

Series expansions for monogenic functions in Clifford algebras and their application

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(Presented by O. Dovgoshey)

Abstract. This paper deals with studying some properties of a monogenic function defined on a vector space with values in the Clifford algebra generated by the space. We provide some expansions of a monogenic function and consider its application to study solutions of second order partial differential equations.

2010 MSC. Primary 30A99; Secondary 15A66.

Key words and phrases. Monogenic function, Clifford algebra, partial differential equation.

1. Introduction

A Clifford algebra is a unital associative algebra, which is the minimal extension of a finite-dimensional vector space V with a quadratic form. In this case the vector space V is called embedded in the Clifford algebra and its orthogonal basis determines the basis of the algebra. We are interested in properties of a monogenic function defined on the embedded vector space with values in Clifford algebra, which is a null solution of the Dirac operator. A monogenic function is one of the basic concept of Clifford analysis [1]. We obtained an expansion of a left monogenic function into series of Fueter type polynomials and since any right monogenic function can be expanded in a similar manner only left monogenic functions are considered. We also study application of the properties of monogenic functions to solving some second-order partial differential equations.

2. Monogenic functions in Clifford algebras

Suppose E^{d+1} is an $(d + 1)$ -dimensional linear space over \mathbb{R} with orthogonal basis e_i , $i = 0, 1, \dots, d$. Suppose E^{d+1} is embedded in the

Received 29.05.2020

Clifford algebra $Cl_{p,q}$, $p + q = d + 1$ with an identity $I = e_0$ and $e_k e_l + e_l e_k = \mathbf{0}$, $k \neq l$, $1 \leq k, l \leq d$. In addition, $e_k^2 = I$ for $1 \leq k \leq p$ and $e_k^2 = -I$ for $p < k \leq d$.

Denote by $D = \sum_{i=0}^d e_i \frac{\partial}{\partial x_i}$ the generalized Cauchy–Riemann operator. It is easily verified that $DD^* = \sum_{i=0}^p \frac{\partial^2}{\partial x_i^2} - \sum_{i=p+1}^d \frac{\partial^2}{\partial x_i^2}$.

We are interested in studying of the following equation

$$DD^* f = 0. \tag{2.1}$$

Definition 2.1. A function $f : E^{d+1} \rightarrow Cl_{p,q}$ is called left monogenic if $Df = 0$, and f is called right monogenic if $fD = 0$.

In the sequel we will consider only left monogenic functions. Any left monogenic function is a solution of the equation $DD^* f = 0$.

For $E^{d+1} \ni \mathbf{x} = \sum_{i=0}^d e_i x_i$ let us introduce the following polynomials:

$$p_k(\mathbf{x}) = x_k - e_k x_0, \quad 1 \leq k \leq d.$$

Polynomials p_k might be called Fueter type polynomials since in the case of the algebra of quaternions $\mathbb{H} = x_0 + ix_1 + jx_2 + kx_3$ polynomials $x_1 - ix_0$, $x_2 - jx_0$, $x_3 - kx_0$ were introduced by Fueter in [2] and are known today as the Fueter polynomials.

An expansion of hyperholomorphic quaternion functions in Fueter polynomials has been reported in [3] and the similar expansion of hyperholomorphic co-quaternion functions was considered in [4]. The relations between monogenic and hyperholomorphic functions was considered in [5].

Theorem 2.1. Let $f \in C^\infty(E^{d+1})$, $n \in \mathbb{N}$, and the function f with its derivatives

$$\frac{\partial^n}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_n}} f(\mathbf{x})$$

be a left monogenic, then there exists real θ such that

$$\begin{aligned} f(\mathbf{x}) &= f(\mathbf{0}) + \sum_{i=1}^d p_i(\mathbf{x}) \frac{\partial}{\partial x_i} f(\mathbf{0}) \\ &+ \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d p_i(\mathbf{x}) p_j(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{0}) + \dots + \frac{1}{(n-1)!} \\ &\times \sum_{i_1=1}^d \sum_{i_2=1}^d \dots \sum_{i_{n-1}=1}^d p_{i_1}(\mathbf{x}) p_{i_2}(\mathbf{x}) \dots p_{i_{n-1}}(\mathbf{x}) \frac{\partial^{n-1}}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_{n-1}}} f(\mathbf{0}) \end{aligned}$$

$$+\frac{1}{n!} \sum_{i_1=1}^d \sum_{i_2=1}^d \cdots \sum_{i_n=1}^d p_{i_1}(\mathbf{x}) p_{i_2}(\mathbf{x}) \dots p_{i_n}(\mathbf{x}) \frac{\partial^n}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_n}} f(\theta_n \mathbf{x}),$$

where $0 \leq \theta_n < 1$.

Proof. Consider function $f(t\mathbf{x})$ as a function of $t \in \mathbb{R}$. Taking into account that $f(\mathbf{x})$ is left monogenic that is $\frac{\partial}{\partial x_0} f(\mathbf{x}) = -\sum_{i=1}^d \mathbf{e}_i \frac{\partial}{\partial x_i} f(\mathbf{x})$, we obtain

$$\frac{d}{dt} f(t\mathbf{x}) = \sum_{i=0}^d x_i \frac{\partial}{\partial x_i} f(t\mathbf{x}) = \sum_{i=1}^d (x_i - \mathbf{e}_i x_0) \frac{\partial}{\partial x_i} f(t\mathbf{x}). \tag{2.2}$$

Putting $t = 0$, we have

$$\frac{d}{dt} f(\mathbf{0}) = \sum_{i=1}^d p_i(\mathbf{x}) \frac{\partial}{\partial x_i} f(\mathbf{0}). \tag{2.3}$$

Now considering that functions

$$\frac{\partial}{\partial x_i} f(\mathbf{x}), \quad i = 1, \dots, n$$

are left monogenic and Eq. (2.2), we get

$$\begin{aligned} \frac{d^2}{dt^2} f(t\mathbf{x}) &= \sum_{i=1}^d (x_i - \mathbf{e}_i x_0) \frac{d}{dt} \left(\frac{\partial}{\partial x_i} f(t\mathbf{x}) \right) \\ &= \sum_{i=1}^d (x_i - \mathbf{e}_i x_0) \sum_{j=1}^d (x_j - \mathbf{e}_j x_0) \frac{\partial^2}{\partial x_j \partial x_i} f(t\mathbf{x}). \end{aligned}$$

Hence, by putting $t = 0$

$$\frac{d^2}{dt^2} f(\mathbf{0}) = \sum_{i=1}^d \sum_{j=1}^d (x_i - \mathbf{e}_i x_0) (x_j - \mathbf{e}_j x_0) \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{0}). \tag{2.4}$$

Much in the same way one can show that

$$\begin{aligned} &\frac{d^n}{dt^n} f(t\mathbf{x}) \\ &= \sum_{i_1=1}^d \sum_{i_2=1}^d \cdots \sum_{i_n=1}^d p_{i_1}(\mathbf{x}) p_{i_2}(\mathbf{x}) \dots p_{i_n}(\mathbf{x}) \frac{\partial^n}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_n}} f(t\mathbf{x}). \end{aligned} \tag{2.5}$$

Now, consider the Taylor series for $f(t\mathbf{x})$ w.r.t. t

$$\begin{aligned}
 f(t\mathbf{x}) &= f(\mathbf{0}) + \frac{d}{dt}f(\mathbf{0})t + \dots & (2.6) \\
 &+ \frac{d^{n-1}}{dt^{n-1}}f(\mathbf{0})\frac{t^{n-1}}{(n-1)!} + \frac{d^n}{dt^n}f(\theta_n\mathbf{x})\frac{t^n}{n!}, \quad 0 \leq \theta_n < 1.
 \end{aligned}$$

Substituting Eqs. (2.3)–(2.5) into Eq. (2.6), we obtain Eq. (2.1).
 If

$$\begin{aligned}
 \frac{1}{n!} \sum_{i_1=1}^d \sum_{i_2=1}^d \cdots \sum_{i_n=1}^d p_{i_1}(\mathbf{x})p_{i_2}(\mathbf{x}) \cdots p_{i_n}(\mathbf{x}) \frac{\partial^n}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_n}} f(\theta_n\mathbf{x}) \rightarrow \mathbf{0}, \\
 n \rightarrow \infty,
 \end{aligned}$$

then it follows from Eq. (2.1) that

$$\begin{aligned}
 f(\mathbf{x}) &= f(\mathbf{0}) \\
 &+ \sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{i_1=1}^d \sum_{i_2=1}^d \cdots \sum_{i_n=1}^d p_{i_1}(\mathbf{x})p_{i_2}(\mathbf{x}) \cdots p_{i_n}(\mathbf{x}) \right. & (2.7) \\
 &\quad \left. \times \frac{\partial^n}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_n}} f(\mathbf{0}) \right).
 \end{aligned}$$

In particular case when

$$\frac{\partial^n}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_n}} f(\mathbf{0}) = a_n$$

for all $i_1, i_2, \dots, i_n \in \{1, 2, \dots, n\}, n \in \mathbb{N}$, we have

$$\begin{aligned}
 &\sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{i_1=1}^d \sum_{i_2=1}^d \cdots \sum_{i_n=1}^d p_{i_1}(\mathbf{x})p_{i_2}(\mathbf{x}) \cdots p_{i_n}(\mathbf{x}) a_n \right) \\
 &= \sum_{n=1}^{\infty} \frac{1}{n!} (p_{i_1}(\mathbf{x}) + p_{i_2}(\mathbf{x}) + \cdots + p_{i_n}(\mathbf{x}))^n a_n \\
 &= \sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{k=1}^d x_k - x_0 \sum_{k=1}^d e_k \right)^n a_n.
 \end{aligned}$$

Thus, in this case

$$f(\mathbf{x}) = a_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{k=1}^d x_k - x_0 \sum_{k=1}^d e_k \right)^n a_n, \quad (2.8)$$

where $a_0 = f(\mathbf{0})$. □

3. Studying of hyperbolic and Laplace equations by using monogenic functions

By using Eqs. (2.7) and (2.8) we can construct solutions of Eq. (2.1). For example, putting in Eq. (2.8) $a_n = n!$ for $1 \leq n \leq m$, $m \in \mathbb{N}$ and $a_n = 0$ for all $n > m$ we get the following solution of Eq. (2.1)

$$f(\mathbf{x}) = 1 + \sum_{n=1}^m \left(\sum_{k=1}^d x_k - x_0 \sum_{k=1}^d e_k \right)^n = \sum_{n=0}^m \left(\sum_{k=1}^d x_k - x_0 \sum_{k=1}^d e_k \right)^n .$$

Taking into account that $\left(\sum_{k=1}^d e_k \right)^{2n} = (p - q)^n$ in $Cl_{p,q}$, we obtain a real solution of Eq. (2.1) as follows

$$\begin{aligned} &u(x_1, x_2, \dots, x_d) \\ &= \sum_{n=0}^m \left(\left(\sum_{k=1}^d x_k \right)^n + C_n^2 \left(\sum_{k=1}^d x_k \right)^{n-2} (p - q)^2 x_0^2 \right. \\ &\quad \left. + C_n^4 \left(\sum_{k=1}^d x_k \right)^{n-4} (p - q)^4 x_0^4 + \dots \right), \end{aligned}$$

where we assume that $C_n^k = 0$ for $n < k$.

We can also construct solutions of Eq. (2.1) in the cases of infinite non-zero terms of series (2.7). Indeed, by putting in Eq. (2.7) $a_n = 1$ for all $n \in \mathbb{N}$, we get the following solution of Eq. (2.8)

$$f(\mathbf{x}) = \exp \left(\sum_{k=1}^d x_k - x_0 \sum_{k=1}^d e_k \right) = \exp \left(\sum_{k=1}^d x_k \right) \exp \left(-x_0 \sum_{k=1}^d e_k \right) .$$

Considering that $\left(\sum_{k=1}^d e_k \right)^{2n} = (p - q)^n$ and the fact that expressions $\left(\sum_{k=1}^d e_k \right)^{2n+1}$ are not real for all $n \in \mathbb{N}$, we obtain the following real solution of Eq. (2.1)

$$u(x_1, x_2, \dots, x_d) = \exp \left(\sum_{k=1}^d x_k \right) \cos(x_0 \sqrt{p - q}) .$$

In the case where $p = 0$ (or $q = 0$) that is, the Clifford algebra is $Cl_{0,d}$ (or $Cl_{d,0}$) from Eq. (2.1) follows that a monogenic function f satisfies the d -dimensional Laplace equation

$$\Delta_d f = \sum_{i=0}^d \frac{\partial^2}{\partial x_i^2} f = 0. \tag{3.1}$$

Consider the case $Cl_{0,3}$ with the embedded vector space E^3 and its basis $I, \mathbf{e}_1, \mathbf{e}_2$, where I is the identity and $\mathbf{e}_k \mathbf{e}_l + \mathbf{e}_l \mathbf{e}_k = -2I\delta_{kl}$, $1 \leq k, l \leq 2$. For this case Eq. (3.1) is the Laplace equation

$$\Delta_3 f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f = 0.$$

Taking into account Eq. (2.8), it is easily seen that

$$f(x, y, z) = \frac{(\mathbf{e}_1 + \mathbf{e}_2)x - y - z}{1 - (\mathbf{e}_1 + \mathbf{e}_2)x + y + z}$$

is monogenic in $E^3 \setminus \{(\mathbf{e}_1 + \mathbf{e}_2)x + y + z = 1\}$.

Let us find the real and imaginary parts of f

$$\begin{aligned} f(x, y, z) &= \frac{(\mathbf{e}_1 + \mathbf{e}_2)x - y - z}{1 - (\mathbf{e}_1 + \mathbf{e}_2)x + y + z} \frac{1 + (\mathbf{e}_1 + \mathbf{e}_2)x + y + z}{1 + (\mathbf{e}_1 + \mathbf{e}_2)x + y + z} \\ &= \frac{y + z - 2x^2 - (y + z)^2}{2x^2 + (1 - y - z)^2} + \frac{x}{2x^2 + (1 - y - z)^2} (\mathbf{e}_1 + \mathbf{e}_2). \end{aligned}$$

Thus, we have the following solutions of Eq. (3.1)

$$u_1(x, y, z) = \frac{y + z - 2x^2 - (y + z)^2}{2x^2 + (1 - y - z)^2}$$

and

$$u_2(x, y, z) = \frac{x}{2x^2 + (1 - y - z)^2}.$$

Now consider the case $Cl_{1,3}$ where the embedded vector space E^4 has the basis $I, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. I is the identity and $\mathbf{e}_k \mathbf{e}_l + \mathbf{e}_l \mathbf{e}_k = -2I\delta_{kl}$, $1 \leq k, l \leq 3$.

For this case Eq. (2.1) is the following form

$$\frac{\partial^2}{\partial t^2} f - \frac{\partial^2}{\partial x^2} f - \frac{\partial^2}{\partial y^2} f - \frac{\partial^2}{\partial z^2} f = 0. \quad (3.2)$$

Consider the function

$$f(w) = \frac{w}{1 - w},$$

where $w = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)t - x - y - z$.

It follows from Eq. (2.7) that for $w \neq 1$ function $f(w)$ is monogenic.

The real and imaginary parts of f can be calculated as follows

$$\begin{aligned}
 f(w) &= \frac{(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)t - x - y - z}{1 - (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)t + x + y + z} \frac{1 + (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)t + x + y + z}{1 + (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)t + x + y + z} \\
 &= \frac{3t^2 - (x + y + z) + x + y + z}{(1 - x - y - z)^2 - 3t^2} + \frac{t(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)}{(1 - x - y - z)^2 - 3t^2}.
 \end{aligned}$$

Hence, we obtain two solutions of Eq. (3.2).

$$u_1(t, x, y, z) = \frac{3t^2 - (x + y + z) + x + y + z}{(1 - x - y - z)^2 - 3t^2}$$

and

$$u_2(t, x, y, z) = \frac{t}{(1 - x - y - z)^2 - 3t^2}.$$

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