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Belonging of Laplace–Stieltjes integrals to convergence classes

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Dedicated to the 80th anniversary of the Corresponding Member of the NAS of Ukraine V. Ya. Gutlyanskii

Abstract. For positive continuous functions α and β increasing to $+\infty$ on $[x_0, +\infty)$ and Laplace–Stieltjes integral $I(\sigma) = \int\limits_0^\infty f(x)e^{x\sigma}dF(x)$, $\sigma \in \mathbb{R}$, a generalized convergence $\alpha\beta$-class is defined by the condition

$$\int\limits_{\sigma_0}^{\infty} \frac{\alpha(\ln I(\sigma))}{\beta(\sigma)} d\sigma < +\infty.$$

Under certain conditions on the functions α, β, f and F , it is proved that the integral I belongs to the generalized convergence $\alpha\beta$-class if and only if $\int\limits_{x_0}^{\infty} \alpha'(x)\beta_1\left(\frac{1}{x}\ln\frac{1}{f(x)}\right) < +\infty$, $\beta_1(x) = \int\limits_x^{+\infty} \frac{d\sigma}{\beta(\sigma)}$. For a positive convex on $(-\infty, +\infty)$ function Φ and integral I, a convergence Φ-class is defined by the condition $\int\limits_{\sigma_0}^{\infty} \frac{\Phi'(\sigma)\ln I(\sigma)}{\Phi^2(\sigma)} d\sigma < +\infty$ and it is proved that under certain conditions on Φ, f and F the integral I belongs the convergence Φ-class if and only if $\int\limits_{x_0}^{\infty} \frac{dx}{\Phi'\left((1/x)\ln(1/f(x))\right)} < +\infty$. Conditions are also found in the fulfilment of which the integral type of Laplace–Stieltjes $\int\limits_0^\infty f(x)g(x\sigma)dF(x)$ belongs to the generalized convergence $\alpha\beta$-class if and only if the function g belongs to this class.

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1. Introduction

For an entire transcendental function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z = re^{i\theta}, \tag{1}$$

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let $M_f(r)=\max\{|f(z)|:\,|z|=r\}$. If f has an order $\varrho\in(0,\,+\infty)$ and minimal type then for the growth characteristic it is used [1, p. 62] a concept of the convergence class (Valiron's convergence class) defined by the condition $\int\limits_1^{\infty}\frac{\ln\,M_f(r)}{r^{\varrho+1}}dr<+\infty$. G. Valiron [2, p. 18] proved that if function (1) belongs to the convergence class then $\sum\limits_{n=1}^{\infty}|a_n|^{\varrho/n}<+\infty$. Direct generalizations of power developments of entire functions are entire (absolutely convergent in $\mathbb{C}$) Dirichlet series

$$D(s)=a_0+\sum_{n=1}^{\infty}a_n\exp{(s\lambda_n)},\quad s=\sigma+it, \tag{2}$$

where (λ_n) is an increasing to $+\infty$ sequence of positive number. We put $M(\sigma,D)=\sup\{|F(\sigma+it)|\,:\,t\in\mathbb{R}\}$, suppose that the R-order ϱ_R of the function D is equal $\varrho\in(0,+\infty)$ and define a convergence class by condition

$$\int_0^{\infty}e^{-\varrho\sigma}\ln\,M(\sigma,D)d\sigma<+\infty. \tag{3}$$

Generalizing Valiron's theorem P. Kamthan [3] showed that if the sequence (λ_n) has a positive finite step (i. e. $0<h\le\lambda_{n+1}-\lambda_n\le H<+\infty$ for $n\ge 0$) and $\kappa_n(F):=\dfrac{\ln|a_n|-\ln|a_{n+1}|}{\lambda_{n+1}-\lambda_n}\uparrow+\infty$ as $n\to\infty$, then in order that condition (3) was executed, it is necessary and sufficient that $\sum\limits_{n=1}^{\infty}|a_n|^{\varrho/\lambda_n}<+\infty$.

O. M. Mulyava [4] removed the condition for the step of the sequence of exponents and proved that if $\ln n=O(\lambda_n)$ as $n\to\infty$ then in order for the relation (3) to be fulfilled, it is necessary and in the case, when $\kappa_n(F)\nearrow+\infty$ as $n\to\infty$, it is sufficient that $\sum\limits_{n=1}^{\infty}(\lambda_n-\lambda_{n-1})|a_n|^{\varrho/\lambda_n}<+\infty$. Moreover, she [4–5] investigated the belonging of entire Dirichlet series to a generalized convergence $\alpha\beta$-class defined by condition

$$\int\limits_{\sigma_0}^{\infty}\frac{\alpha(\ln\,M(\sigma,D))}{\beta(\sigma)}d\sigma<+\infty, \tag{4}$$

where α and β are positive continuous functions increasing to $+\infty$ on $[x_0,+\infty)$. By L^0 we denote a class of positive continuously differentiable functions α increasing to $+\infty$ on $[x_0,+\infty)$ and such that $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$. The following theorem is true (see [4] and [6, p. 13]).

Theorem A. *Let* α *be a concave on* $[x_0, +\infty)$ *function and* $\alpha(e^x) \in L^0$, $\beta \in L^0$, $x\dfrac{\beta'(x)}{\beta(x)} \ge c > 0$ *for all* $x \in [x_0, +\infty)$ *and* $\int\limits_{x_0}^{\infty} \dfrac{\alpha(x)}{\beta(x)} dx < +\infty$. *Suppose that* $\ln n = o(\lambda_n \beta^{-1}(c\alpha(\lambda_n)))$ *as* $n \to \infty$ *for each* $c \in (0, +\infty)$. *Then in order that an entire Dirichlet series belongs to the generalized convergence* $\alpha\beta$*-class, it is necessary and in case, when* $\varkappa_n(F) \nearrow +\infty$ *as* $n \to \infty$, *it is sufficient that*

$$\sum_{n=1}^{\infty} (\alpha(\lambda_n) - \alpha(\lambda_{n-1}))\beta_1 \left(\frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right) < +\infty, \quad \beta_1(x) = \int\limits_{x}^{+\infty} \frac{d\sigma}{\beta(\sigma)}. \quad (5)$$

Another generalization of Valiron convergence class is the convergence Φ-class studied for Dirichlet series in the articles [7–9].

As in [10], let Ω be a class of positive unbounded functions Φ on $(-\infty, +\infty)$ such that the derivative Φ' is positive continuously differentiable and increasing to $+\infty$ on $(-\infty, +\infty)$. For $\Phi \in \Omega$ let φ be the inverse function to Φ' and $\Psi(\sigma) = \sigma - \dfrac{\Phi(\sigma)}{\Phi'(\sigma)}$ be the function associated with Φ in the sense of Newton. Then [10] the function Ψ is continuously differentiable and increasing to $+\infty$ on $(-\infty, +\infty)$ and the function φ is continuously differentiable and increasing to $+\infty$ on $(x_0, +\infty)$. For entire Dirichlet series Φ-class is defined in [7] by the condition

$$\int\limits_{\sigma_0}^{\infty} \frac{\Phi'(\sigma) \ln M(\sigma, D)}{\Phi^2(\sigma)} d\sigma < +\infty. \quad (6)$$

Combining Theorem 1 from [7] and Theorem 1 from [9], we get the following theorem.

Theorem B. *Let* $\Phi \in \Omega$, *the function* $\Phi'(\sigma)/\Phi(\sigma)$ *is nondecreasing on* $[\sigma_0, +\infty)$ *and*

$$0 < h \le \frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2} \le H < +\infty, \quad \sigma \ge \sigma_0. \quad (7)$$

Suppose that

$$\int\limits_{t_0}^{\infty} \frac{\ln n(t)}{t\Phi(\Psi(\varphi(t)))} dt < +\infty, \quad n(t) = \sum_{\lambda_n \le t} 1.$$

In order that (6) *holds, it is necessary and in the case, when* $\kappa_n(F) \nearrow +\infty$ *as* $n \to \infty$, *it is sufficient that*

$$\sum_{n=n_0}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\Phi'\left(\dfrac{1}{\lambda_n} \ln \dfrac{1}{|a_n|}\right)} < +\infty. \tag{8}$$

Here we will obtain analogues of Theorems A and B for Laplace–Stieltjes integrals.

2. Belonging of Laplace-Stieltjes integrals to a generalized convergence $\alpha\beta$-class

Let V be a class of nonnegative nondecreasing unbounded continuous on the right functions F on $[0, +\infty)$. For a nonnegative function f on $[0, +\infty)$ the integral

$$I(\sigma) = \int\limits_0^{\infty} f(x) e^{x\sigma} dF(x), \quad \sigma \in \mathbb{R}, \tag{9}$$

is called of Laplace–Stieltjes [11, p. 7]. Integral (9) is a direct generalization of the ordinary Laplace integral $I(\sigma) = \int_0^{\infty} f(x) e^{x\sigma} dx$ and of Dirichlet series (2) with nonnegative coefficients a_n end exponents λ_n, $0 \le \lambda_n \uparrow +\infty \, (n \to \infty)$, if we choose $F(x) = n(x) = \sum\limits_{\lambda_n \le x} 1$ and $f(\lambda_n) = a_n \ge 0$ for all $n \ge 0$.

Let $\mu(\sigma) = \mu(\sigma, I) = \max\{f(x) e^{x\sigma} : x \ge 0\}$ $(\sigma \in \mathbb{R})$ be the maximum of the integrand, σ_c be the abscissa of convergence of the integral (9) and σ_μ be the abscissa of maximum of the integrand. Then [11, p. 8] $\sigma_\mu = \varliminf\limits_{x \to +\infty} \dfrac{1}{x} \ln \dfrac{1}{f(x)}$ and if either $\ln F(x) = o(x)$ or $\ln F(x) = o(\ln f(x))$ as $x \to +\infty$ then [11, p. 13] $\sigma_c \ge \sigma_\mu$. From whence it follows that if $\ln F(x) = o(x)$ as $x \to +\infty$ and

$$r(x) := \frac{1}{x} \ln \frac{1}{f(x)} \to +\infty, \quad x \to +\infty, \tag{10}$$

then integral (9) converges for all $\sigma \in \mathbb{R}$. Further we assume that (10) holds.

By a definition [11, p. 21] a positive function f has regular variation in regard to F if there exist $a \ge 0$, $b \ge 0$ and $h > 0$ such that $\int\limits_{x-a}^{x+b} f(t) dF(t) \ge h f(x)$ for all $x \ge a$.

Let, as above, α and β be positive continuous functions increasing to $+\infty$ on $[x_0, +\infty)$. We will say that integral (9) belongs to the generalized convergence $\alpha\beta$-class if

$$\int_{\sigma_0}^{\infty} \frac{\alpha(\ln I(\sigma))}{\beta(\sigma)} d\sigma < +\infty, \tag{11}$$

and will find conditions on f under which (11) holds.

Before to pass to the analog of Theorem A we will specify on some property of functions from the class L^0.

Lemma 1. *If $\beta \in L^0$ then for each $\lambda \in [1, +\infty)$ and all $x \geq x_0(\lambda)$ the inequalities $1 \leq \beta(\lambda x)/\beta(x) \leq M(\lambda) < +\infty$ hold.*

Indeed, if $\beta \in L^0$ then [12] β is *RO*-varying and, thus, [13, p. 86] $1 \leq \beta(\lambda x)/\beta(x) \leq M(\lambda) < +\infty$ for each $\lambda \in [1, +\infty)$ and all $x \geq x_0(\lambda)$.

The following analog of Theorem A is true.

Theorem 1. *Let the functions α and β satisfy the conditions of Theorem A. Suppose that $F \in V$, $\ln F(x) = o(x\beta^{-1}(c\alpha(x))$ as $x \to \infty$ for each $c \in (0, +\infty)$ and the function f has regular variation in regard to F. If the function $v(x) := -(\ln f(x))'$ is continuous and increasing to $+\infty$ on $[x_0, +\infty)$ then condition*

$$\int_{x_0}^{\infty} \alpha'(x)\beta_1\left(\frac{1}{x}\ln\frac{1}{f(x)}\right) < +\infty, \quad \beta_1(x) = \int_x^{+\infty} \frac{d\sigma}{\beta(\sigma)}, \tag{12}$$

is necessary and sufficient in order that integral (9) belongs to the generalized convergence $\alpha\beta$-class.

Proof. We begin from the sufficiency. At first we show that conditions (11) and

$$\int_{\sigma_0}^{\infty} \frac{\alpha(\ln \mu(\sigma, I))}{\beta(\sigma)} d\sigma < +\infty \tag{13}$$

are equivalent. Indeed, since f has regular variation in regard to F then [11, p. 75]

$$\ln \mu(\sigma, I) \leq (1 + o(1)) \ln I(\sigma), \quad \sigma \to +\infty. \tag{14}$$

Therefore, (11) implies (13).

On the other hand, from (13) it follows that for all $\sigma \geq \sigma_0$

$$1 \geq \int_{\sigma}^{\infty} \frac{\alpha(\ln \mu(\sigma, I))}{\beta(\sigma)} d\sigma \geq \alpha(\ln \mu(\sigma, I)) \int_{\sigma}^{\infty} \frac{1}{\beta(\sigma)} d\sigma = \alpha(\ln \mu(\sigma, I))\beta_1(\sigma).$$

Therefore, in view of the condition $\beta \in L^0$ for the generalized order $\varrho_{\alpha\beta}(\ln\,\mu)$ we have

$$\varrho_{\alpha\beta}(\ln\,\mu) := \varlimsup_{\sigma\to+\infty}\frac{\alpha(\ln\,\mu(\sigma,I))}{\beta(\sigma)} \le \varlimsup_{\sigma\to+\infty}\frac{1}{\beta_1(\sigma)\beta(\sigma)}$$

$$\le \varlimsup_{\sigma\to+\infty}\frac{1}{\beta(\sigma)\int_\sigma^{\sigma+1}dx/\beta(x)} \le \varlimsup_{\sigma\to+\infty}\frac{\beta(\sigma+1)}{\beta(\sigma)} < +\infty.$$

Since [11, p. 77-78]

$$k_{\alpha\beta}(f) := \varlimsup_{x\to+\infty}\frac{\alpha(x)}{\beta\left(\dfrac{1}{x}\ln\dfrac{1}{f(x)}\right)} \le \varrho_{\alpha\beta}(\ln\,\mu),$$

we have $k_{\alpha\beta}(f) < +\infty$ and, thus, $\ln\,f(x) \le -x\beta^{-1}(\alpha(x)/k))$ for some $k < +\infty$ and all $x \ge x_0(k)$. Therefore, in view of the condition $\ln\,F(x) = o(x\beta^{-1}(c\alpha(x))$ as $x \to \infty$ for each $c \in (0, +\infty)$ we obtain

$$\varlimsup_{x\to+\infty}\frac{\ln\,F(x)}{\ln\,(1/f(x))} = 0$$

and, thus [11, p. 61],

$$I(\sigma) \le K(\varepsilon)\mu\left(\frac{\sigma}{1-\varepsilon}\right)^{1-\varepsilon} \le K(\varepsilon)\mu\left(\frac{\sigma}{1-\varepsilon}\right)$$

for every $\varepsilon \in (0,\,1)$ and all $\sigma \ge \sigma_0(\varepsilon)$. Since $\beta \in L^0$ by Lemma 1 (13) implies (11).

As in [11, p. 24], let $\nu(\sigma)$ be central point of the maximum of integrand. Then [11, p. 26] $\nu(\sigma) \nearrow +\infty$ as $\sigma \to +\infty$ and

$$\ln\mu(\sigma) = \ln\mu(\sigma_0) + \int_{\sigma_0}^{\sigma}\nu(t)dt, \tag{15}$$

from whence we get

$$\ln\mu(\sigma) \ge \ln\mu(\sigma_0) + \int_{\sigma/2}^{\sigma}\nu(t)dt \ge \ln\mu(\sigma_0) + \frac{\sigma}{2}\nu(\sigma/2) \ge \nu(\sigma/2)$$

and

$$\ln\mu(\sigma) \le \ln\mu(\sigma_0) + (\sigma-\sigma_0)\nu(\sigma) \le 2\sigma\nu(\sigma)$$

for all σ enough large. Thus,

$$\alpha(\nu(\sigma/2)) \le \alpha(\ln \mu(\sigma)) \le \alpha(\exp\{\ln 2\sigma + \ln\, \nu(\sigma)\})$$
$$\le \alpha(\exp\{2 \max\{\ln 2\sigma, \ln\, \nu(\sigma)\}\})$$
$$\le c_1 \alpha(\exp\{\max\{\ln 2\sigma, \ln\, \nu(\sigma)\}\}) = c_1 \max\{\alpha(2\sigma), \alpha(\ln\, \nu(\sigma))\}$$
$$= c_1(\alpha(2\sigma) + \alpha(\ln\, \nu(\sigma))), \quad c_1 = \mathrm{const} > 1,$$

i. e. by the conditions $\int\limits_{x_0}^{+\infty} \frac{\alpha(x)}{\beta(x)} dx < +\infty$ and $\beta \in L^0$ it follows that (13) and, thus, (11) holds if and only if

$$\int_{\sigma_0}^{+\infty} \frac{\alpha(\nu(\sigma))}{\beta(\sigma)} d\sigma < +\infty. \tag{16}$$

If the function $v(x) := -(\ln\, f(x))'$ is continuous and increasing to $+\infty$ on $[x_0, +\infty)$ then $\nu(\sigma)$ is unique point of the maximum of the function $\ln\, f(x) + \sigma x$. Moreover, this function $\nu(\sigma)$ is continuous and increasing to $+\infty$ on $[\sigma_0, +\infty)$.

Clearly,

$$\int_{\sigma_0}^{+\infty} \frac{\alpha(\nu(\sigma))}{\beta(\sigma)} d\sigma = - \int_{\sigma_0}^{+\infty} \alpha(\nu(\sigma)) d\beta_1(\sigma)$$
$$= -\alpha(\nu(\sigma))\beta_1(\sigma)\big|_{\sigma_0}^{\infty} + \int_{\sigma_0}^{+\infty} \beta_1(\sigma)\alpha'(\nu(\sigma)) d\nu(\sigma).$$

Since $\alpha(\nu(\sigma))\beta_1(\sigma) > 0$, from whence it follows that (16) holds if and only if

$$\int_{\sigma_0}^{+\infty} \beta_1(\sigma)\alpha'(\nu(\sigma)) d\nu(\sigma) < +\infty. \tag{17}$$

On the other hand, $0 \le \ln\, \mu(\sigma) = \ln\, f(\nu(\sigma)) + \sigma\nu(\sigma)$, i. e.

$$\sigma = \frac{1}{\nu(\sigma)} \ln\, \frac{1}{f(\nu(\sigma))} + \frac{\ln\, \mu(\sigma)}{\nu(\sigma)} \ge \frac{1}{\nu(\sigma)} \ln\, \frac{1}{f(\nu(\sigma))},$$

and since the function β_1 is non-increasing, we have

$$\beta_1(\sigma) \le \beta_1 \left(\frac{1}{\nu(\sigma)} \ln\, \frac{1}{f(\nu(\sigma))} \right).$$

Therefore, if (12) holds then in view of the continuity of $\nu(\sigma)$ we get

$$\int_{\sigma_0}^{+\infty} \beta_1(\sigma)\alpha'(\nu(\sigma))d\nu(\sigma)$$
$$\leq \int_{\sigma_0}^{+\infty} \alpha'(\nu(\sigma))\beta_1\left(\frac{1}{\nu(\sigma)}\ln\frac{1}{f(\nu(\sigma))}\right)d\nu(\sigma) < +\infty.$$

The sufficiency of condition (12) is proved.

Now we prove its necessity. Since $x = \nu(\sigma)$ is a solution of the equation $-v'(x) + \sigma = 0$ then $\sigma = v'(\nu(\sigma))$ and from (12) we get $\int\limits_{\sigma_0}^{\infty} \alpha'(\nu(\sigma))\beta_1(v'(\nu(\sigma)))d\nu(\sigma) < +\infty$, i. e.

$$\int\limits_{\sigma_0}^{\infty} \alpha'(x)\beta_1(v'(x))dx < +\infty. \tag{18}$$

From a theorem proved in [14] it follows that *if $a(x)$ and $b(x)$ are continuous functions on $(0, +\infty)$, $-\infty \leq A < a(x) < B \leq +\infty$, $b(x) \searrow b \geq 0$ as $x \to +\infty$, and for a positive function f on (A, B) the function $f^{1/p}$ is convex on (A, B) then*

$$\int\limits_0^y b(x)f\left(\frac{1}{x}\int\limits_0^x a(t)dt\right)dx \leq \left(\frac{p}{p-1}\right)^p \int\limits_0^y b(x)f(a(x))dx,\ 0 \leq y \leq +\infty. \tag{19}$$

We choose $b(x) = \alpha'(x)$, $a(x) = v'(x)$, $f(x) = \beta_1(x)$ and show that the function $\beta_1^{1/p}$ is convex for some $p > 1$. Indeed,

$$(\beta_1^{1/p}(x))'' = \frac{1}{p}\beta_1^{1/p-2}(x)\left(\beta_1(x)\beta_1''(x) - \frac{p-1}{p}(\beta_1'(x))^2\right),$$

$$\beta_1(x)\beta_1''(x) - \frac{p-1}{p}(\beta_1'(x))^2 = \frac{1}{\beta(x)^2}\left(\beta'(x)\int\limits_x^{\infty}\frac{dr}{\beta(r)} - \frac{p-1}{p}\right)$$

and in view of the condition $x\beta'(x)/\beta(x) \geq h > 0$ for $x \geq x_0$

$$\beta'(x)\int\limits_x^{\infty}\frac{dr}{\beta(r)} \geq \beta'(x)\int\limits_x^{2x}\frac{dr}{\beta(r)} \geq \frac{x\beta'(x)}{\beta(x)} \geq h > 0.$$

Therefore, choosing $p > 1$ such that $h - \dfrac{p-1}{p} \geq 0$, we get the inequality $(\beta_1^{1/p}(x))'' \geq 0$ for $x \geq x_0$, i. e. the function $\beta_1^{1/p}(x)$ is convex and in view of (19) and (18)

$$\int_{x_0}^{\infty}\alpha'(x)\beta_1\left(\frac{1}{x}\int_{x_0}^{x}v'(t)dt\right)dx \le \left(\frac{p}{p-1}\right)^p\int_{x_0}^{\infty}\alpha'(x)\beta_1(v'(x))dx < +\infty. \tag{20}$$

Since

$$\int_{x_0}^{x}v'(t)dt = \ln\frac{1}{f(x))} - \ln\frac{1}{f(x_0)} = (1+o(1))\ln\frac{1}{f(x)}, \quad x\to+\infty,$$

and from the condition $\beta\in L^0$ it follows that $\beta_1(x(1+o(1))) = (1+o(1))\beta_1(x)$ as $x\to+\infty$, (20) implies (12). The proof of Theorem 1 is complete. □

We remark that the condition of the increase of the function $-(\ln f(x))'$ in Theorem 1 is an analogue of the condition of the increase of the sequence $(\varkappa_n(F))$ in Theorem A. This condition was used in the proof of both the sufficiency and necessity of the condition (12) in Theorem 1. At the same time, the nondecreasing of the sequence $(\varkappa_n(F))$ was used only to prove the sufficiency of the condition (5) in Theorem A. However, by reducing Laplace-Stiltjes integral to Dirichlet series and using Theorem A, we can prove the necessity of condition (12) without using the condition of the increase of the function $-(\ln f(x))'$.

Proposition 1. *Let the functions α and β satisfy the conditions of Theorem A and $F\in V$. Suppose that there exists a sequence (λ_n) such that $0=\lambda_0<\lambda_n\uparrow+\infty$, $\lambda_{n+1}-\lambda_n\le H<+\infty$ for all n and*

$$\int_{\lambda_n}^{\lambda_{n+1}} f(t)dF(t) \ge h^* f(x), \quad h^*>0, \tag{21}$$

for all $x\in[\lambda_n,\lambda_{n+1}]$. If $\ln n = o(\lambda_n\beta^{-1}(c\alpha(\lambda_n)))$ as $n\to\infty$ for each $c\in(0,+\infty)$ then (11) implies (12).

Proof. If we write

$$I(\sigma) = \sum_{n=0}^{\infty}\int_{\lambda_n}^{\lambda_{n+1}} f(x)e^{x\sigma}dF(x), \tag{22}$$

then for $\sigma\ge 0$

$$e^{\lambda_n\sigma}\int\limits_{\lambda_n}^{\lambda_{n+1}} f(x)dF(x) \le \int\limits_{\lambda_n}^{\lambda_{n+1}} f(x)e^{x\sigma}dF(x) \le e^{\lambda_{n+1}\sigma}\int\limits_{\lambda_n}^{\lambda_{n+1}} f(x)dF(x)$$

$$\le e^{H\sigma}e^{\lambda_n\sigma}\int\limits_{\lambda_n}^{\lambda_{n+1}} f(x)dF(x).$$

Therefore, if we put $A_n = \int_{\lambda_n}^{\lambda_{n+1}} f(t)dF(t)$ and consider Dirichlet series

$$D(\sigma) = \sum_{n=0}^{\infty} A_n \exp\{\sigma\lambda_n\}, \tag{23}$$

then we obtain the estimates $D(\sigma) \le I(\sigma) \le e^{H\sigma}D(\sigma)$, i. e. $\ln D(\sigma) \le \ln I(\sigma) \le \ln D(\sigma) + H\sigma$ for all $\sigma \ge 0$. Since $\alpha \in L^0$, by Lemma 1

$$\alpha(\ln D(\sigma) + H\sigma) \le \alpha(2\max\{\ln D(\sigma), H\sigma\} \le K\alpha(\max\{\ln D(\sigma), H\sigma\}$$

$$\le K\max\{\alpha(\ln D(\sigma)), \alpha(H\sigma)\} \le K(\alpha(\ln D(\sigma)) + \alpha(H\sigma)),\ K = \text{const} > 0,$$

and in view of the conditions $\beta \in L^0$ and $\int\limits_{x_0}^{\infty} \frac{\alpha(x)}{\beta(x)}dx < +\infty$ the correlation (11) holds if and only if

$$\int\limits_{\sigma_0}^{\infty} \frac{\alpha(\ln D(\sigma))}{\beta(\sigma)}d\sigma < +\infty. \tag{24}$$

But from Theorem A it follows that if condition (24) holds then

$$\sum_{n=1}^{\infty}(\alpha(\lambda_n) - \alpha(\lambda_{n-1}))\beta_1\left(\frac{1}{\lambda_n}\ln\frac{1}{A_n}\right) < +\infty, \quad \beta_1(x) = \int\limits_{x}^{+\infty}\frac{d\sigma}{\beta(\sigma)},$$

i. e.

$$\sum_{n=1}^{\infty}(\alpha(\lambda_n) - \alpha(\lambda_{n-1}))\beta_1\left(\frac{1}{\lambda_n}\ln\frac{1}{\int_{\lambda_n}^{\lambda_{n+1}} f(t)dF(t)}\right) < +\infty. \tag{25}$$

On the other hand,

$$\int\limits_{\lambda_{n_0}}^{\infty} \alpha'(x)\beta_1\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)dx = \sum_{n=n_0}^{\infty}\int\limits_{\lambda_n}^{\lambda_{n+1}} \alpha'(x)\beta_1\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)dx$$

$$\le \sum_{n=n_0}^{\infty}(\alpha(\lambda_{n+1})-\alpha(\lambda_n))\max_{\lambda_n\le x\le\lambda_{n+1}}\beta_1\left(\frac{1}{x}\ln\frac{1}{f(x)}\right).$$

Since the function α is continuously differentiable and $\alpha(e^x)\in L^0$, this function is slowly increasing and, thus, $\frac{x\alpha'(x)}{\alpha(x)}\to 0$ as $x\to+\infty$. From whence it follows that $\alpha'(x)\to 0$ as $x\to+\infty$ and from the concavity of α it follows that $\alpha'(x)\searrow 0$ as $x\to+\infty$. Therefore, $\alpha(\lambda_{n+1})-\alpha(\lambda_n)\le\alpha(\lambda_n)-\alpha(\lambda_{n-1})$ and, thus,

$$\int\limits_{x_0}^{\infty}\alpha'(x)\beta_1\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)dx$$

$$\le \sum_{n=n_0}^{\infty}(\alpha(\lambda_n)-\alpha(\lambda_{n-1}))\max_{\lambda_n\le x\le\lambda_{n+1}}\beta_1\left(\frac{1}{x}\ln\frac{1}{f(x)}\right). \quad (26)$$

Therefore, in view of (25) we need the estimate

$$\max_{\lambda_n\le x\le\lambda_{n+1}}\beta_1\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)\le K_1\beta_1\left(\frac{1}{\lambda_n}\ln\frac{1}{\int_{\lambda_n}^{\lambda_{n+1}}f(t)dF(t)}\right), \quad (27)$$

because (26), (27) and (25) imply (12).

The conditions $\lambda_{n+1}-\lambda_n\le H<+\infty$ and (21) imply

$$\frac{1}{\lambda_n}\ln\frac{1}{\int_{\lambda_n}^{\lambda_{n+1}}f(t)dF(t)}\le\frac{1}{\lambda_n}\ln\frac{1}{h^*f(x)}=\frac{1+o(1))}{x}\ln\frac{1}{f(x)},\quad x\to+\infty$$

and, since β_1 is decreasing function, from whence (27) follows. Proposition 1 is proved. □

3. Belonging of Laplace–Stieltjes integrals to a convergence Φ-class

Let $\Phi\in\Omega$. We will say that integral (9) belongs to the convergence Φ-class if

$$\int\limits_{\sigma_0}^{\infty}\frac{\Phi'(\sigma)\ln I(\sigma)}{\Phi^2(\sigma)}d\sigma<+\infty. \quad (28)$$

To obtain an analogue of Theorem B, we need the following statement.

Lemma 2. *Let $F \in V$, $\Phi \in \Omega$ and the function $\Phi'(\sigma)/\Phi(\sigma)$ is nondecreasing on $[\sigma_0, +\infty)$. If $\ln \mu(\sigma, I) \leq \Phi(\sigma)$ for $\sigma \geq \sigma_0$ and $\ln F(x) = o(x)$ as $x \to +\infty$ then*

$$\ln I(\sigma) \leq \ln \mu(\sigma, I) + \ln F(g(\sigma)) + o(1), \quad \sigma \to +\infty, \tag{29}$$

where $g(\sigma) = \Phi'(\Psi(\sigma + \beta^*(\sigma)))$ and $\beta^*(\sigma) = \Phi(\sigma)/\Phi'(\Psi^{-1}(\sigma))$.

Inequality (29) was obtained in the proof of Theorem 5.1.3 in [11, p. 103–106] (see also [15]).

The following analog of Theorem B is true.

Theorem 2. *Let the function Φ satisfies the conditions of Theorem B. Suppose that $F \in V$, f has regular variation in regard to F and*

$$\int\limits_{x_0}^{\infty} \frac{\ln F(x)}{x\Phi(\Psi(\varphi(x))} dx < +\infty. \tag{30}$$

In order that integral (9) belongs to convergence Φ-class, it is necessary and in the case, when the function $v(x) := -(\ln f(x))'$ is continuous and increasing to $+\infty$ on $[x_0, +\infty)$, it is sufficient that

$$\int\limits_{x_0}^{\infty} \frac{dx}{\Phi'\left(\dfrac{1}{x} \ln \dfrac{1}{f(x)}\right)} < +\infty. \tag{31}$$

Proof. At first we show that the conditions (29) and

$$\int\limits_{\sigma_0}^{\infty} \frac{\Phi'(\sigma) \ln \mu(\sigma, I)}{\Phi^2(\sigma)} d\sigma < +\infty \tag{32}$$

are equivalent.

Indeed, in view of (14) from (28) we get (32).

On the other hand, since $\Psi'(\sigma) = \dfrac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2}$ and the function $\dfrac{\Phi'(\sigma)}{\Phi(\sigma)}$ is non-decreasing, we have $\dfrac{\Phi(\Psi(\varphi(x)))}{\Phi'(\Psi(\varphi(x)))} \geq \dfrac{\Phi(\varphi(x))}{\Phi'(\varphi(x))}$ and by L'Hopital's rule

$$\varliminf_{x\to+\infty}\Phi(\Psi(\varphi(x)))\int\limits_x^\infty\frac{dt}{t\Phi(\Psi(\varphi(t)))}=\varliminf_{x\to+\infty}\frac{\int\limits_x^\infty\frac{dt}{t\Phi(\Psi(\varphi(t)))}}{\frac{1}{\Phi(\Psi(\varphi(x)))}}$$

$$\geq\varliminf_{x\to+\infty}\frac{\Phi(\Psi(\varphi(x)))^2}{x\Phi(\Psi(\varphi(x)))\Phi'(\Psi(\varphi(x)))\Psi'(\varphi(x))\varphi'(x)}$$

$$=\varliminf_{x\to+\infty}\frac{\Phi(\Psi(\varphi(x)))}{\Phi'(\Psi(\varphi(x)))}\frac{(\Phi'(\varphi(x)))^2}{\Phi''(\varphi(x))\Phi(\varphi(x))x\varphi'(x)}$$

$$\geq\varliminf_{x\to+\infty}\frac{\Phi(\varphi(x))}{\Phi'(\varphi(x))}\frac{(\Phi'(\varphi(x)))^2}{\Phi''(\varphi(x))\Phi(\varphi(x))x\varphi'(x)}$$

$$=\varliminf_{x\to+\infty}\frac{\Phi'(\varphi(x))}{\Phi''(\varphi(x))x\varphi'(x)}=1.$$

Therefore, for every $\varepsilon>0$ and all $x\geq x_0(\varepsilon)$ from (30) we get

$$\varepsilon>\int\limits_x^\infty\frac{\ln\, F(t)}{t\Phi(\Psi(\varphi(t)))}dt\geq\ln\, F(x)\int\limits_x^\infty\frac{dt}{t\Phi(\Psi(\varphi(t)))}$$

$$\geq\frac{(1+o(1))\ln\, F(x)}{\Phi(\Psi(\varphi(x)))},\quad x\to+\infty,$$

whence it follows that $\ln\, F(x)=o(\Phi(\Psi(\varphi(x))))$ as $x\to+\infty$. But $\Phi(\Psi(\varphi(x)))\leq\Phi(\varphi(x))=O(\Phi'(\varphi(x)))=O(x)$ as $x\to+\infty$. Therefore, $\ln\, F(x)=o(x)$ as $x\to+\infty$.

From (32) it follows that

$$1>\int\limits_\sigma^\infty\frac{\Phi'(\sigma)\ln\,\mu(\sigma)}{\Phi^2(\sigma)}d\sigma\geq\frac{\ln\,\mu(\sigma)}{\Phi(\sigma)},$$

for all $\sigma\geq\sigma_0$, i. e. $\ln\,\mu(\sigma)\leq\Phi(\sigma)$ for all $\sigma\geq\sigma_0$. Therefore, by Lemma 2 estimate (29) is correct, and it remains for us to prove that

$$J:=\int\limits_{\sigma_0}^\infty\frac{\Phi'(\sigma)\ln\, F(g(\sigma))}{\Phi^2(\sigma)}d\sigma<+\infty,\tag{33}$$

where $g(\sigma)=\Phi'(\Psi^{-1}(\sigma+\beta^*(\sigma)))$ and $\beta^*(\sigma)=\Phi(\sigma)/\Phi'(\Psi^{-1}(\sigma))$.

Since

$$\sigma+\beta^*(\sigma)=\sigma+\frac{\Phi(\sigma)}{\Phi'(\Psi^{-1}(\sigma))}\le\sigma+\frac{\Phi(\Psi^{-1}(\sigma))}{\Phi'(\Psi^{-1}(\sigma))}$$

$$=\sigma+\Psi^{-1}(\sigma)-\Psi(\Psi^{-1}(\sigma))=\Psi^{-1}(\sigma)$$

and the function $\Phi'(\sigma)/\Phi(\sigma)$ is non-decreasing, we have for some $\xi=\xi(\sigma)\in(\Psi(\sigma),\,\sigma)$

$$\ln\,\Phi(\sigma)-\ln\,\Phi(\Psi(\sigma))=\frac{\Phi'(\xi)}{\Phi(\xi)}(\sigma-\Psi(\sigma))=\frac{\Phi'(\xi)}{\Phi(\xi)}\frac{\Phi(\sigma)}{\Phi'(\sigma)}\le 1,$$

i. e. $\Phi(\sigma)\le e\Phi(\Psi(\sigma))$, and

$$J\le\int\limits_{\sigma_0}^{\infty}\frac{\Phi'(\sigma)}{\Phi(\sigma)}\frac{\ln\,F(\Phi'(\Psi^{-1}(\Psi^{-1}(\sigma)))}{\Phi(\sigma)}d\sigma$$

$$\le e\int\limits_{\sigma_0}^{\infty}\frac{\Phi'(\Psi^{-1}(\sigma))}{\Phi(\Psi^{-1}(\sigma))}\frac{\ln\,F(\Phi'(\Psi^{-1}(\Psi^{-1}(\sigma))}{\Phi(\Psi^{-1}(\sigma))}d\sigma$$

$$=e\int\limits_{\sigma_0}^{\infty}\frac{\Phi'(\Psi^{-1}(\sigma))}{\Phi(\Psi^{-1}(\sigma))}\frac{\ln\,F(\Phi'(\Psi^{-1}(\Psi^{-1}(\sigma))}{\Phi(\Psi^{-1}(\sigma)))}\frac{d\Psi^{-1}(\sigma)}{(\Psi^{-1}(\sigma))'}$$

$$=e\int\limits_{\Psi^{-1}(\sigma_0)}^{\infty}\frac{\Phi'(\sigma)}{\Phi(\sigma)}\frac{\ln\,F(\Phi'(\Psi^{-1}(\sigma))}{\Phi(\sigma)}\Psi'(\sigma)d\sigma$$

$$=e\int\limits_{\Psi^{-1}(\sigma_0)}^{\infty}\frac{\Phi'(\sigma)}{\Phi(\sigma)}\frac{\ln\,F(\Phi'(\Psi^{-1}(\sigma))}{\Phi(\sigma)}\frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2}d\sigma$$

$$\le eH\int\limits_{\Psi^{-1}(\sigma_0)}^{\infty}\frac{\Phi'(\sigma)}{\Phi(\sigma)}\frac{\ln\,F(\Phi'(\Psi^{-1}(\sigma))}{\Phi(\sigma)}d\sigma.$$

Since $(\Psi(\varphi(x)))'=\dfrac{\Phi(\varphi(x))}{x^2}$, from whence and (30) we get

$$J \le eH \int\limits_{x_0)}^{\infty} \frac{\Phi'(\Psi(\varphi(x))}{\Phi(\Psi(\varphi(x)))} \frac{\ln F(x)}{\Phi(\Psi(\varphi(x)))} \frac{\Phi(\varphi(x))}{x^2} dx$$

$$\le eH \int\limits_{x_0)}^{\infty} \frac{\Phi'(\varphi(x))}{\Phi(\varphi(x))} \frac{\ln F(x)}{\Phi(\Psi(\varphi(x)))} \frac{\Phi(\varphi(x))}{x^2} dx$$

$$= eH \int\limits_{x_0)}^{\infty} \frac{\ln F(x)}{x\Phi(\Psi(\varphi(x)))} dx < +\infty.$$

i. e. (33) holds and (32) and (28) are equivalent.

Now, suppose that (32) holds. Since the function $\Phi'(\sigma)/\Phi(\sigma)$ is non-decreasing on $[\sigma_0, +\infty)$, the function $\lambda(\sigma) := \dfrac{\Phi'(\sigma) \ln \mu(\sigma)}{\Phi(\sigma)}$ is continuous and increasing to $+\infty$ on $[\sigma_0, +\infty)$. Therefore, there exits a continuous and increasing to $+\infty$ on $[x_0, +\infty)$ function $\sigma(x)$ such that

$$\frac{\Phi'(\sigma(x)) \ln \mu(\sigma(x))}{\Phi(\sigma(x))} = x,$$

and, since in view of (32) $\int\limits_{\sigma_0}^{\infty} \dfrac{\lambda(\sigma)}{\Phi(\sigma)} d\sigma < +\infty$, we have

$$\int\limits_{\sigma_0}^{\infty} \frac{x}{\Phi(\sigma(x))} d\sigma(x) = \int\limits_{\sigma_0}^{\infty} \frac{\lambda(\sigma(x))}{\Phi(\sigma(x))} d\sigma(x) < +\infty.$$

Let $B(\sigma) = \int\limits_{\sigma}^{\infty} \dfrac{dx}{\Phi(x)}$. Using L'Hopital's rule we obtain

$$\lim_{\sigma\to+\infty} B(\sigma)\Phi(\sigma) = \lim_{\sigma\to+\infty} \frac{\int\limits_{\sigma}^{\infty} dx/\Phi(x)}{1/\Phi(\sigma)} = \lim_{\sigma\to+\infty} \frac{\Phi(\sigma)}{\Phi'(\sigma)} < +\infty.$$

Therefore,

$$xB(\sigma(x)) = \frac{\Phi'(\sigma(x)) \ln \mu(\sigma(x))}{\Phi(\sigma(x))} B(\sigma(x))$$

$$= \Phi(\sigma(x))B(\sigma(x)) \frac{\Phi'(\sigma(x)) \ln \mu(\sigma(x))}{\Phi^2(\sigma(x))} = O(1)$$

as $x \to +\infty$ and

$$\int_{\sigma_0}^{\infty} B(\sigma(x))dx = xB(\sigma(x))\Big|_{\sigma_0}^{+\infty} - \int_{\sigma_0}^{\infty} xdB(\sigma(x))$$

$$= \text{const} + \int_{\sigma_0}^{\infty} \frac{x}{\Phi(\sigma(x))} d\sigma(x) < +\infty.$$

From (7) it follows that

$$B(\sigma) = \int_{\sigma}^{\infty} \frac{dx}{\Phi(x)} \geq \frac{1}{H} \int_{\sigma}^{\infty} \frac{\Phi''(x)}{(\Phi'(x))^2} dx = \frac{1}{H\Phi'(\sigma)}$$

and, therefore,

$$\int_{x_0}^{\infty} \frac{dx}{\Phi'(\sigma(x))} < +\infty. \tag{34}$$

Since $\ln\, \mu(\sigma(x)) \geq \ln\, f(x) + x\sigma(x)$, we have

$$\sigma(x) \leq \frac{\ln\, \mu(\sigma(x))}{x} + \frac{1}{x} \ln\, \frac{1}{f(x)} = \frac{\Phi(\sigma(x))}{\Phi'(\sigma(x))} + \frac{1}{x} \ln\, \frac{1}{f(x)},$$

from whence we get

$$\Psi(\sigma(x)) \leq \frac{1}{x} \ln\, \frac{1}{f(x)}.$$

For some $\xi = \xi(\sigma) \in [\Psi(\sigma), \sigma]$ we have

$$0 \leq \ln\, \Phi'(\sigma) - \ln\, \Phi'(\Psi(\sigma)) = \frac{\Phi''(\xi)}{\Phi'(\xi)}(\sigma - \Psi(\sigma))$$

$$= \frac{\Phi''(\xi)}{\Phi'(\xi)} \frac{\Phi(\sigma)}{\Phi'(\sigma} \leq \frac{\Phi''(\xi)\Phi(\xi)}{(\Phi'(\xi))^2} \leq H$$

i. e. $\Phi'(\sigma) = O(\Phi'(\Psi(\sigma)))$ as $\sigma \to +\infty$ and, thus,

$$\Phi'(\sigma(x)) \leq K\Phi'(\Psi(\sigma(x))) \leq K\Phi'\left(\frac{1}{x} \ln\, \frac{1}{f(x)}\right),$$

where $K = \text{const} > 0$. From whence and (34) we get (31). The necessity of (31) is proved.

Now we prove its sufficiency. At first we remark that the condition $v(x) \uparrow +\infty$ as $x \to +\infty$ implies (10), and (7) implies

$$\int_{r_0}^{\infty} \frac{d\sigma}{\Phi(\sigma)} \leq \frac{1}{h} \int_{\sigma_0}^{\infty} \frac{\Phi''(\sigma)d\sigma}{(\Phi'(\sigma))^2} = \frac{1}{h\Phi'(\sigma_0)} < +\infty.$$

Therefore, $B(x) = \int\limits_x^\infty \frac{d\sigma}{\Phi(\sigma)} \downarrow 0$ as $x \to +\infty$, $B(x) \le \frac{1}{h\Phi'(x)}$ and in view of (32)

$$\int\limits_{x_0}^{\infty} B\left(\frac{1}{x}\ln\frac{1}{f(x)}\right)dx < +\infty. \tag{35}$$

As above, let $\nu(\sigma)$ be central point of the maximum of integrand. Since $v(x) := -(\ln f(x))'$ is continuous and increasing to $+\infty$ on $[x_0, +\infty)$, as in the proof Theorem 1 we obtain $\sigma \ge \frac{1}{\nu(\sigma)}\ln\frac{1}{f(\nu(\sigma))}$, and in view of (35)

$$\int\limits_{\sigma_0}^{\infty} B(\sigma)d\nu(\sigma) \le \int\limits_{\sigma_0}^{\infty} B\left(\frac{1}{\nu(\sigma)}\ln\frac{1}{f(\nu(\sigma))}\right)d\nu(\sigma) < +\infty. \tag{36}$$

But

$$+\infty > \int\limits_{\sigma_0}^{\infty} B(\sigma)d\nu(\sigma) = B(\sigma)\nu(\sigma)\Big|_{r_0}^{\infty} - \int\limits_{\sigma_0}^{\infty} \nu(\sigma)B'(\sigma)d\sigma$$

$$\ge \text{const} + \int\limits_{\sigma_0}^{\infty} \frac{\nu(\sigma)}{\Phi(\sigma)}d\sigma.$$

Therefore,

$$\int\limits_{\sigma_0}^{\infty} \frac{\Phi'(\sigma)\ln\mu(\sigma)}{\Phi^2(\sigma)}d\sigma = \int\limits_{\sigma_0}^{\infty} \ln\mu(\sigma)d\left(-\frac{1}{\Phi(\sigma)}\right)$$

$$= \text{const} + \int\limits_{\sigma_0}^{\infty}\left(\int\limits_{\sigma_0}^{\sigma}\nu(x)dx\right)d\left(-\frac{1}{\Phi(\sigma)}\right)$$

$$= \text{const} + \int\limits_{\sigma_0}^{\infty} \frac{\nu(\sigma)}{\Phi(\sigma)}d\sigma < +\infty,$$

i. e. (32) holds. The proof of Theorem 2 is complete. □

Choosing $\Phi(\sigma) = e^{\varrho\sigma^p}$ ($\varrho > 0$, $p \ge 1$) for $\sigma \ge 0$, from Theorem 2 we obtain the following statement.

Corollary 1. *Let $F \in V$, $\int\limits_{x_0}^{\infty} \frac{\ln^{(p-1)/p} x}{x^2}\ln F(x)dx < +\infty$ and f has regular variation in regard to F. In order that $\int\limits_{x_0}^{\infty} \sigma^{p-1}e^{-\varrho\sigma}\ln I(\sigma)d\sigma < +\infty$,* *it is necessary and in the case, when the function* $v(x) := -(\ln f(x))'$ *is continuous and increasing to* $+\infty$ *on* $[x_0, +\infty)$*, it is sufficient that* $\int\limits_{x_0}^{\infty} \frac{dx}{r(x)^{p-1}\exp\{\varrho r(x)^p\}} < +\infty$*, where* $r(x)$ *is defined by equality* (10)*.*

Indeed, since $\Phi'(\sigma) = e^{\varrho\sigma^p} p\varrho\sigma^{p-1}$, the function $\Phi'(\sigma)/\Phi(\sigma) = p\varrho\sigma^{p-1}$ is non-decreasing on $[\sigma_0, +\infty)$ and $\frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2} = 1 + \frac{p-1}{\varrho p\sigma^p}$, i. e. (7) holds and Φ satisfies the condition of Theorem 2. Clearly, $e^{\varrho\varphi(x)^p} p\varrho\varphi(x)^{p-1} \equiv x$, i. e. $e^{\varrho\varphi(x)^p} = \frac{x}{p\varrho\varphi(x)^{p-1}}$. Therefore, $\Phi(\varphi(x)) = e^{\varrho\varphi(x)^p} = \frac{x}{p\varrho\varphi(x)^{p-1}}$ and

$$\Phi(\Psi(\varphi(x))) = \Phi\left(\varphi(x) - \frac{\Phi(\varphi(x))}{x}\right) = \Phi\left(\varphi(x) - \frac{1}{p\varrho\varphi(x)^{p-1}}\right)$$

$$= \exp\left\{\varrho\left(\varphi(x) - \frac{1}{p\varrho\varphi(x)^{p-1}}\right)^p\right\} = \exp\left\{\varrho\varphi(x)^p\left(1 - \frac{1}{p\varrho\varphi(x)^p}\right)^p\right\}$$

$$= \exp\left\{\varrho\varphi(x)^p\left(1 - \frac{p}{p\varrho\varphi(x)^p} + O(\varphi(x)^{-2p})\right)\right\}$$

$$= \exp\left\{\varrho\varphi(x)^p - (1 + o(1))\right\}$$

$$= \frac{1+o(1)}{e}\exp\left\{\varrho\varphi(x)^p\right\} = \frac{1+o(1)}{e}\frac{x}{p\varrho\varphi(x)^{p-1}}, \quad x \to +\infty.$$

But $\varphi(x) = (1+o(1))\left(\frac{\ln x}{\varrho}\right)^{1/p}$. Therefore,

$$\Phi(\Psi(\varphi(x))) = \frac{(1+o(1))x}{ep\varrho^{1/p}\ln^{(p-1)/p} x}$$

as $x \to +\infty$ and the condition $\int\limits_{x_0}^{\infty} \frac{\ln^{(p-1)/p} x}{x^2} \ln F(x)dx < +\infty$ implies (30). Since now the conditions

$$\int\limits_{x_0}^{\infty} \sigma^{p-1}e^{-\varrho\sigma}\ln I(\sigma)d\sigma < +\infty$$

and $\int\limits_{x_0}^{\infty} \frac{dx}{r(x)^{p-1}\exp\{\varrho r(x)^p\}} < +\infty$ are equivalent to conditions (28) and (31) respectively, Corollary 1 is proved. □

Choosing $p = 1$, from Corollary 1 we obtain the following statement.

Corollary 2. *Let $F \in V$, $\int\limits_{x_0}^{\infty} \frac{\ln F(x)}{x^2} dx < +\infty$ and f has regular variation in regard to F. In order that $\int\limits_{x_0}^{\infty} e^{-\varrho\sigma} \ln I(\sigma) d\sigma < +\infty$, it is necessary and in the case, when the function $v(x) := -(\ln f(x))'$ is continuous and increasing to $+\infty$ on $[x_0, +\infty)$, it is sufficient that $\int\limits_{x_0}^{\infty} f(x)^{\varrho/x} dx < +\infty$.*

4. Belonging of Laplace-Stieltjes-type integrals to a generalized convergence $\alpha\beta$-class

As above, let $F \in V$, the function g is positive, increasing and continuous on $[0, +\infty)$, and positive on $[0, +\infty)$ function f is such that the Laplace–Stietjes-type integral

$$\overline{I}(r) = \int\limits_{0}^{\infty} f(x) g(rx) dF(x) \tag{37}$$

exists for every $r \in [0, +\infty)$. If $g(x) = e^x$ then $\overline{I}(r) = I(r)$. The growth of function $\overline{I}$ with respect to g investigated in [16].

We assume that $g(z) = \sum\limits_{k=0}^{\infty} g_k z^k$ is an entire transcendental function and $g_k \geq 0$ for all $k \geq 0$ and suppose that $x_0 > 1$ is such that $\int\limits_{1}^{x_0} f(x) dF(x) \geq c > 0$. Then

$$\overline{I}(r) \geq \int\limits_{1}^{x_0} f(x) g(rx) dF(x) \geq g(r)c. \tag{38}$$

On the other hand, since function g is transcendental and $g_k \geq 0$ for all $k \geq 0$ then the function $\ln M_g(r) = \ln g(r)$ is logarithmically convex and, thus,

$$\Gamma_g(r) := \frac{d \ln g(r)}{d \ln r} \nearrow +\infty, \quad r \to +\infty.$$

Therefore, for $\tau > 0$ and $r \geq 1$ we have

$$\ln g((1+\tau)xr) - \ln g(rx) = \int\limits_{rx}^{(1+\tau)rx} \Gamma_g(t) d \ln t$$

$$\geq \Gamma_g(rx) \ln (1+\tau) \geq \Gamma_g(x) \ln (1+\tau)$$

and, thus, if $\mu_{\overline{I}}(r)=\max\{f(x)g(rx):\, x\ge 0\}$ is the maximum of integrand, $\ln F(x)\le q\Gamma_g(x)$ and $\ln(1+\tau)>q$ then

$$\overline{I}(r)=\int\limits_0^\infty f(x)g((1+\tau)rx)\frac{g(rx)}{g((1+\tau)rx)}dF(x)$$

$$\le \mu_{\overline{I}}((1+\tau)r)\int\limits_0^\infty \frac{g(rx)}{g((r+\tau)x)}dF(x)\le \mu_{\overline{I}}((1+\tau)r)\int\limits_0^\infty e^{-\Gamma_f(x)\ln(1+\tau)}dF(x)$$

$$=\mu_{\overline{I}}((1+\tau)r)\left(T_1+\ln(1+\tau)\int\limits_0^\infty F(x)e^{-\Gamma_g(x)\ln(1+\tau)}d\Gamma_g(x)\right)$$

$$\le \mu_{\overline{I}}((1+\tau)r)\left(T_1+\ln(1+\tau)\int\limits_0^\infty e^{-\Gamma_g(x)((\ln(1+\tau)-q)}d\Gamma_g(x)\right)$$

$$\le T_2\mu_{\overline{I}}((1+\tau)r), \qquad (39)$$

where $T_j=\text{const}>0$. Also we have

$$\mu_{\overline{I}}(r)=\max\left\{f(x)\sum_{k=0}^\infty g_k(xr)^k:\, x\ge 0\right\}$$

$$\le \sum_{k=0}^\infty \max\{f(x)x^k:\, x\ge 0\}g_kr^k=G(r):=\sum_{k=0}^\infty \mu_J(k)g_kr^k, \qquad (40)$$

where $\mu_J(\sigma)=\max\{f(x)e^{\sigma\ln x}:\, x\ge 0\}$ is maximum of integrand of $J(\sigma)=\int\limits_0^\infty f(x)e^{\sigma\ln x}dF(x)$.

Now we can prove the following theorem.

Theorem 3. *Let α be a concave function on $[x_0,+\infty)$ and $\alpha(e^x)\in L^0$, β be positive continuously differentiable increasing to $+\infty$ on $[x_0,+\infty)$ function such that*

$$0<h\le\left(\frac{x\beta'(x)}{\beta(x)}-1\right)\ln x\le H<+\infty,\quad x\ge x_0, \qquad (41)$$

and $\int\limits_{x_0}^\infty \frac{\alpha(\ln x)}{\beta(x)}dx<+\infty$. Suppose that $\ln F(x)=O(\Gamma_g(x)$ as $x\to+\infty$, $\frac{\mu_J(k)g_k}{\mu_J(k+1)g_{k+1}}\nearrow+\infty$ as $k\to\infty$ and

$$\sum_{k=1}^\infty(\alpha(k)-\alpha(k-1))\beta_1\left((g_k\mu_J(k))^{-1/k}\right)<+\infty,\quad \beta_1(x)=\int\limits_x^{+\infty}\frac{d\sigma}{\beta(\sigma)}. \qquad (42)$$

Then integral (37) *belongs to the generalized convergence $\alpha\beta$-class if and only if the function g belongs to the generalized convergence $\alpha\beta$-class.*

Proof. From (38) it follows that if $\int\limits_{r_0}^{\infty}\frac{\alpha(\ln \overline{I}(r))}{\beta(r)}dr < +\infty$ then $\int\limits_{r_0}^{\infty}\frac{\alpha(\ln g(r))}{\beta(r)}dr < +\infty$.

On the other hand, from (39) and (40) it follows that $\overline{I}(r) \le T_2 G((1+\tau)r)$ and, therefore, $\int\limits_{r_0}^{\infty}\frac{\alpha(\ln \overline{I}(r))}{\beta(r)}dr < +\infty$, if $\int\limits_{r_0}^{\infty}\frac{\alpha(\ln G(r))}{\beta(r)}dr < +\infty$. Putting $r = e^{\sigma}$ in the power series $G(r) = \sum\limits_{k=0}^{\infty} G_k r^k$, $G_k = \mu_J(k) g_k$, we obtain the Dirichlet series $D(\sigma) = G(e^{\sigma}) = \sum\limits_{k=0}^{\infty} G_k e^{k\sigma}$. Then $\int\limits_{r_0}^{\infty}\frac{\alpha(\ln G(r))}{\beta(r)}dr < +\infty$ if and only if $\int\limits_{\sigma_0}^{\infty}\frac{\alpha(\ln D(\sigma))}{\overline{\beta}(\sigma)}d\sigma < +\infty$, where $\overline{\beta}(\sigma) = e^{-\sigma}\beta(e^{\sigma})$. Since $\frac{\sigma\overline{\beta}'(\sigma)}{\overline{\beta}(\sigma)} = \sigma\left(\frac{e^{\sigma}\beta'(e^{\sigma})}{\beta(e^{\sigma})} - 1\right)$, from (41) it follows that $0 < h \le \frac{\sigma\overline{\beta}'(\sigma)}{\overline{\beta}(\sigma)} \le H < +\infty$. Easy to show that if $\frac{x\beta' x}{\beta(x)} = O(1)$ as $x \to +\infty$ then $\beta \in L^0$. Thus, $\overline{\beta} \in L^0$.Finally, if $\int\limits_{x_0}^{\infty}\frac{\alpha(\ln x)}{\beta(x)}dx < +\infty$ then $\int\limits_{\sigma_0}^{\infty}\frac{\alpha(\sigma)}{\overline{\beta}(\sigma)}d\sigma < +\infty$. Thus, the functions of α and $\overline{\beta}$ (instead of β) satisfy the conditions of the theorem A.

Since $\lambda_k = k$ and $G_k/G_{k+1} \nearrow +\infty$, by Theorem A $\int\limits_{\sigma_0}^{\infty}\frac{\alpha(\ln D(\sigma))}{\overline{\beta}(\sigma)}d\sigma < +\infty$ if and only if

$$\sum_{k=1}^{\infty}(\alpha(k) - \alpha(k-1))\overline{\beta}_1\left(\frac{1}{k}\ln\frac{1}{G_k}\right) < +\infty, \quad \overline{\beta}_1(x) = \int\limits_{x}^{+\infty}\frac{d\sigma}{\overline{\beta}(\sigma)}. \quad (43)$$

Since

$$\overline{\beta}_1(x) = \int\limits_{x}^{+\infty}\frac{d\sigma}{e^{-\sigma}\beta(e^{\sigma})} = \int\limits_{e^x}^{+\infty}\frac{dt}{\beta(t)} = \beta_1(e^x),$$

conditions (42) and (43) are equivalent. Theorem 3 is proved. □

In conclusion, we show that if, for example,

$$\ln f(x) \le -\alpha^{-1}\left(\frac{1}{\overline{\beta}_1(x)}\right)\ln x, \quad x \ge x_0, \quad (44)$$

then condition (42) can be replaced by the condition

$$\sum_{k=1}^{\infty}(\alpha(k)-\alpha(k-1))\beta_1\left(g_k^{-1/k}\right)<+\infty,\quad \beta_1(x)=\int\limits_x^{+\infty}\frac{d\sigma}{\beta(\sigma)}. \tag{45}$$

Indeed, if $\int\limits_{r_0}^{\infty}\frac{\alpha(\ln g(r))}{\beta(r)}dr<+\infty$ then

$$1\geq\int\limits_r^{\infty}\frac{\alpha(\ln g(t))}{\beta(t)}dt\geq\alpha(\ln g(r))\beta_1(r)$$

for all sufficiently large values of r, from whence $\ln g(r)\leq\alpha^{-1}(1/\beta_1(r))$. Therefore, by Cauchy inequality for all k we have

$$\ln g_k\leq\ln g(r)-k\ln r\leq\alpha^{-1}\left(\frac{1}{\beta_1(r)}\right)-k\ln r,\quad r\geq r_0^*.$$

Choosing $r=\beta_1^{-1}(1/\alpha(k))$ we obtain $\ln g_k\leq k-k\beta_1^{-1}(1/\alpha(k))$, i. e.

$$\frac{1}{k}\ln\frac{1}{g_k}\geq\beta_1^{-1}\left(\frac{1}{\alpha(k)}\right)-1,\quad k\geq k_0. \tag{46}$$

On the other hand, (44) implies

$$\ln\mu_J(\sigma)=\max\{\ln f(x)+\sigma\ln x:\,x\geq 0\}$$

$$\leq\max\left\{\max\{\ln f(x)+\sigma\ln x:\,0\leq x\geq x_0\},\max\left\{-\alpha^{-1}\left(\frac{1}{\overline{\beta}_1(x)}\right)\ln x+\sigma\ln x:\,x\geq x_0\right\}\right\}$$

$$=\max\left\{\max\{O(\sigma),\max\left\{\left(\sigma-\alpha^{-1}\left(\frac{1}{\overline{\beta}_1(x)}\right)\right)\ln x:\,x\geq x_0\right\}\right\},\ \sigma\to+\infty. \tag{47}$$

If $\nu_J(\sigma)$ is central point of the maximum of integrand of $J(\sigma)$ then, since $\frac{\ln\mu_J(\sigma)}{\sigma}\to+\infty$ as $\sigma\to+\infty$, from (47) we get $\sigma-\alpha^{-1}\left(\frac{1}{\overline{\beta}_1(\nu_J(\sigma))}\right)\geq 0$, i. e. $\nu_J(\sigma)\leq\overline{\beta}_1^{-1}\left(\frac{1}{\alpha(\sigma)}\right)$ for $\sigma\geq\sigma_0$, because the function $\overline{\beta}_1$ is decreasing. Therefore, using (15), we obtain

$$\ln\mu_J(\sigma)\leq(1+o(1))\int_{\sigma_0}^{\sigma}\overline{\beta}_1^{-1}\left(\frac{1}{\alpha(t)}\right)dt\leq(1+o(1))\sigma\overline{\beta}_1^{-1}\left(\frac{1}{\alpha(\sigma)}\right),\ \sigma\to+\infty. \tag{48}$$

Since $\overline{\beta}_1(x)=\beta_1(e^x)$, we have $\overline{\beta}_1^{-1}(t)=\ln\, \beta_1^{-1}(t)$, i.e. $\overline{\beta}_1^{-1}(t)/\beta_1^{-1}(t)\to 0$ as $t\downarrow 0$. Therefore, in view of (46) and (48)

$$\frac{\ln\, \mu_J(k)}{\ln\,(1/g_k)}\le \frac{k\overline{\beta}_1^{-1}(1/\alpha(k))}{k\beta_1^{-1}(1/\alpha(k))-k}\to 0,\ k\to\infty$$

and, thus,

$$\frac{1}{k}\ln\,\frac{1}{g_k\mu_J(k)}=\frac{1}{k}\ln\,\frac{1}{g_k}\left(1-\frac{\ln\, \mu_J(k)}{\ln\,(1/g_k)}\right)=\frac{1+o(1)}{k}\ln\,\frac{1}{g_k},\quad k\to\infty.$$

From hence it follows that

$$\beta_1\left((g_k\mu_J(k))^{-1/k}\right)=\overline{\beta}_1\left(\frac{1}{k}\ln\,\frac{1}{g_k\mu_J(k)}\right)=\overline{\beta}_1\left(\frac{1+o(1)}{k}\ln\,\frac{1}{g_k}\right)$$

$$=(1+o(1))\overline{\beta}_1\left(\frac{1}{k}\ln\,\frac{1}{g_k}\right)=(1+o(1))\beta_1\left(g_k^{-1/k}\right),\quad k\to\infty,$$

because (as in the proof of Theorem 3) $\overline{\beta}\in L^0$ and, thus, $\overline{\beta}_1((1+o(1))x)=(1+o(1))\overline{\beta}_1(x)$ as $x\to+\infty$. Therefore, conditions (42) and (45) are equivalent provided (44) holds.

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