1. *For any quasicircle L of the indicated form, its quasireflection coefficient q_L and Fredholm eigenvalue ρ_L are given by*

$$q_L = 1/\rho_L = |1 - |a||,$$

where $|\alpha|$ is defined similar to (23) by angles between the subintervals of L. The same is valid for images $\gamma(L)$ under the Moebius maps $\gamma \in PSL(2, \mathbb{C})$.

4. Connection with complex geometry of universal Teichmüller space

4.1. Another reason why the convex polygons are interesting for quasiconformal theory is their close geometric connection with the geometry of universal Teichmüller space.

1. Introductory remarks. There is an interesting still unsolved completely question on shape of holomorphic embeddings of Teichmüller spaces stated in [9]:

For an arbitrary finitely or infinitely generated Fuchsian group Γ is the Bers embedding of its Teichmüller space $\mathbf{T}(\Gamma)$ starlike?

Recall that in this embedding $\mathbf{T}(\Gamma)$ is represented as a bounded domain formed by the Schwarzian derivatives S_w of holomorphic univalent functions $w(z)$ in the lower half-plane $U^* = \{z : \Im z < 0\})$ (or in the disk) admitting quasiconformal extensions to the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ compatible with the group Γ acting on U^*.

It was shown in [36] that universal Teichmüller space $\mathbf{T} = \mathbf{T}(\mathbf{1})$ has points which cannot be joined to a distinguished point even by curves of a considerably general form, in particular, by polygonal lines with the same finite number of rectilinear segments. The proof relies on the existence of conformally rigid domains established by Thurston in [101] (see also [7]).

This implies, in particular, that universal Teichmüller space is not starlike with respect to any of its points, and there exist points $\varphi \in \mathbf{T}$ for which the line interval $\{t\varphi : 0 < t < 1\}$ contains the points from

$\mathbf{B} \setminus \mathbf{S}$, where $\mathbf{B} = \mathbf{B}(U^*)$ is the Banach space of hyperbolically bounded holomorphic functions in the half-plane U^* with norm

$$\|\varphi\|_{\mathbf{B}} = 4 \sup_{U^*} y^2 |\varphi(z)|$$

and $\mathbf{S}$ denotes the set of all Schwarzian derivatives of univalent functions on U^*. These points correspond to holomorphic functions on U^* which are only locally univalent.

Toki [102] extended the result on the nonstarlikeness of the space $\mathbf{T}$ to Teichmüller spaces of Riemann surfaces that contain hyperbolic disks of arbitrary large radius, in particular, for the spaces corresponding to Fuchsian groups of second kind. The crucial point in the proof of [102] is the same as in [36]

On the other hand, it was established in [37] that also all finite dimensional Teichmüller spaces $\mathbf{T}(\Gamma)$ of high enough dimensions are not starlike.

The nonstarlikeness causes obstructions to some problems in the Teichmüller space theory and its applications to geometric complex analysis.

The argument exploited in the proof of Theorems 5 and 7 provide much simpler constructive proof that the universal Teichmüller space is not starlike, representing explicitly the functions which violate this property. It reveals completely different underlying geometric features.

Pick unbounded convex rectilinear polygon P_n with finite vertices $A_1, \dots, A_{n-1}$ and $A_n = \infty$. Denote the exterior angles at A_j by $\pi\alpha_j$ so that $\pi < \alpha_j < 2\pi$, $j = 1, \dots, n-1$. Then, similar to (28), the conformal map f_n of the lower half-plane $H^* = \{z : \Im z < 0\}$ onto the complementary polygon $P_n^* = \widehat{\mathbb{C}} \setminus \overline{P_n}$ is represented by the Schwarz–Christoffel integral

$$f_n(z) = d_1 \int\limits_0^z (\xi - a_1)^{\alpha_1 - 1} (\xi - a_2)^{\alpha_2 - 1} \dots (\xi - a_{n-1})^{\alpha_{n-1} - 1} d\xi + d_0,$$

with $a_j = f_n^{-1}(A_j) \in \mathbb{R}$ and complex constants d_0, d_1; here $f_n^{-1}(\infty) = \infty$. Its Schwarzian derivative is given by

$$S_{f_n}(z) = b'_{f_n}(z) - \frac{1}{2} b^2_{f_n}(z) = \sum_1^{n-1} \frac{C_j}{(z - a_j)^2} - \sum_{j,l=1}^{n-1} \frac{C_{jl}}{(z - a_j)(z - a_l)}, \quad (33)$$

where $b_f = f''/f'$ and

$$C_j = -(\alpha_j - 1) - (\alpha_j - 1)^2/2 < 0, \quad C_{jl} = (\alpha_j - 1)(\alpha_l - 1) > 0.$$

It defines a point of the universal Teichmüller space $\mathbf{T}$ modeled as a bounded domain in the space $\mathbf{B}(H^*)$ of hyperbolically bounded holomorphic functions on H^* with norm $\|\varphi\|_{\mathbf{B}(H^*)} = \sup_{H^*} |z - \overline{z}|^2 |\varphi(z)|$.

Denote by r_0 the positive root of the equation

$$\frac{1}{2}\Big[\sum_1^{n-1}(\alpha_j - 1)^2 + \sum_{j,l=1}^{n-1}(\alpha_j - 1)(\alpha_l - 1)\Big] r^2 - \sum_1^{n-1}(\alpha_j - 1)\ r - 2 = 0,$$

and put $S_{f_n,t} = t b'_{f_n} - b^2_{f_n}/2,\ t > 0$. Then for appropriate α_j, we have

Theorem 8. [45] *For any convex polygon P_n, the Schwarzians rS_{f_n,r_0} define for any $0 < r < r_0$ a univalent function $w_r : H^* \to \mathbb{C}$ whose harmonic Beltrami coefficient $\nu_r(z) = -(r/2)y^2 S_{f_n,r_0}(\overline{z})$ in H is extremal in its equivalence class, and*

$$k(w_r) = \varkappa(w_r) = \frac{r}{2}\|S_{f_n,r_0}\|_{\mathbf{B}(H^*)}. \tag{34}$$

By the Ahlfors–Weill theorem [6], every $\varphi \in \mathbf{B}(H^*)$ with $\|\varphi\|_{\mathbf{B}(H^*)} < 1/2$ is the Schwarzian derivative S_W of a univalent function W in H^*, and this function has quasiconformal extension onto the upper half-plane $H = \{z :\ \Im z > 0\}$ with Beltrami coefficient of the form

$$\mu_\varphi(z) = -2y^2\varphi(\overline{z}), \quad \varphi = S_f\ (z = x + iy \in H^*)$$

called harmonic. Theorem 8 yields that any w_r with $r < r_0$ does not admit extremal quasiconformal extensions of Teichmüller type, and in view of extremality of harmonic coefficients $\mu_{S_{w_r}}$ the Schwarzians S_{w_r} for some r between r_0 and 1 must lie outside of the space $\mathbf{T}$; so this space is not a starlike domain in $\mathbf{B}(H^*)$.

4.2. There are unbounded convex polygons P_n for which the equalities (34) are valid in the strengthened form

$$k(f_n) = \varkappa(f_n) = \frac{1}{2}\|S_{f_n}\|_{\mathbf{B}(H^*)} \tag{35}$$

for all $r \le 1$, completing the bounds (21).

We illustrate this on the case of triangles. Let P_3 be a triangle with vertices $A_1, A_2 \in \mathbb{R}$ and $A_3 = \infty$ and exterior angles $\alpha_1,\ \alpha_2, \alpha_3$. The logarithmic derivative of conformal map $f_3 :\ H^* \to P_3^*$ has the form

$$b_{f_3}(z) = \frac{\alpha_1 - 1}{z - a_1} + \frac{\alpha_2 - 1}{z - a_2}$$

with $a_j = f_3^{-1}(A_j) \in \mathbb{R},\ j = 1, 2$, and similar to (34),

$$S_{f_3}(z) = \frac{C_1}{(z - a_1)^2} + \frac{C_2}{(z - a_2)^2} - \frac{C_{12}}{(z - a_1)(z - a_2)}$$

with

$$C_j = -(\alpha_j - 1) - \frac{1}{2}(\alpha_j - 1)^2 = -\frac{\alpha_j^2 + 1}{2} < 0, \quad j = 1, 2;$$

$$C_{12} = (\alpha_1 - 1)(\alpha_2 - 1) > 0.$$

If the angles of P_3^* satisfy $\alpha_1, \alpha_2 < |a_3|$, where $-\pi\alpha_3$ is the angle at A_3, the arguments from [45] yield that the harmonic Beltrami coefficient $\mu_{S_{f_3}}$ satisfies (35).

4.3. In fact, the requirement of convexity for P_n can be weakened, as it follows from the results of [50].

Surprisingly, this construction is closely connected also with the weighted bounded rational approximation in sup norm.

5. Quasiconformal features and fredholm eigenvalues of bounded convex pokygons

5.1. Affine deformations and Grunsky norm

As it was mentioned above, there exist bounded convex domains even with analytic boundaries L whose conformal mapping functions have different Grunsky and Teichmüller norms, and therefore, $\rho_L < 1/q_L$.

The aim of this chapter is to provide the classes of bounded convex domains, especially polygons, for which these norms are equal and give explicitly the values of the associate curve functionals $k(f)$, $\varkappa(f)$, q_L, ρ_L.

One of the interesting questions is *whether the equality of Teichmüller and Grunsky norms is preserved under the affine deformations*

$$g^c(w) = c_1 w + c_2 \overline{w} + c_3$$

with $c = c_2/c_1$, $|c| < 1$ (as well as of more general maps) of quasidisks.

In the case of unbounded convex domains, this follows from Theorem 5. We establish this here for bounded domains D.

More precisely, we consider the maps g^c, which are conformal in the complementary domain $D^* = \widehat{\mathbb{C}} \setminus \overline{D}$ and have in D a constant quasiconformal dilatation c, regarding such maps as the **affine deformations** and the collection of domains $g^c(D)$ as the affine class of D.

If f is a quasiconformal automorphism of $\widehat{\mathbb{C}}$ conformal in $\mathbb{D}^*$ mapping the disk $\mathbb{D}$ onto a domain D, then for a fixed c the maps $g^c|D \circ f$ and $(g^c \circ f)|\mathbb{D}$ differ by a conformal map $h: \ D \to g^c(D)$ and hence have in the disk $\mathbb{D}$ the same Beltrami coefficient.

Note that the inequality $|c| < 1$ equivalent to $|c_2| < |c_1|$ follows immediately from the orientation preserving under this map and its composition with conformal map by forming the corresponding affine deformation (which arises after extension the constant Beltrami coefficient c by zero to the complementary domain).

The following theorem solves the problem positively.

Theorem 9. *For any function $f \in \Sigma^0$ with $\varkappa(f) = k(f)$ mapping the disk $\mathbb{D}^*$ onto the complement of a bounded domain (quasidisk) D and any affine deformation g^c of this domain (with $|qc| < 1$), we have the equality*

$$\varkappa(g^c \circ f) = k(g^c \circ f). \tag{36}$$

Theorems 9 essentially increases the set of quasicircles $L \subset \widehat{\mathbb{C}}$ for which $\rho_L = 1/q_L$ giving simultaneously the explicit values of these curve functionals. Even for quadrilaterals, this fact was known until now only for some special types of them (for rectangles [41,55–57] and for rectilinear or circular quadrilaterals having a common tangent circle [103]).

5.2. Scheme of the proof of Theorem 9

The proof follows the lines of Theorem 1.1 in [44] and is divided into several lemmas.

First, we establish some auxiliary results characterizing the homotopy disk of a map with $\varkappa(f) = k(f)$.

Take the generic homotopy function

$$f_t(z) = tf(z/t) = z + b_0 t + b_1 t^2 z^{-1} + b_2 t^3 z^{-2} + \cdots : \ \mathbb{D}^* \times \mathbb{D} \to \widehat{\mathbb{C}}.$$

Then $S_{f_t}(z) = t^{-2} S_f(t^{-1} z)$ and this point-wise map determines a holomorphic map $\chi_f(t) = S_{f_t}(\cdot) : \ \mathbb{D} \to \mathbf{T}$ so that the homotopy disks $\mathbb{D}(S_f) = \chi_f(\mathbb{D})$ foliate the space $\mathbf{T}$. Note also that

$$\alpha_{mn}(f_t) = \alpha_{mn}(f) t^{m+n},$$

and if $F(z) = 1/f(1/z)$ maps the unit disk onto a convex domain, then all level lines $f(|z| = r)$ for $z \in \mathbb{D}^*$ are starlike.

Lemma 1. *If the homotopy function f_t of $f \in \Sigma^0$ satisfy $\varkappa(f_{t_0}) = k(f_{t_0})$ for some $0 < t_0 < 1$, then the equality $\varkappa(f_t) = k(f_t)$ holds for all $|t| \le t_0$ and the homotopy disk $\mathbb{D}(S_{f_t})$ has no critical points t with $0 < |t| < t_0$.*

Take the univalent extension f_1 of f to a maximal disk $\mathbb{D}_b^* = \{z \in \widehat{\mathbb{C}} : |z| > b\}, \ (0 < b < 1)$ and define

$$f^*(z) = b^{-1} f_1(bz) \in \Sigma^0, \quad |z| > 1.$$

Its Beltrami coefficient in $\mathbb{D}$ is defined by holomorphic quadratic differentials $\psi \in A_1^2$ of the form (19), and we have the holomorphic map, for a fixed $\mathbf{x}^b = (x_n^b) \in l^2$,

$$h_{\mathbf{x}^b}(S_{f_t^*}) = \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn}(f^*) x_m^b x_n^b (bt)^{m+n} \tag{37}$$

of the disk $\mathbb{D}(S_{f^*})$ into $\mathbb{D}$. In view of our assumption on f, the series (37) is convergent in some wider disk $\{|t| < a\}(a > 1)$.

Using the map (37), we pull back the hyperbolic metric $\lambda_{\mathbb{D}}(t) = |dt|/(1 - |t|^2)$ to the disk $\mathbb{D}(S_{F_1})$ (parametrized by t) and define on this disk the conformal metric $ds = \lambda_{\widetilde{h}_{\mathbf{x}}}(t)|dt|$ with

$$\lambda_{\widetilde{h}_{\mathbf{x}^b}}(t) = (h_{\mathbf{x}^a} \circ \chi_{f_1})^* \lambda_{\mathbb{D}} = \frac{|\widetilde{h}'_{\mathbf{x}^b}(t)||dt|}{1 - |\widetilde{h}_{\mathbf{x}^b}(t)|^2}. \tag{38}$$

of Gaussian curvature -4 at noncritical points. In fact, this is the supporting metric at $t = a$ for the upper envelope $\lambda_{\varkappa} = \sup_{\mathbf{x} \in S(l^2)} \lambda_{\widetilde{h}_{\mathbf{x}^b}}(t)$ of metrics (38) followed by its upper semicontinuous regularization

$$\lambda_{\varkappa}(t) \mapsto \lambda_{\varkappa}^*(t) = \limsup_{t' \to t} \lambda_{\varkappa}(t')$$

(supporting means that $\lambda_{\widetilde{h}_{\mathbf{x}^b}}(a) = \lambda_{\varkappa}(a)$ and $\lambda_{\widetilde{h}_{\mathbf{x}^b}}(t) < \lambda_{\varkappa}(t)$ in a neighborhood of a).

The metric $\lambda_{\varkappa}(t)$ is logarithmically subharmonic on $\mathbb{D}$ and its **generalized Laplacian**

$$\Delta u(t) = 4 \liminf_{r \to 0} \frac{1}{r^2} \Big\{ \frac{1}{2\pi} \int_0^{2\pi} u(t + re^{i\theta}) d\theta - \lambda(t) \Big\}$$

satisfies

$$\Delta \log \lambda_{\varkappa} \geq 4\lambda_{\varkappa}^2$$

(while for $\lambda_{\widetilde{h}_{\mathbf{x}^b}}$ we have at its noncritical points $\Delta \log \lambda_{\widetilde{h}_{\mathbf{x}^b}} = 4\lambda^2_{\widetilde{h}_{\mathbf{x}^b}}$).

As was mentioned above, the Grunsky coefficients define on the tangent bundle $\mathcal{T}(\mathbf{T})$ a new Finsler structure $F_{\varkappa}(\varphi, v)$ dominated by the infinitesimal Teichmüller metric $F(\varphi, v)$. This structure generates on any embedded holomorphic disk $\gamma(\mathbb{D}) \subset \mathbf{T}$ the corresponding Finsler metric $\lambda_\gamma(t) = F_{\varkappa}(\gamma(t), \gamma'(t))$ and reconstructs the Grunsky norm by integration along the Teichmüller disks:

Lemma 2. [44] *On any extremal Teichmüller disk $\mathbb{D}(\mu_0) = \{\phi_{\mathbf{T}}(t\mu_0) : t \in \mathbb{D}\}$ (and its isometric images in $\mathbf{T}$), we have the equality*

$$\tanh^{-1}[\varkappa(f^{r\mu_0})] = \int_0^r \lambda_{\varkappa}(t) dt.$$

Taking into account that the disk $\mathbb{D}(S_f)$ touches at the point $\varphi = S_{f_a}$ the Teichmüller disk centered at the origin of $\mathbf{T}$ and passing through this point and that the metric $\lambda_\varkappa$ does not depend on the tangent unit vectors whose initial points are the points of $\mathbb{D}(S_f)$, one obtains from Lemma 2 and the equality $\varkappa(f_a) = k(f_a)$ that also

$$\lambda_\varkappa(a) = \lambda_\mathcal{K}(a). \tag{39}$$

The following lemma is a needed reformulation of Theorem 3.

Lemma 3. [44] *The infinitesimal forms $\mathcal{K}_\mathbf{T}(\varphi, v)$ and $F_\mathbf{T}(\varphi, v)$ of both Kobayashi and Teichmüller metrics on the tangent bundle $\mathcal{T}(\mathbf{T})$ of $\mathbf{T}$ are continuous logarithmically plurisubharmonic in $\varphi \in \mathbf{T}$ and have constant holomorphic sectional curvature $\kappa_\mathcal{K}(\varphi, v) = -4$.*

We compare the metric $\lambda_{\widetilde{h}_{\mathbf{x}b}}$ with $\lambda_\mathcal{K}$ using Lemmas 2, 3 and Minda's maximum principle given by

Lemma 4. [72] *If a function $u: \ D \to [-\infty, +\infty)$ is upper semicontinuous in a domain $D \subset \mathbb{C}$ and its (generalized) Laplacian satisfies the inequality $\Delta u(z) \ge K u(z)$ with some positive constant K at any point $z \in D$, where $u(z) > -\infty$, and if*

$$\limsup_{z\to\zeta} u(z) \le 0 \quad \textit{for all } \zeta \in \partial D,$$

then either $u(z) < 0$ for all $z \in D$ or else $u(z) = 0$ for all $z \in \Omega$.

Lemma 4 and the equality (39) imply that the metrics $\lambda_{\widetilde{h}_{\mathbf{x}b}}, \lambda_\varkappa, \lambda_\mathcal{K}$ must be equal in the entire disk $\mathbb{D}(S_F)$, which yields by Lemma 2 the equality

$$\varkappa(f_r) = k(f_r) = \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn}(F_1) r^{m+n} x_m^r x_n^r \Big|$$

for all $r = |t| \in (0, 1)$ (with $(x_n^r) \in S(l^2)$ depending on r) and that for any $f \in \Sigma^0$ with $\varkappa(f) = k(f)$ *its homotopy disk $\mathbb{D}(S_F)$ has only a singularity at the origin of* $\mathbf{T}$.

We may now investigate the action of affine deformations on the set of functions $f \in \Sigma^0$ with equal Grunsky and Teichmüller norms.

Lemma 5. *For any affine deformation g^c of a convex domain D with expansion $g^c(w) = w + b_0^c + b_1^c w^{-1} + \dots$ near $w = \infty$, we have*

$$b_1^c = \frac{S_{g^c}(\infty)}{6} = \frac{1}{6} \lim_{z\to\infty} w^4 S_{g^c}(w) \ne 0,$$

and for sufficiently small $|c|$ *all composite maps*

$$W_{f,c}(z) = g^c \circ f(z) = z + \widehat{b}_0^c + \widehat{b}_1^c z^{-1} + \dots, \quad f \in \Sigma^0,$$

also satisfy $\widehat{b}_1^c \neq 0$.

Finally, we use the following important result of Kühnau [55].

Lemma 6. *For any function* $f(z) = z + b_0 + b_1 z^{-1} + \dots \in \Sigma^0$ *with* $b_1 \neq 0$, *the extremal quasiconformal extensions of the homotopy functions* f_t *to* $\mathbb{D}$ *are defined for sufficiently small* $|t| \leq r_0 = r_0(f)$ $(r_0 > 0)$ *by nonvanishing holomorphic quadratic differentials, and therefore,* $\varkappa(f_t) = k(f_t)$.

Using these lemmas, one establishes the equalities $\lambda_\varkappa = \lambda_\mathcal{K}$ on the disk $\mathbb{D}(S_{W_{f,c}})$ and

$$\varkappa(W_{F,c}) = k(W_{F,c}). \tag{40}$$

The final step of the proof is to extend the last equality to all c with $|c| < 1$.

Applying again the chain rule for Beltrami coefficients μ, ν from the unit ball in $L_\infty(\mathbb{C})$,

$$w^\mu \circ w^\nu = w^\tau \quad \text{with} \quad \tau = (\nu + \widetilde{\mu})/(1 + \overline{\nu}\widetilde{\mu})$$

and $\widetilde{\mu}(z) = \mu(w^\nu(z))\overline{w_z^\nu}/w_z^\nu$ (so for ν fixed, τ depends holomorphically on μ in L_∞ norm) and defining the corresponding functions (37), one gets now the holomorphic functions of $c \in \mathbb{D}$. Then, constructing in a similar way the corresponding Finsler metrics

$$\lambda_{\widetilde{h}_\mathbf{x}}(c) = |\widetilde{h}'_\mathbf{x}(c)||dc|/(1 - |\widetilde{h}_\mathbf{x}(c)|^2), \quad |c| < 1.$$

and taking their upper envelope $\lambda_\varkappa(c)$ and its upper semicontinuous regularization, one obtains a subharmonic metric of Gaussian curvature $\kappa_{\lambda_\varkappa} \leq -4$ on the nonsingular disk $\{|c| < 1\}$. One can repeat for this metric all the above arguments using the already established equality (40) for small $|c|$.

5.3. Generalization

The arguments in the proof of Theorem 9 are extended almost straightforwardly to more general case:

Theorem 10. *Let* $F \in \Sigma^0$ *and* $\varkappa(F) = k(F)$. *Let* h *be a holomorphic map* $\mathbb{D} \to \mathbf{T}$ *without critical points in* $\mathbb{D}$ *and* $h(0) = S_F$. *Denote by* $\mathbf{g}^c$ *the univalent solution of the Schwarzian equation*

$$S_\mathbf{g} = (h(c) \circ H)(H')^2 + S_H,$$

where $H(w) = F^{-1}(w)$, *on the domain* $F(\mathbb{D}^*)$. *Then, for any* $c \in \mathbb{D}$, *the composition* $\mathbf{g}^c \circ F$ *also satisfies* $\varkappa(\mathbf{g}^c \circ F) = k(\mathbf{g}^c \circ F)$.

Note that by the lambda lemma for holomorphic motions, the map h determines a holomorphic disk in the ball of Beltrami coefficients on $F(\mathbb{D})$, which yields, together with assumptions of the theorem, that for small $|c|$,

$$\mathbf{g}^c(w) = w + b_0^c + b_1^c w^{-1} + \dots \quad \text{as} \quad w \to \infty$$

with $b_1^c \neq 0$. This was an essential point in the proof.

5.4. Bounded polygons: a counterexample to Theorem 5

The case of bounded convex polygons has an intrinsic interest, in view of the following negative fact underlying the features and contrasting Theorem 5.

Theorem 11. *There exist bounded rectilinear convex polygons* P_n *with sufficiently large number of sides such that*

$$\rho_{\partial P_n} < 1/q_{\partial P_n}.$$

It follows simply from Theorem 9 that if a polygon P_n, whose edges are quasiconformal arcs, satisfies $\rho_{\partial P_n} = 1/q_{\partial P_n}$ then this equality is preserved for all its affine images. In particular, this is valid for all rectilinear polygons obtained by affine maps from polygons with edges having a common tangent ellipse (which includes the regular n-gons).

Theorem 11 naturally gives raise to the question whether the property $\rho_{\partial P_n} = 1/q_{\partial P_n}$ is valid for all bounded convex polygons with sufficiently small number of sides.

In the case of triangles this immediately follows from Theorem 7 as well as from Werner's result.

Noting that the affinity preserves parallelism and moves the lines to lines, one concludes from Theorem 9 that the equality $\rho_{\partial P_4} = 1/q_{\partial P_4}$ holds in particular for quadrilaterals P_4 obtained by affine transformations from quadrilaterals which are symmetric with respect to one of diagonals and for quadrilaterals whose sides have common tangent outwardly ellipse (in particular, for all parallelograms and trapezoids). For the same reasons, it holds also for hexagons with axial symmetry having two opposite sides parallel to this axes.

In fact, Theorem 9 allows us to establish much stronger result answering the question positively for quadrilaterals.

Theorem 12. *For every rectilinear convex quadrilateral* P_4, *we have*

$$\varkappa(f) = k(f) = q_{\partial P_4} = 1/\rho_{\partial P_4}, \tag{41}$$

where f is the appropriately normalized conformal map of $\mathbb{D}^*$ onto P_4^*.

The proof of this theorem essentially relies on Theorem 9 and on result of [41] that the equalities (41) are valid for all rectangles, and hence for their affine transformations.

Fix such a quadrilateral $P_4^0 = A_1^0 A_2^0 A_3^0 A_4^0$ and consider the collection $\mathcal{P}^0$ of quadrilaterals $P_4 = A_1^0 A_2^0 A_3^0 A_4$ with the same first three vertices and variable A_4; the corresponding A_4 runs over a subset E of the trice punctured sphere $\widehat{\mathbb{C}} \setminus \{A_1^0, A_2^0, A_3^0\}$.

The collection $\mathcal{P}^0$ contains the trapezoids, for which we have the equalities (41) by Theorem 9 (and consequently, the infinitesimal equality (39) at the corresponding points a).

Similar to the proof of Theorem 7, one obtains in the universal Teichmüller space $\mathbf{T}$ a holomorphic disk Ω extending the real analytic curve filled by the Schwarzians which correspond to the values $t = A$ on E. On this disk, one can construct, similar to (38), the corresponding metric $\lambda_{\varkappa}$. Lemmas 4-6 again imply that this metric must coincide at all points of Ω with the dominant infinitesimal Teichmüller-Kobayashi metric $\lambda_{\mathcal{K}}$ of $\mathbf{T}$. Together with Lemma 2, this provides the global equalities (41) for all points of the disk Ω (and hence for the prescribed quadrilateral P_4^0).

6. Reflections across finite collections of quasintervals

6.1.

There are only a few exact estimates of the reflection coefficients of quasiconformal arcs (quasiintervals) and some their sharp upper bounds presented in [38,39]. The most of these bounds have been obtained using the classical Bernstein-Walsh-Siciak theorem which quantitatively connects holomorphic extension of a function defined on a compact $K \Subset \mathbb{C}^n$ with the speed of its polynomial approximation. Another approach was applied by Kühnau in [58,59,62,63]. In particular, using somewhat modification of Teichmüller's Verschiebungssatz [100], he established in [63] the reflection coefficient of the set E which consists of the interval $[-2i, 2i]$ and a separate point $t > 0$. All these result are presented in [43].

6.2. Reflections across the finite collections of quasiintervals

Theorems 5 and 7 open a new way in solving this problem following the lines of the first example after Theorem 5. Namely, given a finite union

$$L = \bigcup L_1 \bigcup L_2 \cdots \bigcup L_N$$

of smooth curvelinear quasiintervals (possibly mutually separated) such that L can be extended without adding new vertices (angular points) to a quasicircle $L_0 \supset L$ containing $z = \infty$ and bounding a convex polygon P_N which satisfies the assumptions of Theorem 5 or a polygon considered in Theorem 7, then by these theorems, *the reflection coefficient of the set L equals*

$$q_L = |1 - |a||, \tag{42}$$

where α is defined for L_0 similar to (23).

The main point here is to get a **convex** (or sufficiently close to convex, as in Theorem 7) polygon, because the initial and final arcs of components L_j can be smoothly extended and then rounded off.

Note also that adding to L a finite number of appropriately located isolated points $z_1, \dots z_m$ does not change the reflection coefficient (42).

7. Some open problems

1. The first problem is the following question of Kühnau (personal communication):

Does the reflection coefficient of a rectangle $\mathcal{R}$ be a monotone nondecreasing function of its conformal module $\mu_{\mathcal{R}}$ (the ratio of the vertical and horizontal side lengths) ?

The results of Kühnau and Werner for the rectangles $\mathcal{R}$ state that if the module $\mu(\mathcal{R})$ satisfies $1 \le \mu(\mathcal{R}) < 1.037$, then

$$q_{\partial\mathcal{R}} = 1/\rho_{\partial\mathcal{R}} = 1/2;$$

if $\mu(\mathcal{R}) > 2.76$, then $q_{\partial\mathcal{R}} > 1/2$ (see [57, 103]).

On the other hand, the reflection coefficients of long rectangles are close to 1, because the limit half-strip is not a quasidisk.

2. *To what extent can be generalized the above theorems to convex polygons with infinite number of vertexes?*

The existence of an obstruction follows from Theorem 11.

Let us mention also that the underlying fact of many extension results is the Ahlfors–Weill theorem ensuring quasiconformal extension across quasiconformal curves with sufficiently small **B**-norm of exterior conformal mapping function.

3. *Similar question on estimating quasiconformal reflection coefficients and Fredholm eigenvalues for quasiintervals formed by infinite collections of rectilinear intervals.*

In particular, find the reflection coefficients and Fredholm eigenvalues of arbitrary reclinear polygons whose neighbor sides are orthogonal.

4. *Let L be a finite collection of parallel linear segments either located on a line or intersecting a line under the same angle. Find the reflection coefficient q_L.*

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