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A note on meromorphic functions with finite order and of bounded l-index

Andriy Bandura

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Abstract. We present a generalization of concept of bounded l-index for meromorphic functions of finite order. Using known results for entire functions of bounded l-index we obtain similar propositions for meromorphic functions. There are presented analogs of Hayman's Theorem and logarithmic criterion for this class. The propositions are widely used to investigate l-index boundedness of entire solutions of differential equations. Taking this into account we raise a general problem of generalization of some results from theory of entire functions of bounded l-index by meromorphic functions of finite order and their applications to meromorphic solutions of differential equations. There are deduced sufficient conditions providing l-index boundedness of meromoprhic solutions of finite order for the Riccati differential equation. Also we proved that the Weierstrass $\wp$-function has bounded l-index with $l(z) = |z|$.

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1. Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic function. The goal of the paper is to introduce a concept of bounded index for meromorphic functions of finite order and to study properties of functions from this class. There are considered some applications of theory of bounded index to study properties of analytic solutions of differential equations and of some infinite products. In particularly, we find conditions providing l-index boundedness of all meromorphic solutions of Riccati's differential equation. Moreover, we prove that the Weierstrass $\wp$-function has bounded l-index with $l(z) = |z|$.

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We remind some classic notations from the theory of meromorphic functions of one variable (see also [12, 14, 17, 18]). The number of poles of $f(z)$ in the disc $\{|z| \le r\}$ will be denoted by $n(r, f)$. The order ρ of meromorphic function f is defined as

$$\rho = \overline{\lim_{r\to\infty}} \frac{\ln^+ T(r, f)}{\ln r}$$

where $a^+ = \max\{a, 0\}$, $T(r, f) = m(r, f) + N(r, f)$,

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(re^{i\varphi})| d\varphi,$$

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \ln r.$$

The meromorphic function f admits a representation as quotient $f(z) = \frac{f_1(z)}{f_2(z)}$, where $f_1(z)$ and $f_2(z)$ are entire functions without common zeros. Let $u(z) = \max\{|f_1(z)|, |f_2(z)|\}$. Despite the non-uniqueness of representation the function f as quotient of entire functions we have [12, p. 17] $T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \ln u(re^{i\phi}) d\varphi$.

A number $p \in \mathbb{N} \cup \{0\}$ is called a genus of the sequence $(a_n)_{n\in\mathbb{N}}$ if $\sum_{n=1}^{\infty} \frac{1}{|a_n|^p} = \infty$ and $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty$.

For the meromorphic functions of finite order it is known a convenient representation.

Theorem 1.1 (Hadamard, [12, p. 57]). *Let $f(z)$ be a meromorphic function of finite order ρ, and let $(a_n)_{n\in\mathbb{N}}$ and $(b_m)_{m\in\mathbb{N}}$ be the sequences of zeros and poles of the function $f(z)$, which are different from $z = 0$. Let p_1 be the genus of (a_n) and p_2 be the genus of (b_m). Suppose that in the neighborhood of $z = 0$ the function $f(z)$ has a representation $f(z) = c_s z^s + c_{s+1} z^{s+1} + \ldots$ with $c_s \neq 0$. Then*

$$f(z) = z^s e^{P(z)} \frac{\prod_{a_n} E(z/a_n, p_1)}{\prod_{b_n} E(z/b_n, p_2)}, \tag{1.1}$$

where $P(z)$ is a polynomial, whose degree q does not exceed $[\rho]$ and

$$E(z, p) = \begin{cases} 1 - z, & \text{if } p = 0, \\ (1 - z) \exp\{z + z^2/2 + \ldots + z^p/p\}, & \text{if } p \ge 1. \end{cases}$$

Taking into account of Theorem 1.1, we suppose that

$$f_1(z) = z^s e^{P(z)} \prod_{a_n} E(z/a_n, p_1), \ f_2(z) = \prod_{b_n} E(z/b_n, p_2).$$

There are known many differential equations which have only meromorphic solutions. For example, Riccati's equation, Painlevé's equations, Briot-Bouquet's equation and etc. Despite the second century of their exploration these equations are still interesting for many mathematicians [8, 13, 16]. At the same time, for differential equations with entire solutions there is well-developed theory of functions of bounded index. The concept has few advantages in the comparison with the traditional approaches to study the properties of entire solutions of differential equations. In particular, if an entire solution has a bounded index then it immediately yields its growth estimates, a uniform in a some sense distribution of its zeros, a certain regular behavior of the solution, etc [4,20,24,25]. The concept of bounded index allows to describe a local behavior of analytic functions. Another approach to study local behavior is developed by Donetsk school of mathematicians. Between them are papers of E.A. Sevost'yanov [30, 31], R.R. Salimov [28, 29], where they consider quasiconformal mappings.

2. Main definition and base propositions

Let $l : \mathbb{C} \to \mathbb{R}_+$ be a fixed positive continuous function, where $\mathbb{R}_+ = (0, +\infty)$. An entire function f is said to be of bounded $l-$index [19] if there exists an integer m, independent of z, such that for all p and all $z \in \mathbb{C}$ $\frac{|f^{(p)}(z)|}{l^p(z)p!} \leq \max\{\frac{|f^{(s)}(z)|}{l^s(z)s!} : 0 \leq s \leq m\}$. The least such integer m is called the l-index of f and is denoted by $N(f;l)$. If $l(z) \equiv 1$ then the function f is of bounded index [21].

In this paper, we propose the following generalization of the concept of bounded l-index. We say that a meromorphic function f of finite order *is of bounded l-index* if the entire functions f_1 and f_2 are of bounded l-index where the functions f_1, f_2 are defined above. And the l-index of the function f is defined as $N(f, l) = \max\{N(f_1; l), N(f; l_2)\}$.

The definition is correct [25] because if an entire function f is of bounded l-index then f is of bounded l^*-index for any $l^* \geq l$. If $f_2(z)$ is a polynomial and $l \equiv 1$ then the definition matches with the definition of index boundedness for meromorphic function with finite number of poles proposed by R. Roy and S. Shah [23].

Obviously, the positivity and continuity are weak restrictions by the function l. Therefore, we will assume some additional conditions. Let Q be a class of positive continuous functions $l : \mathbb{C} \to \mathbb{R}_+$ such that

$$\lambda(r) = \sup_{t_1,t_2\in\mathbb{C}} \left\{ \frac{l(t_1)}{l(t_2)} : |t_1 - t_2| < \frac{r}{\min\{l(t_1), l(t_2)\}} \right\}$$

is finite for all $r \geq 0$.

If $l = l(|z|)$ then the condition '$\lambda(r)$ is finite' means that $l(r + O(\frac{1}{l(r)})) = O(l(r))$ as $r \to +\infty$. Therefore, this class Q is very wide. All elementary functions and their compositions belong to this class. For example, the iterated exponent also is from Q. And if an entire function f has bounded l-index with $l \notin Q$ then the function f is of bounded l^*-index for any $l^* \geq l$. And in the worst case we can choose the function l^* as the iterated exponent.

Using the supposed definition and some properties of entire functions of bounded l-index we immediately obtain the following theorems for meromorphic functions of bounded l-index.

Proposition 2.1. *Let $l \in Q$, $f = \frac{f_1}{f_2}$ be a meromorphic function of finite order, f_1, f_2 are defined above. If there exists $r > 0$ $n_0 \in \mathbb{Z}_+$ and $P_0 \geq 1$ such that for every $z_0 \in \mathbb{C}$ and some $k_j = k_j(z_0) \in \mathbb{Z}_+$ with $0 \leq k_j \leq n_0$, $j \in \{1, 2\}$*

$$\max\{|f_j^{(k_j)}(z)| : \ |z - z_0| \leq \frac{r}{l(z_0)}\} \leq P_0 |f_j^{(k_j)}(z_0)|, \ j \in \{1, 2\} \qquad (2.1)$$

then f has bounded l-index.

Proposition 2.2. *Let $l \in Q$, $f = \frac{f_1}{f_2}$ be a meromorphic function of finite order, f_1, f_2 are defined above. If f has bounded l-index then for every $r > 0$ there exists $n_0 \in \mathbb{Z}_+$ and $P_0 \geq 1$ such that for every $z_0 \in \mathbb{C}$ and some $k_j = k_j(z_0) \in \mathbb{Z}_+$ with $0 \leq k_j \leq n_0$, $j \in \{1, 2\}$ inequality* (2.1) *holds.*

Proposition 2.3. *Let $l \in Q$, $f = \frac{f_1}{f_2}$ be a meromorphic function of finite order, f_1, f_2 are defined above. If there exists r_{1j}, r_{2j} with $0 < r_{1j} < r_{2j}$ and $P_1 \geq 1$ such that for every $z_0 \in \mathbb{C}$*

$$\max\left\{|f_j(z)| : |z - z_0| = \frac{r_{2j}}{l(z_0)}\right\} \leq$$
$$\leq P_1 \max\left\{|f_j(z)| : |z - z_0| = \frac{r_{1j}}{l(z_0)}\right\}, \ j \in \{1, 2\}$$

then f has bounded l-index.

Proposition 2.4. *Let $l \in Q$, $f = \frac{f_1}{f_2}$ be a meromorphic function of finite order, f_1, f_2 are defined above. If f has bounded l-index then for every r_1, r_2 with $0 < r_1 < r_2$ there exists $P_1 \geq 1$ such that for every $z_0 \in \mathbb{C}$*

$$\max\left\{|f_j(z)| : |z - z_0| = \frac{r_2}{l(z_0)}\right\} \leq$$
$$\leq P_1 \max\left\{|f_j(z)| : |z - z_0| = \frac{r_1}{l(z_0)}\right\}, \ j \in \{1, 2\}$$

Proposition 2.5. *Let $l \in Q$, $f = \frac{f_1}{f_2}$ be a meromorphic function of finite order, f_1, f_2 are defined above. If there exist $R > 0$, $P_2 \geq 1$ and $\eta \in (0, R)$ such that for every $z_0 \in \mathbb{C}$ and some $r_j = r_j(z_0) \in [\eta, R]$*

$$\max\left\{|f_j(z)| : |z - z_0| = \frac{r_j}{l(z_0)}\right\} \leq$$
$$\leq P_2 \min\left\{|f_j(z)| : |z - z_0| = \frac{r_j}{l(z_0)}\right\}, \ j \in \{1, 2\} \tag{2.2}$$

then f has bounded l-index.

Proposition 2.6. *Let $l \in Q$, $f = \frac{f_1}{f_2}$ be a meromorphic function of finite order, f_1, f_2 are defined above. If the function f has bounded l-index then for every $R > 0$ there exists $P_2 \geq 1$ and $\eta \in (0, R)$ such that for every $z_0 \in \mathbb{C}$ and some $r_j = r_j(z_0) \in [\eta, R]$ inequality* (2.2) *is fulfilled.*

Proposition 2.7. *Let $l \in Q$, $f = \frac{f_1}{f_2}$ be a meromorphic function of finite order, f_1, f_2 are defined above. The meromorphic function f is of bounded l-index if and only if there exist numbers $p_j \in \mathbb{Z}_+$ and $C > 0$ such that*

$$\frac{|f_j^{(p_j+1)}(z)|}{l^{p_j+1}(z)} \leq C \max\left\{\frac{|f_j^{(k)}(z)|}{l^k(z)} : 0 \leq k \leq p_j\right\}, \ j \in \{1, 2\}.$$

Let us denote $n(r, z, 1/f) = \sum_{|a_k - z| < r} 1$ be a counting zero function, $n(r, z, f) = \sum_{|b_k - z| < r} 1$ be a counting pole function, where z is a fixed point, $(a_k)_{k \in \mathbb{N}}$ is a zero sequence of the function f, $(b_k)_{k \in \mathbb{N}}$ is a pole sequence of the function f. Let us write $G_r(f_1) = \bigcup_n \{z : |z - a_n| < \frac{r}{l(a_n)}\}$ and $G_r(f_2) = \bigcup_n \{z : |z - b_n| < \frac{r}{l(b_n)}\}$.

Proposition 2.8. *Suppose $l \in Q$, $f = \frac{f_1}{f_2}$ be a meromorphic function of finite order, f_1, f_2 are defined above. If the meromorphic function f has bounded l-index then*

1) *for any $r > 0$ there exists $P = P(r) > 0$ such that $|f_j'(z)/f_j(z)| \leq Pl(z)$ for each $z \in \mathbb{C} \setminus G_r(f_j)$, $j \in \{1, 2\}$;*

2) *for any $r > 0$ there exists $\tilde{n} = \tilde{n}(r) \in \mathbb{Z}_+$ such that $n(r/l(z), z, 1/f) \leq \tilde{n}$ and $n(r/l(z), z, f) \leq \tilde{n}$ for each $z \in \mathbb{C}$.*

Denote $n(r, f) = \sup\limits_{z \in \mathbb{C}} n(r, z, f)$, $n(r, 1/f) = \sup\limits_{z \in \mathbb{C}} n(r, z, 1/f)$, $n(r) = \sup\{n(r, f), n(r, 1/f)\}$.

Proposition 2.9. *Suppose $l \in Q$, $f = \frac{f_1}{f_2}$ be a meromorphic function of finite order, f_1, f_2 are defined above. If*

1) $n(r) \equiv 0$ *or there exists* $r_1 > 0$ $n(r_1) \in [1; \infty)$

2) there exist $r_2 \in (0, \frac{r_1/\lambda(r_1)}{2n(r_1)})$ *(for* $n(r) \equiv 0$ $r_2 = 0$*) and* $P > 0$ *such that for each* $z \in \mathbb{C}\backslash G_{r_2}(f_j)$, $j \in \{1, 2\}$, *the inequality* $\frac{|f_j'(z)|}{|f_j(z)|} \leq Pl(z)$ *holds, where* r_1 *is chosen from 1),*

then the function f *has bounded* l*-index.*

Proposition 2.1 follows from sufficiency of Theorem 4 in [25], Proposition 2.2 follows from necessity of Theorem 2 in [25], Proposition 2.4 follows from necessity of Theorem 5 in [25], Proposition 2.6 follows from necessity of Theorem 6 in [25], Proposition 2.7 follows from Theorem 1 in [26], and Proposition 2.8 follows from necessity of Theorem 1 in [25]. However, Proposition 2.3 follows from Theorem 4 in [5], Proposition 2.5 follows from Theorem 6 in [5], Proposition 2.9 follows from Theorem 8 in [5].

For entire functions of bounded index, Propositions 2.1–2.6, 2.8–2.9 were deduced by G. H. Fricke [10,11], and Proposition 2.7 was obtained by W. Hayman [15]. Later M. M. Sheremeta and A. D. Kuzyk [25,26] generalized them for entire functions of bounded l-index. Recently, A. I. Bandura and O. B. Skaskiv [1,5] weakened sufficient conditions in these theorems. Here we formulate the sufficient conditions and the necessary conditions as different propositions, but not as criteria. Propositions 2.1–2.9 describe the local behavior of meromorhic functions of bounded l-index. Moreover, the analogs of Propositions 2.7, 2.8, 2.9 for the various classes of holomorphic functions are very often used to examine properties of entire and analytic solutions of differential equations and systems of partial differential equations [3, 4, 22]. They help to prove l-index boundedness of these solutions. Thus, it leads to the following question:

Problem 1. *Let* $l \in Q$*. What are conditions by the meromorphic coefficients of the higher order linear differential equation*

$$p_0(z)w^{(n)} + p_1(z)w^{(n-1)} + \ldots + p_n(z)w = h(z)$$

providing l*-index boundedness of every meromorphic solutions of finite order?*

If h is a meromorphic function with a finite number of poles and of bounded index ($l \equiv 1$) and p_j are polynomials then the equation is studied by S. Shah and R. Roy [23]. Particularly, they proved that if $\deg P_0 \geq \max_{1 \leq i \leq n} P_i$ then every meromorphic solution of the equation is of bounded index.

Of course, Propositions 2.1–2.9 generate a more general problem

Problem 2. *Is it possible to deduce analogs of all results from the theory of entire functions of bounded l-index for the meromorphic functions of finite order?*

3. Riccati's differential equation

It is known [13] that all solutions of the Riccati differential equation with entire coefficients

$$f'(z) = a_0(z) + a_1(z)f(z) + a_2(z)f^2(z), \quad a_2(z) \not\equiv 0, \tag{3.1}$$

are meromorphic. Moreover, every pair of entire functions (P, T) satisfying the system of differential equations

$$\begin{cases} P' = a_0(z)T + a_1(z)P, \\ T' = -a_2(z)P, \end{cases} \tag{3.2}$$

is a meromorphic solution of (3.1). Therefore, the following statement is valid.

Theorem 3.1. *Let $l \in Q$. Assume that a_0, a_1, a_2 are entire functions of bounded l-index and there exists $M > 0$ such that for all $z \in \mathbb{C}$ $|a_j(z)| \le Ml(z)$, $j \in \{1, 2, 3\}$. Then every meromorphic solution $f(z) = \frac{P(z)}{T(z)}$ of* (3.1) *is of bounded l-index, where P, T are entire solutions of system* (3.2) *with finite order.*

Proof. To prove the theorem we need some notations from [7]. There was proposed a concept of analytic curve in disc having bounded l-index. We only consider entire curves.

Let $m \in \mathbb{N}$ and $F = (f_1, f_2, \dots, f_m)$ be an entire curve in D_R, i.e. $F : D_R \to \mathbb{C}^m$ is a vector-valued function, where each function f_j is entire. We put $F^{(n)} = (f_1^{(n)}, f_2^{(n)}, \dots, f_m^{(n)})$, and let $\|F(z)\|_S = \max\{|f_j(z)| : 1 \le j \le m\}$ be the sup-norm.

An analytic curve F is said [7] to be of bounded l-index by sup-norm if there exists $N \in \mathbb{Z}_+$ such that

$$\frac{\|F^{(n)}(z)\|_S}{n! l^n(|z|)} \le \max\left\{ \frac{\|F^{(k)}(z)\|_S}{k! l^k(|z|)} : 0 \le k \le N \right\} \tag{3.3}$$

for all $n \in \mathbb{Z}_+$ and $z \in D_R$. The least such integer N is called the l-index by sup-norm and is denoted by $N_S(l; F)$.

Bordulyak M. and Sheremeta M. obtained some proposition for analytic curves in the disc of arbitrary radius $R \in (0, +\infty]$. Below we formulate the corresponding consequence for entire curves.

Remind that $G_r(f) = \bigcup_n \{z : |z - a_n| < \frac{r}{l(a_n)}\}$ where $(a_n)_{n\in\mathbb{N}}$ is a zero sequence of the function f.

Lemma 3.1 ([7, Theorem 4]). *Let $l \in Q$. Assume that an entire curve $F(z) = \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix}$ satisfies the differential equation*

$$W' + Q(z)W = 0, \tag{3.4}$$

where $Q(z) = \begin{pmatrix} a_1(z) & a_2(z) \\ a_3(z) & a_4(z) \end{pmatrix}$. Let a_j, $j = \{1, 2, 3, 4\}$, be meromorphic functions of the form $a_j = \frac{A_j}{B_j}$, where A_j, B_j are entire functions of bounded l-index. If for every $r > 0$ there exists $M = M(r) > 0$ such that for each $z \in \mathbb{C} \setminus \bigcup_{j=1}^{4} G_r(B_j)$ and for every $j \in \{1, 2, 3, 4\}$

$$|a_j(z)| \le Ml(z) \tag{3.5}$$

then f_1, f_2, and, therefore, F are of bounded l-index in D_R.

For system (3.2) we have $W = \begin{pmatrix} P(z) \\ T(z) \end{pmatrix}$, $Q(z) = \begin{pmatrix} a_1(z) & a_0(z) \\ -a_2(z) & 0 \end{pmatrix}$, all $B_j(z) \equiv 1$, $a_j(z) \equiv A_j(z)$ and $G_r(B_j) = \emptyset$.

Then by Lemma 3.1 the functions $P(z)$ and $Q(z)$ are of bounded l-index. Therefore, the meromorphic function $f(z) = \frac{P(z)}{Q(z)}$ is of bounded l-index. □

4. Weierstrass $\wp$-function

In this section, we demostrate an application of the concept of bounded index to study properties of some known meromorphic function.

Let us consider the Weierstrass $\wp$-function [27, p.434]

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left\{ \frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)} \right\}$$

where $Im\,\frac{\omega_2}{\omega_1} > 0$ and the summation extends over all integer values (positive, negative and zero) of m and n, simultaneous zero values of m and n are excepted.

For this function it is known [27, p.451] that for any z, $y \in \mathbb{C}$ one has

$$\wp(z) - \wp(y) = -\frac{\sigma(z+y)\sigma(z-y)}{\sigma^2(z)\sigma^2(y)}, \tag{4.1}$$

where $\sigma(z)$ is the Weierstrass σ-function.

The function $\wp$ has two zeros (modulo periods ω_1 and ω_2). Also it has second-order pole at each point 0 and $2m\omega_1 - 2n\omega_2$. It is very difficult to express the zeros of $\wp$ by closed formula, except for special values of the modulus (e.g. when the period lattice is the Gaussian integers). An expression was found, by Zagier and Eichler [9]. Let y_0 be zero of the function $\wp$. Then from (4.1) with $y = y_0$ we deduce that

$$\wp(z) = -\frac{\sigma(z+y_0)\sigma(z-y_0)}{\sigma^2(z)\sigma^2(y_0)}.$$

The function σ is entire function of bounded l-index with $l(z) = |z|$ (see [6]). It is known [25, Corollary 1] that the product of entire functions of bounded l-index is also a function of bounded l-index. Therefore, we conclude that the functions $\sigma(z+y_0)\sigma(z-y_0)$ and $\sigma^2(z)\sigma^2(y_0)$ have bounded l-index. In view of definition of bounded l-index for meromorphic function the function$\wp$ is also of bounded l-index. Therefore, we have proved the following proposition.

Proposition 4.1. *The Weierstrass $\wp$-function has bounded l-index with $l(z) = |z|$.*

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References

[1] Bandura, A.I., Skaskiv, O.B. (2017). Directional logarithmic derivative and the distribution of zeros of an entire function of bounded L-index along the direction. *Ukrain. Mat. J., 69*(3), 500–508.

[2] Bandura, A., Skaskiv, O. (2017). Functions analytic in a unit ball of bounded L-index in joint variables. *Ukr. Math. Bull. 14*(1), 1–15; transl. in (2017). *J. Math. Sci., 227*(1), 1–12.

[3] Bandura, A.I., Skaskiv, O.B. (2017). Analytic functions in the unit ball of bounded $\mathbf{L}$-index: asymptotic and local properties.*Mat. Stud., 48*(1), 37–73.

[4] Bandura, A., Skaskiv, O., Filevych, P. (2017). Properties of entire solutions of some linear PDE's. *J. Appl. Math. Comput. Mech., 16*(2), 17–28.

[5] Bandura, A.I. (2017). Some improvements of criteria of L-index boundedness in direction. *Mat. Stud., 47*(1), 27–32.

[6] Bordulyak, M.T. (2013). On l-index boundedness of the Weierstrass σ-function. *Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform., 63*(1), 49–56.

[7] Bordulyak, M.T., Sheremeta, M.M. (2011). Boundedness of l-index of analytic curves. *Mat. Stud., 36*(2), 152–161.

[8] Ciechanowicz, E., Filipuk, G. (2016). Meromorphic solutions of $P_{3,34}$ and their value distribution. *Annales Academiæ Scientiarum Fennicæ. Mathematica, 41*, 617–638.

[9] Eichler, M., Zagier, D. (1982). On the zeros of the Weierstrass $\wp$-Function. *Mathematische Annalen., 258*(4), 399–407.

[10] Fricke, G.H. (1975). Entire functions of locally slow growth. *J. Anal. Math., 28*(1), 101–122.

[11] Fricke, G.H. (1973). Functions of bounded index and their logarithmic derivatives. *Math. Ann., 206*, 215–223.

[12] Goldberg, A.A., Ostrovskii, I.V. (2008). *Value Distribution of meromorphic functions*. Providence, AMS. Translations of Mathematical monographs, vol. 236.

[13] Gromak, V.I., Laine, I., Shimomura, S. (2008). *Painlevé Differential Equations in the Complex Plane,* in: De Gruyter Studies in Mathematics, V. 28, Walter de Gruyter.

[14] Hanyak, M.O., Kondratyuk, A.A. (2007). Meromorphic functions in m-punctured complex planes. *Mat. Stud., 27*(1), 53–69.

[15] Hayman, W.K. (1973). Differential inequalities and local valency. *Pacific J. Math., 44*(1), 117–137.

[16] Hinkkanen, A., Laine, I. (2001). Solutions of a modified third Painlevé equation are meromorphic. *J. Anal. Math., 85*(1), 323–337.

[17] Kondratyuk, A.A. (1988). *Fourier series and meromorphic functions.* Lvov: Vyshcha shkola (in Russian).

[18] Krystiyanyn, A.Ya., Kondratyuk, A.A. (2005). On the Nevanlinna theory for meromorphic functions on annuli I. *Mat. Stud., 23*(1), 19–30.

[19] Kuzyk, A.D., Sheremeta, M.N. (1986). Entire functions of bounded l-distribution of values. *Math. notes, 39*(1), 3–8.

[20] Kuzyk, A.D., Sheremeta, M.N. (1990). On entire functions, satisfying linear differential equations. *Diff. equations, 26*(10), 1716–1722 (in Russian).

[21] Lepson, B. (1968). Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index. *Proc. Sympos. Pure Math., 11*, 298–307.

[22] Petrechko, N. (2017). Bounded **L**-index in joint variables and analytic solutions of some systems of PDE's in bidisc. *Visn. Lviv Univ. Ser. Mech. Math., 83*, 100–108.

[23] Roy, R., Shah, S.M. (1983). Meromorphic functions satisfying a differential equation. In: *Value Distribution Theory and Its Applications. Contemporary Mathematics, 25*, 131–139.

[24] Shah, S.M. (1983). Entire solutions of linear differential equations and bounds for growth and index numbers. *Proc. Sect. A: Mathematics, Royal Soc. Edinburgh, 93A*, 49–60.

[25] Sheremeta, M.N., Kuzyk, A.D. (1992). Logarithmic derivative and zeros of an entire function of bounded l-index. *Sib. Math. J., 33*(2), 304–312.

[26] Sheremeta, M.N. (1992). Entire functions and Dirichlet series of bounded l-index. *Russian Math. (Iz. VUZ), 36*(9), 76–82.

[27] Whittaker, E.T., Watson, G.N. (1996). *A course of modern analysis.* 4th ed., Reprinted Campridge Unviersity Press.

[28] Afanasieva, E.S., Ryazanov, V.I., Salimov, R.R. (2012). On mappings in the Orlicz-Sobolev classes on Riemannian manifolds. *J. Math. Sci., 181*(1), 1–17.

[29] Kovtonyuk, D., Petkov, I., Ryazanov, V., Salimov, R. (2014). On the Dirichlet problem for the Beltrami equation. *J. d'Analyse Mathematique, 122*(1), 113–141.

[30] Sevost'yanov, E.A. (2009). Generalization of one Poletskii lemma to classes of space mappings. *Ukr. Math. J., 61*(7), 1151–1157.

[31] Sevost'yanov, E.A., Skvortsov, S.A. (2018). On the Convergence of Mappings in Metric Spaces with Direct and Inverse Modulus Conditions. *Ukr. Math. J., 70*(7), 1097–1114.

Contact information

Andriy Bandura

Department of Advanced Mathematics,
Ivano-Frankivsk National Technical
University of Oil and Gas,
Ivano-Frankivsk, Ukraine
E-Mail: andriykopanytsia@gmail.com