

# Asymptotic estimates for the best uniform approximations of classes of convolution of periodic functions of high smoothness

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(Presented by V. Gutlyanskii)

*Dedicated to the memory of  
Professor S.B. Stechkin and Professor S.A. Telyakovskii*

**Abstract.** We find two-sides estimates for the best uniform approximations of classes of convolutions of  $2\pi$ -periodic functions from unit ball of the space  $L_p$ ,  $1 \leq p < \infty$ , with fixed kernels, modules of Fourier coefficients of which satisfy the condition  $\sum_{k=n+1}^{\infty} \psi(k) < \psi(n)$ . In the case of

$\sum_{k=n+1}^{\infty} \psi(k) = o(1)\psi(n)$  the obtained estimates become the asymptotic equalities.

**2010 MSC.** 42A10.

**Key words and phrases.** Best approximation, Fourier sum, Weyl–Nagy class,  $(\psi, \beta)$ -integral, asymptotic equality, Kolmogorov–Nicol’skii problem.

Let  $L_p$ ,  $1 \leq p < \infty$ , be the space of  $2\pi$ -periodic functions  $f$  summable to the power  $p$  on  $[-\pi, \pi)$ , in which the norm is given by the formula

$$\|f\|_{L_p} = \|f\|_p = \left( \int_{-\pi}^{\pi} |f(t)|^p dt \right)^{1/p},$$

$L_{\infty}$  be the space of measurable and essentially bounded  $2\pi$ -periodic functions  $f$  with the norm

$$\|f\|_{L_{\infty}} = \|f\|_{\infty} = \operatorname{ess\,sup}_t |f(t)|,$$

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Received 04.07.2020

*This work was partially supported by the Grant H2020-MSCA-RISE-2019, project number 873071 (SOMPATY: Spectral Optimization: From Mathematics to Physics and Advanced Technology).*

and  $C$  be the space of continuous  $2\pi$ -periodic functions  $f$ , in which the norm is defined by the equality

$$\|f\|_C = \max_t |f(t)|.$$

Denote by  $C_{\bar{\beta},p}^\psi$ ,  $1 \leq p \leq \infty$ , the set of all  $2\pi$ -periodic functions  $f$ , representable as convolution

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(x-t)\Psi_{\bar{\beta}}(t)dt, \quad a_0 \in \mathbb{R}, \quad \varphi \in B_p^0, \quad (1)$$

$$B_p^0 = \{\varphi \in L_p : \|\varphi\|_p \leq 1, \varphi \perp 1\},$$

with a fixed generated kernel  $\Psi_{\bar{\beta}} \in L_{p'}$ ,  $1/p+1/p' = 1$ , the Fourier series of which has the form

$$S[\Psi_{\bar{\beta}}](t) = \sum_{k=1}^{\infty} \psi(k) \cos\left(kt - \frac{\beta_k \pi}{2}\right), \quad \beta_k \in \mathbb{R}, \quad \psi(k) \geq 0. \quad (2)$$

A function  $f$  in the representation (1) is called  $(\psi, \bar{\beta})$ -integral of the function  $\varphi$  and is denoted by  $\mathcal{J}_{\bar{\beta}}^\psi \varphi$  ( $f = \mathcal{J}_{\bar{\beta}}^\psi \varphi$ ). If  $\psi(k) \neq 0$ ,  $k \in \mathbb{N}$ , then the function  $\varphi$  in the representation (1) is called  $(\psi, \bar{\beta})$ -derivative of the function  $f$  and is denoted by  $f_{\bar{\beta}}^\psi$  ( $\varphi = f_{\bar{\beta}}^\psi$ ). The concepts of  $(\psi, \bar{\beta})$ -integral and  $(\psi, \bar{\beta})$ -derivative was introduced by Stepanets (see, e.g., [31, 32]). Since  $\varphi \in L_p$  and  $\Psi_{\bar{\beta}} \in L_{p'}$ , then (see. [32, Proposition 3.9.2.]) the function  $f$  of the form (1) is a continuous function, i.e.  $C_{\bar{\beta},p}^\psi \subset C$ .

In the case  $\beta_k \equiv \beta$ ,  $\beta \in \mathbb{R}$ , the classes  $C_{\bar{\beta},p}^\psi$  are denoted by  $C_{\beta,p}^\psi$ .

For  $\psi(k) = k^{-r}$ ,  $r > 0$ , the classes  $C_{\bar{\beta},p}^\psi$  та  $C_{\beta,p}^\psi$  are denoted by  $W_{\bar{\beta},p}^r$  and  $W_{\beta,p}^r$ , respectively. The  $W_{\beta,p}^r$  are the well-known Weyl–Nagy classes (see, e.g., [12, 29, 31, 32]). In other words  $W_{\beta,p}^r$ ,  $1 \leq p \leq \infty$ , are the classes of  $2\pi$ -periodic functions  $f$ , representable as convolutions of the form

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(x-t)B_{r,\beta}(t)dt, \quad a_0 \in \mathbb{R}, \quad (3)$$

the Weyl–Nagy kernels  $B_{r,\beta}(\cdot)$  of the form

$$B_{r,\beta}(t) = \sum_{k=1}^{\infty} k^{-r} \cos\left(kt - \frac{\beta \pi}{2}\right), \quad r > 0, \quad \beta \in \mathbb{R}, \quad (4)$$

with functions  $\varphi \in B_p^0$ . The function  $\varphi$  in the formula (3) is called the Weyl–Nagy derivative of the function  $f$  and is denoted by  $f_\beta^r$ .

If  $r \in \mathbb{N}$  and  $\beta = r$ , then the functions  $B_{r,\beta}(\cdot)$  of the form (4) are the well-known Bernoulli kernels and the corresponding classes  $W_{\beta,p}^r$  coincide with the well-known classes  $W_p^r$ , which consist of  $2\pi$ -periodic functions  $f$  with absolutely continuous derivatives  $f^{(k)}$  up to  $(r - 1)$ -th order inclusive and such that  $\|f^{(r)}\|_p \leq 1$ . In addition, for almost everywhere  $x \in \mathbb{R}$   $f^{(r)}(x) = f_r^r(x) = \varphi(x)$ , where  $\varphi$  is the function from (3).

For  $\psi(k) = e^{-\alpha k^r}$ ,  $\alpha > 0$ ,  $r > 0$ , the classes  $C_{\beta,p}^\psi$  are denoted by  $C_{\beta,p}^{\alpha,r}$ . In the case of  $r = 1$ ,  $\beta_k \equiv \beta$ ,  $\beta \in \mathbb{R}$ , and  $p = \infty$  the sets  $C_{\beta,p}^{\alpha,r}$  are well-known classes of the Poisson integrals  $C_{\beta,\infty}^{\alpha,1}$  (see, e.g., [30–32]).

If  $f \in C$  by  $E_n(f)_C$  we denote the best uniform approximation of the function  $f$  by elements of the subspace  $\mathcal{T}_{2n-1}$  of trigonometric polynomials  $T_{n-1}$  of the order  $n - 1$ :

$$T_{n-1}(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{n-1} (\alpha_k \cos kx + \beta_k \sin kx), \quad \alpha_k, \beta_k \in \mathbb{R}.$$

Let  $\mathfrak{N}$  be the some functional class from the space  $C$  ( $\mathfrak{N} \subset C$ ). Then the quantity

$$E_n(\mathfrak{N})_C = \sup_{f \in \mathfrak{N}} E_n(f)_C = \sup_{f \in \mathfrak{N}} \inf_{T_{n-1} \in \mathcal{T}_{2n-1}} \|f(\cdot) - T_{n-1}(\cdot)\|_C \quad (5)$$

is called the best uniform approximation of the class  $\mathfrak{N}$  by elements of the subspace  $\mathcal{T}_{2n-1}$  of trigonometric polynomials  $T_{n-1}$  of the order  $n - 1$ .

At present, the exact values for the quantities of the form (5) are known for important functional classes  $\mathfrak{N}$ . In particular, thanks to the articles of Favard [7, 8], Akhiezer and Krein [1], Nikol'skii [14], Dzyadyk [4, 5], Stechkin [29] and Sun [34] the exact values of the best uniform approximations of the Weyl–Nagy classes  $W_{\beta,\infty}^r$  are found for arbitrary  $r > 0$  and  $\beta \in \mathbb{R}$ .

For the classes of the Poisson integrals  $C_{\beta,\infty}^{\alpha,1}$  the exact values of the form (5) are also known for all  $\alpha > 0$  and  $\beta \in \mathbb{R}$  thanks to the articles of Krein [11], Bushanskij [3] and Shevaldin [28] (see also [2, 32]).

The exact values of the best approximations  $E_n(\mathfrak{N})_C$  were obtained in a number of other cases (see, e.g., [16–20, 22, 32]).

In the general case, the problem of finding of the exact values of the best uniform approximations of the classes  $C_{\beta,p}^\psi$  for  $1 \leq p \leq \infty$  remains open, and therefore, the investigation of the asymptotic behavior of the quantities  $E_n(C_{\beta,p}^\psi)_C$  as  $n \rightarrow \infty$  is relevant.

In this paper we investigate the problem of finding of the asymptotic equalities for the quantities (5) as  $n \rightarrow \infty$  in the case, when the classes  $\mathfrak{N}$  are the classes  $C_{\beta,p}^\psi$ ,  $1 \leq p \leq \infty$ , and the sequences  $\psi(k)$  decrease to zero very rapidly, in particular, when

$$\sum_{k=n+1}^{\infty} \psi(k) = o(1)\psi(n). \tag{6}$$

This work can be considered a continuation of the authors' research [21, 24, 25], in which the asymptotics of the best uniform approximations of classes of the generalized  $(\psi, \bar{\beta})$ -integrals were investigated.

Note that in the case of  $p = \infty$  the asymptotic equalities and even the exact values of the quantities  $E_n(C_{\beta,\infty}^\psi)_C$  are known under certain restrictions on  $\psi(k)$  (see, e.g., [22, 33]).

For a fixed  $\mathfrak{N} \subset C$  denote by  $\mathcal{E}_n(\mathfrak{N})_C$  the quantity

$$\mathcal{E}_n(\mathfrak{N})_C = \sup_{f \in \mathfrak{N}} \|f(\cdot) - S_{n-1}(f; \cdot)\|_C, \tag{7}$$

where  $S_{n-1}(f; \cdot)$  is the partial Fourier sum of order  $n - 1$  of the function  $f$ .

Since

$$E_n(\mathfrak{N})_C \leq \mathcal{E}_n(\mathfrak{N})_C, \quad \mathfrak{N} \subset C, \tag{8}$$

then the quantities (7) are naturally used for upper bounds for the best approximations of the classes  $\mathfrak{N}$ .

The problem of finding of the asymptotic equalities for the quantities of the form (7) as  $n \rightarrow \infty$  is called the Kolmogorov–Nicol'skii problem for the Fourier sums. The Kolmogorov–Nicol'skii problem has a rich history. Reviews on the history of this problem can be found e.g. in the monographs [31, 32].

For characteristics of the form (7) on the Weyl–Nagy classes  $W_{\beta,\infty}^r$  ( $\mathfrak{N} = W_{\beta,\infty}^r$ ) the following asymptotic formula holds

$$\mathcal{E}_n(W_{\beta,\infty}^r)_C = \frac{4 \ln n}{\pi n^r} + \mathcal{O}\left(\frac{1}{n^r}\right), \quad r > 0, \quad \beta \in \mathbb{R}. \tag{9}$$

For  $r \in \mathbb{N}$  and  $\beta = r$  this estimate was obtained by Kolmogorov [9], for arbitrary  $r > 0$  by Pinkevich [15] and Nicol'skii [13]. In the general case the estimate (9) follows from results of Efimov [6] and Telyakovskii [35].

In these works the parameters  $r$  and  $\beta$  of the Weyl–Nagy classes were assumed to be fixed, and the question about the dependence of the remainder term in the estimates (9) on these parameters was not considered. The character of the dependence on  $r$  and  $\beta$  of the remainder

term in the estimate (9) was investigated by Sokolov, Selivanova, Natanson, Telyakovskii, Stechkin and other authors (see [26, 30, 36, 37] and the references therein).

In the work of Stechkin [30] the asymptotic behavior of the quantities  $\mathcal{E}_n(W_{\beta, \infty}^r)_C$  was completely investigated as  $n \rightarrow \infty$  and  $r \rightarrow \infty$ . Namely, he proved that for arbitrary  $r \geq 1$  and  $\beta \in \mathbb{R}$  the following equality takes place

$$\mathcal{E}_n(W_{\beta, \infty}^r)_C = \frac{1}{n^r} \left( \frac{8}{\pi^2} \mathbf{K}(e^{-r/n}) + \mathcal{O}(1) \frac{1}{r} \right), \tag{10}$$

where

$$\mathbf{K}(q) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - q^2 \sin^2 t}} \tag{11}$$

is a complete elliptic integral of the first kind, and  $\mathcal{O}(1)$  is a quantity uniformly bounded with respect to  $r, n$  and  $\beta$ .

In addition, Stechkin [30, Theorem 4] proved that for rapidly growing  $r$  the remainder in the equality (10) can be improved. Namely, for arbitrary  $r \geq n + 1$  and  $\beta \in \mathbb{R}$  the following formula holds

$$\mathcal{E}_n(W_{\beta, \infty}^r)_C = \frac{1}{n^r} \left( \frac{4}{\pi} + \mathcal{O}(1) \left( 1 + \frac{1}{n} \right)^{-r} \right), \tag{12}$$

where  $\mathcal{O}(1)$  is a quantity uniformly bounded with respect to  $r, n$  and  $\beta$ . If  $r/n \rightarrow \infty$  then the estimate (12) becomes the asymptotic equality.

In the works of Telyakovskii [36, 37] it was shown that the second term in the formula (12) can be replaced by a smaller one, namely, we can write  $\mathcal{O}(1)(1 + 2/n)^{-r}$  instead of  $\mathcal{O}(1)(1 + 1/n)^{-r}$ , and it is also the estimate (12) sharper by separating out the following terms of the asymptotics.

In the work of authors [26], in particular, the formula of Stechkin (12) was generalized on classes  $W_{\bar{\beta}, p}^r$ . Namely, it is proved that if  $1 \leq p \leq \infty$ ,  $n \in \mathbb{N}$  and  $\bar{\beta} = \{\beta_k\}_{k=1}^\infty$  is arbitrary sequence of real numbers, then for  $r \geq n + 1$  the following estimate holds

$$\mathcal{E}_n(W_{\bar{\beta}, p}^r)_C = n^{-r} \left( \frac{\|\cos t\|_{p'}}{\pi} + \mathcal{O}(1) \left( 1 + \frac{1}{n} \right)^{-r} \right), \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{13}$$

where  $\mathcal{O}(1)$  is a quantity uniformly bounded in all parameters.

For the classes of the generalized Poisson integrals  $C_{\beta, p}^{\alpha, r}$  and for all fixed  $\alpha > 0, r > 0, \beta \in \mathbb{R}$  and  $1 \leq p \leq \infty$  the asymptotic equalities for the quantities (7) as  $n \rightarrow \infty$  are known due to works of Nikol'skii [14], Stechkin [30], Stepanets [31, 32], Telyakovskii [36, 37], Serdyuk and Stepanyuk [27].

As for the classes  $C_{\bar{\beta},p}^\psi$ , for the rapidly decreasing sequences  $\psi(k)$  (in particular, when the condition (6) is satisfied) the asymptotics of the quantities  $\mathcal{E}_n(C_{\bar{\beta},p}^\psi)_C$  are known for all  $1 \leq p \leq \infty$  and  $\beta_k \in \mathbb{R}$  (see [23, 27, 31, 32, 36, 37]). Therefore, the main goal of this work is focused on finding lower estimates for the best approximations  $E_n(C_{\bar{\beta},p}^\psi)_C$ ,  $1 \leq p \leq \infty$ .

In present paper we find two-sides estimates for the quantities  $E_n(C_{\bar{\beta},p}^\psi)_C$  and  $\mathcal{E}_n(C_{\bar{\beta},p}^\psi)_C$ , from which, in particular, it follows that under condition (6)

$$E_n(C_{\bar{\beta},p}^\psi)_C \sim \mathcal{E}_n(C_{\bar{\beta},p}^\psi)_C \sim \frac{\|\cos t\|_{p'}}{\pi} \psi(n), \tag{14}$$

where  $1 \leq p \leq \infty$ ,  $\beta_k \in \mathbb{R}$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ , and  $A(n) \sim B(n)$  as  $n \rightarrow \infty$  means that  $\lim_{n \rightarrow \infty} A(n)/B(n) = 1$ .

It should be noted that the asymptotic equality (14) is a manifestation of the high-smoothness effect, which occurs when the first harmonic in the remainder of the Fourier series after the  $(n - 1)$ -th member of the generating kernels  $\Psi_{\bar{\beta}}(t)$  of the form (2) is a dominant in estimating of the  $L_{p'}$ -norm of the specified remainder of the series. When the sequences  $\psi(k)$  decrease to zero not so fast, the equalities  $E_n(C_{\bar{\beta},p}^\psi)_C$  and  $\mathcal{E}_n(C_{\bar{\beta},p}^\psi)_C$  for  $1 \leq p < \infty$  do not asymptotically coincide with each other and for  $p = \infty$  can be different even in order.

Let  $n \in \mathbb{N}$ . In what follows, we will require that the sequence of modules of the Fourier coefficients of the generated kernel  $\Psi_{\bar{\beta}}(t)$  satisfies the condition

$$\sum_{k=n+1}^{\infty} \psi(k) < \psi(n). \tag{15}$$

**Theorem 1.** *For arbitrary  $\{\beta_k\}_{k=1}^{\infty}$ ,  $\beta_k \in \mathbb{R}$ ,  $1 < p \leq \infty$ ,  $n \in \mathbb{N}$  and  $\psi(k) \geq 0$ , which satisfy the condition (15), the following inequality holds*

$$E_n(C_{\bar{\beta},p}^\psi)_C \geq \frac{\|\cos t\|_{p'}}{\pi} \left( \psi(n) - \sum_{k=n+1}^{\infty} \psi(k) \right), \tag{16}$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$ .

*Proof.* Let us consider a function

$$\varphi_{n,p}(t) = \|\cos t\|_{p'}^{1-p'} |\cos nt|^{p'-1} \text{sign } \cos nt. \tag{17}$$

Since  $\varphi_{n,p} \perp 1$  and for  $1 < p < \infty$

$$\begin{aligned} \left( \int_{-\pi}^{\pi} |\varphi_{n,p}(t)|^p dt \right)^{\frac{1}{p}} &= \|\cos t\|_{p'}^{1-p'} \left( \int_{-\pi}^{\pi} |\cos nt|^{p(p'-1)} dt \right)^{\frac{1}{p}} \\ &= \|\cos t\|_{p'}^{1-p'} \left( \int_{-\pi}^{\pi} |\cos nt|^{p'} dt \right)^{\frac{1}{p}} \\ &= \|\cos t\|_{p'}^{1-p'} \left( \left( \int_{-\pi}^{\pi} |\cos nt|^{p'} dt \right)^{\frac{1}{p'}} \right)^{\frac{p'}{p}} = \|\cos t\|_{p'}^{1-p'} \|\cos t\|_{p'}^{\frac{p'}{p}} = 1, \end{aligned}$$

then  $\|\varphi_{n,p}\|_p = 1$  and therefore  $\varphi_{n,p} \in B_p^0$ ,  $1 < p < \infty$ .

We also put  $\varphi_{n,\infty}(t) = \text{sign } \cos nt$ . Obviously that  $\varphi_{n,\infty} \in B_\infty^0$ .

Let us consider a function

$$f_{n,p,\bar{\beta}}(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi_{n,p}(x-t) \Psi_{\bar{\beta}}(t) dt$$

from the class  $C_{\bar{\beta},p}^\psi$ ,  $1 < p \leq \infty$ . We have

$$\begin{aligned} f_{n,p,\bar{\beta}}(x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi_{n,p}(x-t) \sum_{k=1}^{\infty} \psi(k) \cos \left( kt - \frac{\beta_k \pi}{2} \right) dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi_{n,p}(x-t) \sum_{k=1}^{n-1} \psi(k) \cos \left( kt - \frac{\beta_k \pi}{2} \right) dt \\ &\quad + \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi_{n,p}(x-t) \sum_{k=n}^{\infty} \psi(k) \cos \left( kt - \frac{\beta_k \pi}{2} \right) dt. \end{aligned} \tag{18}$$

Since  $\varphi_{n,p}(t)$  is the  $\frac{2\pi}{n}$ -periodic function, then according to [10, Proposition 4.1.2] it is orthogonal to all trigonometric polynomials of order  $n - 1$ . Therefore, from (18) we obtain

$$\begin{aligned}
 f_{n,p,\bar{\beta}}(x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi_{n,p}(x-t) \sum_{k=n}^{\infty} \psi(k) \cos\left(kt - \frac{\beta_k\pi}{2}\right) dt \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi_{n,p}(t) \psi(n) \cos\left(n(x-t) - \frac{\beta_n\pi}{2}\right) \\
 &+ \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi_{n,p}(x-t) \sum_{k=n+1}^{\infty} \psi(k) \cos\left(kt - \frac{\beta_k\pi}{2}\right) dt =: F_1(x) + F_2(x). \quad (19)
 \end{aligned}$$

By virtue of Hölder’s inequality and the fact that  $\varphi_{n,p} \in B_p^0$ ,  $1 < p \leq \infty$ , we obtain the estimate

$$\begin{aligned}
 \|F_2\|_C &\leq \frac{1}{\pi} \|\varphi_{n,p}\|_p \left\| \sum_{k=n+1}^{\infty} \psi(k) \cos\left(kt - \frac{\beta_k\pi}{2}\right) \right\|_{p'} \\
 &= \frac{1}{\pi} \left\| \sum_{k=n+1}^{\infty} \psi(k) \cos\left(kt - \frac{\beta_k\pi}{2}\right) \right\|_{p'} \leq \frac{\|\cos t\|_{p'}}{\pi} \sum_{k=n+1}^{\infty} \psi(k). \quad (20)
 \end{aligned}$$

Taking into account that

$$\begin{aligned}
 F_1(x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \|\cos t\|_{p'}^{1-p'} |\cos nt|^{p'-1} \text{sign } \cos nt \psi(n) \cos\left(n(x-t) - \frac{\beta_n\pi}{2}\right) dt \\
 &= \frac{\|\cos t\|_{p'}^{1-p'}}{\pi} \psi(n) \int_{-\pi}^{\pi} |\cos nt|^{p'-1} \text{sign } \cos nt \\
 &\quad \times \left( \cos\left(nx - \frac{\beta_n\pi}{2}\right) \cos nt + \sin\left(nx - \frac{\beta_n\pi}{2}\right) \sin nt \right) dt,
 \end{aligned}$$

we consider on the period  $\left[\frac{\beta_n\pi}{2n}, 2\pi + \frac{\beta_n\pi}{2n}\right)$  the set of  $2n$  points  $\frac{\beta_n\pi}{2n} = x_0 < x_1 < \dots < x_{2n-1} < 2\pi + \frac{\beta_n\pi}{2n}$  of the form

$$x_m = \frac{\beta_n\pi}{2n} + \frac{m\pi}{n}, \quad m = \overline{0, 2n-1}. \quad (21)$$

Let us show that

$$F_1(x_m) = \frac{(-1)^m \|\cos t\|_{p'}}{\pi} \psi(n), \quad m = \overline{0, 2n-1}. \quad (22)$$



Indeed, as for any  $t \in [-\pi, \pi)$

$$\begin{aligned} & |\cos nt|^{p'-1} \operatorname{sign} \cos nt \cos \left( n(x_m - t) - \frac{\beta_n \pi}{2} \right) \\ &= |\cos nt|^{p'-1} \operatorname{sign} \cos nt \cos \left( \frac{\beta_n \pi}{2} + m\pi - nt - \frac{\beta_n \pi}{2} \right) \\ &= (-1)^m |\cos nt|^{p'-1} \operatorname{sign} \cos nt \cos nt = (-1)^m |\cos nt|^{p'}, \end{aligned}$$

then

$$\begin{aligned} F_1(x_m) &= \\ &= \frac{\|\cos t\|_{p'}^{1-p'}}{\pi} \psi(n) \int_{-\pi}^{\pi} |\cos nt|^{p'-1} \operatorname{sign} \cos nt \cos \left( n(x_m - t) - \frac{\beta_n \pi}{2} \right) dt \\ &= \frac{\|\cos t\|_{p'}^{1-p'}}{\pi} \psi(n) (-1)^m \int_{-\pi}^{\pi} |\cos nt|^{p'} dt = \frac{(-1)^m \|\cos t\|_{p'}}{\pi} \psi(n). \end{aligned}$$

From (15), (20) and (22) it follows that

$$|F_1(x_m)| = \frac{\|\cos t\|_{p'}}{\pi} \psi(n) > \frac{\|\cos t\|_{p'}}{\pi} \sum_{k=n+1}^{\infty} \psi(k) \geq \|F_2\|_C \geq |F_2(x_m)|,$$

and hence for  $f_{n,p,\bar{\beta}}(x_m) = F_1(x_m) + F_2(x_m)$  the following relations hold

$$\operatorname{sign} f_{n,p,\bar{\beta}}(x_m) = \operatorname{sign} F_1(x_m) = (-1)^m, \quad m = 0, 1, \dots, 2n - 1, \quad (23)$$

and

$$\begin{aligned} |f_{n,p,\bar{\beta}}(x_m)| &\geq |F_1(x_m)| - |F_2(x_m)| \geq |F_1(x_m)| - \|F_2\|_C \\ &\geq \frac{\|\cos t\|_{p'}}{\pi} \left( \psi(n) - \sum_{k=n+1}^{\infty} \psi(k) \right). \end{aligned} \quad (24)$$

Then by virtue of Valle Poussin's theorem (see, i.e., [31, Theorem 6.2.2])

$$E_n(f_{n,p,\bar{\beta}})_C \geq \min_{m=0,1,\dots,2n-1} |f_{n,p,\bar{\beta}}(x_m)| \geq \frac{\|\cos t\|_{p'}}{\pi} \left( \psi(n) - \sum_{k=n+1}^{\infty} \psi(k) \right).$$

□

**Theorem 2.** For arbitrary  $\{\beta_k\}_{k=1}^{\infty}$ ,  $\beta_k \in \mathbb{R}$ ,  $n \in \mathbb{N}$  and  $\psi(k) \geq 0$ , which satisfy the condition (15), the following inequality holds

$$E_n(C_{\bar{\beta},1}^{\psi})_C \geq \frac{1}{\pi} \left( \psi(n) - \sum_{k=n+1}^{\infty} \psi(k) \right). \quad (25)$$

*Proof.* Let us consider a function

$$\varphi_{n,1}(\delta, t) = \begin{cases} \frac{(-1)^m}{2\delta}, & t \in \left( \frac{m\pi}{n} - \frac{\delta}{2n}, \frac{m\pi}{n} + \frac{\delta}{2n} \right), m \in \mathbb{Z}, \\ 0, & t \in \mathbb{R} \setminus \bigcup_{m \in \mathbb{Z}} \left( \frac{m\pi}{n} - \frac{\delta}{2n}, \frac{m\pi}{n} + \frac{\delta}{2n} \right), m \in \mathbb{Z}, \end{cases} \tag{26}$$

where  $0 < \delta < \frac{\pi}{2}$ .

According to the definition

$$\text{sign } \varphi_{n,1}(\delta, t) = \text{sign } \cos nt, \quad t \in \left( \frac{m\pi}{n} - \frac{\delta}{2n}, \frac{m\pi}{n} + \frac{\delta}{2n} \right), m \in \mathbb{Z}, \tag{27}$$

and the function  $\varphi_{n,1}(\delta, t)$  is  $\frac{2\pi}{n}$ -periodic function. Besides

$$\|\varphi_{n,1}(\delta, \cdot)\|_1 = \sum_{m=0}^{2n-1} \frac{1}{2\delta} \frac{2\delta}{2n} = 1, \tag{28}$$

that is for any  $0 < \delta < \frac{\pi}{2}$   $\varphi_{n,1}(\delta, \cdot) \in B_1^0$ .

Let us consider a function

$$f_{n,1,\bar{\beta}}(\delta, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi_{n,1}(\delta, x - t) \Psi_{\bar{\beta}}(t) dt \tag{29}$$

from the class  $C_{\bar{\beta},1}^{\psi}$ . Since  $\varphi_{n,1}(\delta, t) \perp \sum_{k=1}^{n-1} \psi(k) \cos \left( kt - \frac{\beta_k \pi}{2} \right)$ , then

$$f_{n,1,\bar{\beta}}(\delta, x) = F_1(\delta, x) + F_2(\delta, x),$$

where

$$\begin{aligned} F_1(\delta, x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi_{n,1}(\delta, x - t) \psi(n) \cos \left( nt - \frac{\beta_n \pi}{2} \right) dt \\ &= \frac{1}{\pi} \psi(n) \int_{-\pi}^{\pi} \varphi_{n,1}(\delta, t) \cos \left( n(x - t) - \frac{\beta_n \pi}{2} \right) dt, \end{aligned} \tag{30}$$

$$F_2(\delta, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi_{n,1}(\delta, x - t) \sum_{k=n+1}^{\infty} \psi(k) \cos \left( kt - \frac{\beta_k \pi}{2} \right) dt. \tag{31}$$

By virtue of Hölder’s inequality and the equality (28)

$$\begin{aligned} \|F_2(\delta, \cdot)\|_C &\leq \frac{1}{\pi} \|\varphi_{n,1}(\delta, \cdot)\|_1 \left\| \sum_{k=n+1}^{\infty} \psi(k) \cos\left(kt - \frac{\beta_k \pi}{2}\right) \right\|_{\infty} \\ &\leq \frac{1}{\pi} \sum_{k=n+1}^{\infty} \psi(k). \end{aligned} \tag{32}$$

Let us consider on the segment  $\left[\frac{\beta_n \pi}{2n}, 2\pi + \frac{\beta_n \pi}{2n}\right)$  the following point set

$$x_m = \frac{\beta_n \pi}{2n} + \frac{m\pi}{n}, \quad m = \overline{0, 2n-1}.$$

Let us show that

$$F_1(\delta, x_m) = \frac{(-1)^m 2}{\pi} \frac{\delta}{\delta} \sin \frac{\delta}{2} \psi(n), \quad m = \overline{0, 2n-1}. \tag{33}$$

Indeed, since

$$\begin{aligned} &\varphi_{n,1}(\delta, t) \cos\left(n(x_m - t) - \frac{\beta_n \pi}{2}\right) \\ &= \varphi_{n,1}(\delta, t) \cos\left(n\left(\frac{\beta_n \pi}{2n} + \frac{m\pi}{n} - t\right) - \frac{\beta_n \pi}{2}\right) \\ &= \varphi_{n,1}(\delta, t) \cos(m\pi - nt) = (-1)^m \varphi_{n,1}(\delta, t) \cos nt, \end{aligned}$$

Then, in view of (27), (28) and (30),

$$\begin{aligned} F_1(x_m) &= \frac{1}{\pi} \psi(n) \int_{-\pi}^{\pi} (-1)^m \varphi_{n,1}(\delta, t) \cos nt dt \\ &= \frac{(-1)^m}{\pi} \psi(n) \sum_{m=0}^{2n-1} \int_{-\frac{\pi}{2n} + \frac{m\pi}{2n}}^{\frac{\pi}{2n} + \frac{m\pi}{2n}} \varphi_{n,1}(\delta, t) \cos nt dt \\ &= \frac{(-1)^m}{\pi} \psi(n) \sum_{m=0}^{2n-1} \int_{-\frac{\delta}{2n} + \frac{m\pi}{2n}}^{\frac{\delta}{2n} + \frac{m\pi}{2n}} \frac{(-1)^m}{2\delta} \cos nt dt \\ &= \frac{(-1)^m}{\pi} \psi(n) \sum_{m=0}^{2n-1} \frac{1}{2\delta} \int_{-\frac{\delta}{2n}}^{\frac{\delta}{2n}} \cos nt dt \end{aligned}$$

$$= \frac{(-1)^m}{\pi} \psi(n) \frac{2n}{2\delta n} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \cos t dt = \frac{(-1)^m}{\pi} \psi(n) \frac{2}{\delta} \sin \frac{\delta}{2}.$$

Choose  $\delta$  so small that

$$\frac{2}{\delta} \sin \frac{\delta}{2} \psi(n) > \sum_{k=n+1}^{\infty} \psi(k). \tag{34}$$

For such  $\delta$ , taking into account (32), (33) and (34),

$$|F_1(\delta, x_m)| = \frac{2}{\delta} \sin \frac{\delta}{2} \psi(n) > \frac{1}{\pi} \sum_{k=n+1}^{\infty} \psi(k) \geq \|F_2(\delta, \cdot)\|_C \geq |F_2(\delta, x_m)|,$$

and hence for  $f_{n,p,\bar{\beta}}(\delta, x_m) = F_1(\delta, x_m) + F_2(\delta, x_m)$  the following relations hold

$$\text{sign} f_{n,p,\bar{\beta}}(\delta, x_m) = \text{sign} F_1(\delta, x_m) = (-1)^m, \quad m = 0, 1, \dots, 2n - 1, \tag{35}$$

and

$$\begin{aligned} |f_{n,p,\bar{\beta}}(\delta, x_m)| &\geq |F_1(\delta, x_m)| - |F_2(\delta, x_m)| \geq |F_1(\delta, x_m)| - \|F_2(\delta, \cdot)\|_C \\ &\geq \frac{1}{\pi} \left( \frac{2}{\delta} \sin \frac{\delta}{2} \psi(n) - \sum_{k=n+1}^{\infty} \psi(k) \right). \end{aligned} \tag{36}$$

Then by virtue of Valle Poussin's theorem (see, i.e., [31, Theorem 6.2.2])

$$\begin{aligned} E_n(f_{n,p,\bar{\beta}}(\delta, \cdot))_C &\geq \min_{m=0,1,\dots,2n-1} |f_{n,p,\bar{\beta}}(\delta, x_m)| \\ &\geq \frac{1}{\pi} \left( \frac{2}{\delta} \sin \frac{\delta}{2} \psi(n) - \sum_{k=n+1}^{\infty} \psi(k) \right), \end{aligned} \tag{37}$$

where  $\delta$  satisfy the condition (34), and therefore, by virtue of the belonging  $f_{n,p,\bar{\beta}}(\delta, \cdot) \in C_{\bar{\beta},1}^\psi$ , the following inequality is true

$$E_n(C_{\bar{\beta},1}^\psi)_C \geq \frac{1}{\pi} \left( \frac{2}{\delta} \sin \frac{\delta}{2} \psi(n) - \sum_{k=n+1}^{\infty} \psi(k) \right). \tag{38}$$

Taking the limit as  $\delta \rightarrow 0$  in the inequality (38), we obtain (25). □

The main statement of the work is the following theorem.

**Theorem 3.** For arbitrary  $\{\beta_k\}_{k=1}^\infty$ ,  $\beta_k \in \mathbb{R}$ ,  $1 \leq p \leq \infty$ ,  $n \in \mathbb{N}$  and  $\psi(k) \geq 0$ , which satisfy the condition (15), the following inequalities hold

$$\begin{aligned} \frac{\|\cos t\|_{p'}}{\pi} \left( \psi(n) - \sum_{k=n+1}^\infty \psi(k) \right) &\leq E_n(C_{\beta,p}^\psi)_C \leq \mathcal{E}_n(C_{\beta,p}^\psi)_C \\ &\leq \frac{\|\cos t\|_{p'}}{\pi} \left( \psi(n) + \sum_{k=n+1}^\infty \psi(k) \right), \end{aligned} \tag{39}$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$ .

If  $\psi(k)$  satisfies the condition (6), then the following asymptotic equalities hold

$$\left. \begin{aligned} \mathcal{E}_n(C_{\beta,p}^\psi)_C \\ E_n(C_{\beta,p}^\psi)_C \end{aligned} \right\} = \frac{\|\cos t\|_{p'}}{\pi} \psi(n) + \mathcal{O}(1) \sum_{k=n+1}^\infty \psi(k), \tag{40}$$

where  $\mathcal{O}(1)$  are the quantities uniformly bounded in all parameters.

Note that the asymptotic equality (40) for the quantities  $\mathcal{E}_n(C_{\beta,p}^\psi)_C$  is established in paper [23].

*Proof.* Taking into account Theorems 1 and 2, it suffices to verify the validity of the last inequality in (39).

By virtue of Hölder’s inequality for arbitrary  $f \in C_{\beta,p}^\psi$ ,  $1 \leq p \leq \infty$ ,

$$\begin{aligned} \|f - S_{n-1}(f)\|_C &\leq \frac{1}{\pi} \|\varphi\|_p \left\| \sum_{k=n}^\infty \psi(k) \cos \left( nt - \frac{\beta_k \pi}{2} \right) \right\|_{p'} \\ &\leq \frac{1}{\pi} \sum_{k=n}^\infty \psi(k) \left\| \cos \left( nt - \frac{\beta_k \pi}{2} \right) \right\|_{p'} = \frac{\|\cos t\|_{p'}}{\pi} \sum_{k=n}^\infty \psi(k). \end{aligned}$$

Thus,

$$\mathcal{E}_n(C_{\beta,p}^\psi)_C \leq \frac{\|\cos t\|_{p'}}{\pi} \sum_{k=n}^\infty \psi(k). \tag{41}$$

Inequalities (39) follow from (16), (25) and (41).

To verify the validity of the asymptotic equalities (40) just go to the limit as  $n \rightarrow \infty$  in (39) and take into account the condition (6).  $\square$

Note that the condition (6) is satisfied if  $\psi(k)$  satisfies the condition  $D_0$  :

$$\lim_{k \rightarrow \infty} \frac{\psi(k+1)}{\psi(k)} = 0. \tag{42}$$

We give the corollaries of Theorem 3 in some important special cases.

**Theorem 4.** *Let  $1 \leq p \leq \infty$ ,  $r > 1$ ,  $n \in \mathbb{N}$  and  $\{\beta_k\}_{k=1}^\infty$  be an arbitrary sequence of real numbers. Then for  $r \geq n + 1$  such that*

$$\left(1 + \frac{1}{n}\right)^{-r} < \left(2 + \frac{1}{n}\right)^{-1} \tag{43}$$

the following inequalities hold

$$\begin{aligned} \frac{\|\cos t\|_{p'}}{\pi} n^{-r} \left(1 - \frac{2 + \frac{1}{n}}{\left(1 + \frac{1}{n}\right)^r}\right) &\leq E_n(W_{\beta,p}^r)_C \leq \mathcal{E}_n(W_{\beta,p}^r)_C \\ &\leq \frac{\|\cos t\|_{p'}}{\pi} n^{-r} \left(1 + \frac{2 + \frac{1}{n}}{\left(1 + \frac{1}{n}\right)^r}\right), \end{aligned} \tag{44}$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$ .

*Proof.* Put  $\psi(k) = k^{-r}$ ,  $r > 1$ , and make sure that the condition (15) is folowed from the inequality (43).

Since for arbitrary  $n \in \mathbb{N}$  and  $r \geq n + 1$

$$\begin{aligned} \sum_{k=n+1}^\infty \frac{1}{k^r} &< \frac{1}{(n+1)^r} + \int_{n+1}^\infty \frac{dt}{t^r} = \frac{1}{(n+1)^r} + \frac{1}{(r-1)(n+1)^{r-1}} \\ &= \frac{1}{(n+1)^r} \frac{r+n}{r-1} \leq \frac{1}{n^r} \frac{1}{\left(1 + \frac{1}{n}\right)^r} \frac{2r-1}{r-1} \leq \frac{1}{n^r} \frac{1}{\left(1 + \frac{1}{n}\right)^r} \left(2 + \frac{1}{n}\right), \end{aligned} \tag{45}$$

then under the condition (43) we obtain

$$\sum_{k=n+1}^\infty \frac{1}{k^r} < \frac{1}{n^r}.$$

Then, applying Theorem 3 for  $\psi(k) = k^{-r}$ , from (39) and (45) we obtain (44). □

Note that if the condition

$$\lim_{n \rightarrow \infty} \frac{r}{n} = \infty, \tag{46}$$

is satisfied, then the condition (43) is also satisfied for sufficiently large  $n$ , because

$$\left(1 + \frac{1}{n}\right)^{-r} = \left(\left(1 + \frac{1}{n}\right)^{n+1}\right)^{-\frac{r}{n+1}} \leq e^{-\frac{r}{n+1}} \rightarrow 0, \quad n \rightarrow \infty.$$

And therefore

$$\frac{2 + \frac{1}{n}}{\left(1 + \frac{1}{n}\right)^r} \rightarrow 0 \quad \text{при} \quad n \rightarrow \infty.$$

Taking the limit as  $n \rightarrow \infty$  in the inequalities (44), we obtain asymptotic equalities for the quantities  $E_n(W_{\bar{\beta},p}^r)_C$  and  $\mathcal{E}_n(W_{\bar{\beta},p}^r)_C$ .

**Theorem 5.** *Let  $1 \leq p \leq \infty$ ,  $r > 1$ ,  $n \in \mathbb{N}$  and  $\{\beta_k\}_{k=1}^\infty$  be an arbitrary sequence of real numbers. If the condition (46) is satisfied, then the following asymptotic equalities hold*

$$\left. \begin{aligned} \mathcal{E}_n(W_{\bar{\beta},p}^r)_C \\ E_n(W_{\bar{\beta},p}^r)_C \end{aligned} \right\} = \frac{1}{n^r} \left( \frac{\|\cos t\|_{p'}}{\pi} + \mathcal{O}(1) \left(1 + \frac{1}{n}\right)^{-r} \right), \quad (47)$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $\mathcal{O}(1)$  are the quantities uniformly bounded in all parameters.

The asymptotic equality (47) for the quantities  $\mathcal{E}_n(C_{\bar{\beta},p}^\psi)_C$  is established in [26].

As it showed in [31, P. 163–164]

$$\sum_{k=n+1}^\infty e^{-\alpha k^r} < e^{-\alpha n^r} \left(1 + \frac{1}{\alpha r n^{r-1}}\right) e^{-\alpha r n^{r-1}}, \quad r > 1, \alpha > 0, n \in \mathbb{N}. \quad (48)$$

So, if the condition

$$\left(1 + \frac{1}{\alpha r n^{r-1}}\right) e^{-\alpha r n^{r-1}} < 1, \quad r > 1, \alpha > 0, \quad (49)$$

is satisfied, then

$$\sum_{k=n+1}^\infty e^{-\alpha k^r} < e^{-\alpha n^r},$$

and therefore from Theorem 3 for the classes  $C_{\bar{\beta},p}^{\alpha,r}$  we obtain the following statement.

**Theorem 6.** *Let  $1 \leq p \leq \infty$ ,  $r > 1$ ,  $\alpha > 0$ ,  $n \in \mathbb{N}$  and  $\{\beta_k\}_{k=1}^\infty$  be an arbitrary sequence of real numbers. If the inequality (49) is satisfied, then the following relations hold*

$$\begin{aligned} & \frac{\|\cos t\|_{p'}}{\pi} e^{-\alpha n^r} \left(1 - \left(1 + \frac{1}{\alpha r n^{r-1}}\right) e^{-\alpha r n^{r-1}}\right) \leq E_n(C_{\bar{\beta},p}^{\alpha,r})_C \\ & \leq \mathcal{E}_n(C_{\bar{\beta},p}^{\alpha,r})_C \leq \frac{\|\cos t\|_{p'}}{\pi} e^{-\alpha n^r} \left(1 + \left(1 + \frac{1}{\alpha r n^{r-1}}\right) e^{-\alpha r n^{r-1}}\right), \end{aligned}$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$ .

As  $n \rightarrow \infty$  the following asymptotic equalities hold

$$\left. \begin{array}{l} \mathcal{E}_n(C_{\bar{\beta}, p}^{\alpha, r})_C \\ E_n(C_{\bar{\beta}, p}^{\alpha, r})_C \end{array} \right\} = e^{-\alpha n^r} \left( \frac{\|\cos t\|_{p'}}{\pi} + \mathcal{O}(1) \left( 1 + \left( 1 + \frac{1}{\alpha n^{r-1}} \right) e^{-\alpha n^{r-1}} \right) \right), \quad (50)$$

where  $\mathcal{O}(1)$  are the quantities uniformly bounded in all parameters.

Note that for  $p = \infty$  the asymptotic equality (50) for the quantities  $\mathcal{E}_n(C_{\bar{\beta}, p}^{\alpha, r})_C$  was obtained by Stepanets [31] and for  $1 \leq p < \infty$  in [23].

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