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Quasisymmetric mappings and their generalizations on Riemannian manifolds

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Dedicated to the 80th anniversary of the Corresponding Member of the NAS of Ukraine V. Ya. Gutlyanskii

Abstract. We study the connection between η-quasisymmetric homomorphisms and K-quasiconformal mappings on n-dimensional smooth connected Riemannian manifolds. The main results of our research are presented in Theorems 2.6 and 2.7. Several conditions to the boundary behavior of η-quasisymmetric homomorphisms between two arbitrary domains with weakly flat boundaries and compact closures, QED and uniform domains on the Riemannian manifolds are also obtained in view of the above relations. In addition, the quasiballs, c-locally connected domains and the corresponding results are also considered.

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1. Introduction

The modern aspects of Geometric Function Theory related to quasisymmetric and quasiconformal mappings have been studied by many mathematicians from several points of view. It is known that such kind of mappings have interesting applications to the Ahlfors regular Loewner spaces, to the problems of Riemannian surfaces (Ahlfors), to the modulus of Riemannian surfaces (Teichmüller), to the classification of simply connected Riemannian surfaces (Volkovyskii), etc. In this paper we restrict ourselves to studying of Riemannian manifolds.

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The Riemannian manifolds theory. *A Riemannian manifold* $(\mathbb{M}, g)$ is defined as a smooth n-dimensional connected manifold ($n \geq 2$) endowed with a Riemannian metric, i.e., a scalar product on each tangent space $T_p\mathbb{M}$, which depends smoothly on the base point p. Let $x = (x^1, \ldots, x^n)$ be local coordinates. *A Riemannian metric* is a positive definite symmetric tensor field $g = g_{ij}(x)$ defined on the local coordinates and obeying the transition rule $g_{ij}(x) = h_{kl}(f(x))\frac{\partial f^k}{\partial x^i}\frac{\partial f^l}{\partial x^j}$, where, as usual, $k, l = 1, \ldots, n$ are the so-called dummy indices over which the summation is performed. In what follows, the $g_{ij}(x)$ are assumed to be smooth. Note that $detg_{ij} > 0$, because g_{ij} is positive definite.

Let now $[a, b]$ be a closed interval in $\mathbb{R}$ and let $\gamma : [a, b] \to \mathbb{M}$ be a piecewise smooth curve. *The length* of γ in local coordinates $(x^1(\gamma(t)), ..., x^n(\gamma(t)))$ is defined by

$$L(\gamma) := \int\limits_a^b \sqrt{g_{ij}(x(\gamma(t)))\ \dot{x}^i(t)\dot{x}^j(t)}\ dt,$$

where $\dot{x}^i(t) := \frac{d}{dt}(x^i(\gamma(t)))$. The length is invariant under reparametrization (see, e.g., in [15]).

The geodesic distance $d(p_1, p_2)$ between points p_1 and p_2 is the infimum of the lengths of piecewise smooth curves joining p_1 and p_2 in $\mathbb{M}$. This distance function satisfies the usual axioms of metric space.

The volume element on $\mathbb{M}$ is determined by the invariant form in local coordinates

$$dv_g = \sqrt{detg_{ij}}\ dx^1 ... dx^n.$$

It is invariant on $\mathbb{M}$ (see [15]). Note that $g_{ij}(x)$ are defined only in a coordinate chart, where we may write $g = \Sigma_{i,j} g_{ij} dx^i \otimes dx^j$.

For our purposes, the following fundamental facts are important (see, e.g., Lemma 5.10 and Corollary 6.11 in [17]). For any point p of a Riemannian manifold $\mathbb{M}$, there exist its neighborhoods U and the corresponding local coordinates in these neighborhoods, for which the geodesic spheres centered at the indicated point are associated with Euclidean spheres of the same radii centered at the origin of coordinates and a bundle of geodesic curves originating from this point is associated with a bundle of beams originating from the origin of coordinates. It is customary to call these neighborhoods and coordinates *normal*.

From now on, D and D' are domains in smooth connected Riemannian manifolds $(\mathbb{M}, g)$ and $(\mathbb{M}', g')$ with geodesic distances d and d', respectively.

Fix $x \in \mathbb{M}$. Let $\exp_x : T_x\mathbb{M} \to \mathbb{M}$ be a diffeomorphism of a neighborhood V of the origin in $T_x\mathbb{M}$ and $\exp_x(V) = U$, where U is a neighborhood of x on $\mathbb{M}$. The map $\exp_x$ is called the *exponential map*. It has the following properties:

(i) For each $Y \in T_x\mathbb{M}$, the geodesic γ_Y is given by $\exp_x(Y) := \gamma_Y(1)$, where γ_Y denotes the unique geodesic from x with $Y = \frac{d\gamma}{dt}(0)$;

(ii) $\exp_x$ is C^∞ on $T_x\mathbb{M} \setminus \{0\}$;

(iii) The differential of $\exp_x$ at the origin is the identity, cf. [17].

If $B_r(0)$ is such that $\overline{B}_r(0) \subset V$, we call $\exp_x B_r(0) = B_r(x) = B(x, r)$ the *normal ball (or geodesic ball)* with center p and radius r.

The next important for us result can be found in [5, p. 189].

Theorem 1.1. *A connected Riemannian manifold is a metric space with the metric $d(p, q)$ equals the infimum of the lengths of piecewise smooth curves from p to q. Its metric space topology and manifold topology agree.*

The following concept is extremely useful, since it allows us to extend local Euclidean constructions of a manifold to global ones.

A partition of unity. A *partition of unity*, subordinated to the cover $\{U_i\}_{i\in I}$ of a manifold $\mathbb{M}$, is a collection $\varphi_i : \mathbb{M} \to \mathbb{R}$ of C^∞ functions (where I is an arbitrary index set, not assumed countable) such that

(i) $0 \le \varphi_i(x) \le 1$ for any point x of $\mathbb{M}$;

(ii) the collection of supports $\{\operatorname{supp} \varphi_i\}_{i\in I}$ is locally finite, i.e. for any $x \in \mathbb{M}$ there is a neighborhood which intersects with a finite number of sets of this collection;

(iii) $\sum_i \varphi_i \equiv 1$ for any point x of $\mathbb{M}$;

(iv) $\{\operatorname{supp} \varphi_i\} \subset \{U_i\}$ for all i.

Recall also that a topological space is *paracompact* if every open cover has an open locally finite refinement (cf. [26]).

The next proposition is known (see [21, Corollary 1 on p. 979]).

Proposition 1.2. *Every metric space is paracompact.*

Formulate also the next result from [26, p. 9].

Lemma 1.3. *Let X be a topological space which is locally compact (each point has at least one compact neighborhood), Hausdorff, and second countable (manifolds, for example). Then X is paracompact. In fact, each open cover has a countable, locally finite refinement consisting of open sets with compact closures.*

The space (X, d, μ) is called α-regular by Ahlfors with $\alpha > 1$, if there exists a constant $C \ge 1$ such that

$$C^{-1}r^\alpha \le \mu(B(x_0, r)) \le Cr^\alpha$$

for all balls $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$ which have radius $r <$ diam$X$ and are centered at the point $x_0 \in X$. Is is known that the α-regular spaces have a Hausdorff dimension α (see, e.g., [11, p. 61]). The space (X, d, μ) is called regular by Ahlfors, if it is α-regular by Ahlfors for some $\alpha \in (1, \infty)$.

Recall now the statement, whose proof is given in [1].

Lemma 1.4. *The Hausdorff dimension of domains on smooth Riemannian manifolds* $(\mathbb{M}, g)$ *with respect to a geodesic distance coincides with the topological dimension* n*. In addition, any smooth Riemannian manifolds are locally* n*-regular by Ahlfors.*

2. Quasisymmetric homeomorphisms and quasiconformal mappings

The notion of quasisymmetry was introduced by L. V. Ahlfors and A. Beurling. They studied the questions on continuation of homeomorphisms $f : \mathbb{R}^1 \to \mathbb{R}^1$ to quasiconformal mappings of the upper half-plane onto itself (see, e.g., [23, p. 128]). Their studies gave the necessary and sufficient condition imposed on a function. This condition was used, in turn, by J. A. Kelingos for the determination of quasisymmetric functions. Then H. Renggli dealt with the theory of quasisymmetric mappings in the two-dimensional case and considered mappings satisfying the condition of boundedness of distortions of triangles. O. Lehto and K. I. Virtanen considered quasisymmetries for increasing embeddings $f : \triangle \to \mathbb{R}^1$, where $\triangle \subset \mathbb{R}^1$ is an interval, such that for some constant H, the inequality $|f(a) - f(x)| \leq H|f(b) - f(x)|$ with $|a - x| \leq |b - x|$ holds. Later, the Finnish mathematicians P. Tukia and J. Väisälä noticed that the definition given by H. Renggli can be extended to the case of general metric spaces, which allowed them to assign the class of η-quasisymmetric mappings. Some needed references can be found in [3].

Definition 2.1. *Let* $\eta : [0, \infty) \to [0, \infty)$ *be a homeomorphism. A homeomorphism* $f : D \to D'$ *is called* η*-quasisymmetric (abbr.,* η*-QS homeomorphism) if the inequality*

$$\frac{d'(f(x), f(y))}{d'(f(x), f(z))} \leq \eta\left(\frac{d(x, y)}{d(x, z)}\right) \tag{2.1}$$

holds for any triple $x, y, z \in D$, $x \neq z$*, compare, for example, with* [23].

Recall now the definition of quasiconformality following [16].

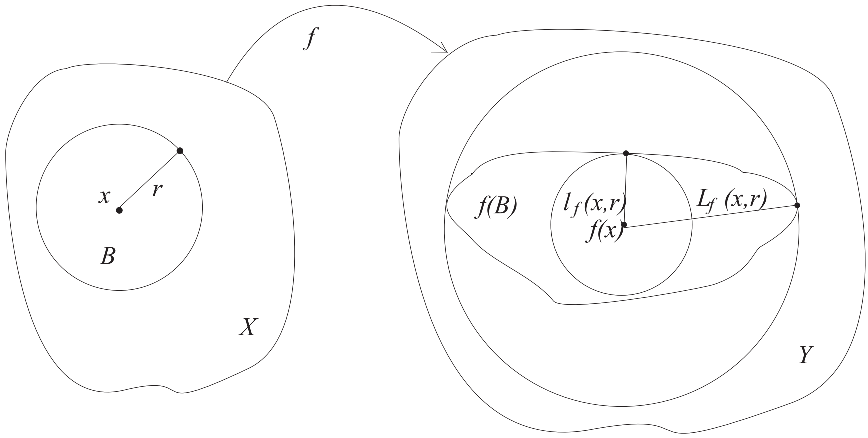

Figure 1.

Given a homeomorphism f from a metric space X to a metric space Y, then for $x \in X$ and $r > 0$ set

$$L_f(x,r) := \sup\{d'(f(x), f(y)) : d(x,y) \leq r\},$$

$$l_f(x,r) := \inf\{d'(f(x), f(y)) : d(x,y) \geq r\}$$

and

$$H_f(x,r) := \frac{L_f(x,r)}{l_f(x,r)}, \tag{2.2}$$

see the Figure 1.

Definition 2.2. *A homeomorphism $f : X \to Y$ is quasiconformal if there exists a constant $K < \infty$ such that*

$$H_f(x) := \limsup_{r \to 0} H_f(x,r) \leq K \tag{2.3}$$

for all $x \in X$. We then say that f is K-quasiconformal (abbr., K-qc homeomorphism), cf. [12].

For the reader convenience, formulate now some auxiliary results.

Theorem 2.3. [22, p. 527] *Let X and Y be locally compact, connected, α-regular metric spaces ($\alpha > 1$) and let $f : X \to Y$ be an η-quasisymmetric homeomorphism. There exists a constant C depending only on η, α and the regularity constants of X and Y so that*

$$\frac{1}{C} M_\alpha(\Gamma) \leq M_\alpha(f(\Gamma)) \leq C M_\alpha(\Gamma)$$

for all curve families Γ in X.

Here $M_\alpha(\Gamma)$ denotes the α-modulus of the curve family Γ (the formal definition see in Section 3).

Lemma 2.4. [16, p. 101] *Suppose that $n \geq 2$, that G and G' are open sets in $\mathbb{R}^n$ and that $f : G \to G'$ is a homeomorphism. If f is locally η-QS, then f is K-qc with $K = \eta(1)^{n-1}$. If f is K-qc and $B(x, ar) \subset G$ for some $a > 1$, then $f \mid B(x, r)$ is η-QS with η depending only on K and a.*

Theorem 2.5. [11, p. 92] *A homeomorphism $f : D \to D'$ between domains in $\mathbb{R}^n$, $n \geq 2$, is K-qc if and only if there is η such that f is η-QS in each ball $B\left(x, \frac{1}{2}dist(x, \partial D)\right)$ for $x \in D$. The statement is quantitative involving K, η, and the dimension n.*

The following theorem gives us the connection between η-quasisymmetric homeomorphisms and K-quasiconformal mappings on Riemannian manifolds.

Theorem 2.6. *Let D and D' be domains in smooth connected Riemannian manifolds $(\mathbb{M}, g)$ and $(\mathbb{M}', g')$ with geodesic distances d and d', respectively. Then the following statements hold.*

(i) *If a homeomorphism $f : D \to D^{'}$ is an η-QS, then f is a K-qc mapping.*

(ii) *If a homeomorphism $f : D \to D^{'}$ is a K-qc mapping, then f is a locally η-QS homeomorphism in D.*

Proof. (i) Let D and $D^{'}$ be two arbitrary domains in $\mathbb{M}$ and $\mathbb{M}^{'}$, respectively, and let $f : D \to D^{'}$ be an η-QS homeomorphism. Since the Riemannian manifolds are connected and locally n-regular by Ahlfors, see Lemma 1.4, we use Theorem 1.1 and move to metric spaces. Then, using Theorem 2.3, we claim that f is a K-qc mapping in D.

(ii) Consider further $\mathbb{M}$ as paracompact, in view of Theorem 1.1 and Proposition 1.2. Choose an open atlas $\{U_i\}_{i\in I}$ in $\mathbb{M}$. It is well known that each open atlas $\{U_i\}_{i\in I}$ in $\mathbb{M}$ has a countable locally finite refinement (atlas) $\{V_k\}$, $1 \leq k \leq N$, in $\mathbb{M}$ (see Lemma 1.3). Let $\{\varphi_k\}$ be a partition of unity subordinated to this atlas such that $\{\operatorname{supp} \varphi_k\} \subset \{V_k\}$, by virtue of the paracompactness of $\mathbb{M}$ (see, e.g., Prop. 3.4.4 in [6]). Since f is a K-quasiconformal mapping on D, f is a locally K-quasiconformal mapping on any chart $V_{k_1}, \ldots, V_{k_l}$. Note that the number of charts $V_{k_1}, ..., V_{k_l}$, $1 \leq l \leq N$, is finite, since the atlas is a locally finite cover. Choose an arbitrary normal neighborhood $U(x_0) \subset D$ whose closure $\overline{U}(x_0)$ is compact, where $U(x_0) = \exp_{x_0} W$ and $W \subset T_{x_0}\mathbb{M}$, then $U(x_0)$ intersects only finitely many $V_{k_1}, \ldots, V_{k_l}$, by virtue of the locally finiteness of $\{V_k\}$.

Note that f is a K-quasiconformal mapping in any ball $B(x_0, 2r) \subset U(x_0)$, $r = \lambda\, d(x, \partial D)$, $0 < \lambda < 1/2$. Then, by Lemma 2.4 (for $a = 2$), we see that f is an η-QS homeomorphism, with η depending only on K, in the ball $B(x_0, r)$ (cf. Theorem 2.5). We obtain that f is a locally η-QS homeomorphism in D. □

Note that such approach can be successfully extended to wider classes of mappings in both metric terms [9] and modulur ones [10].

Let $f : D \to D'$ be a homeomorphism. Define the *volume derivative*

$$\mu'_f(x) := \lim_{r\to 0} \frac{H^n(f(\overline{B}(x,r)))}{H^n(B(x,r))}, \tag{2.4}$$

which exists almost everywhere in D, belongs to $L^1_{loc}(D)$ and

$$\int_E \mu'_f(x) dH^n(x) \le H^n(f(E)) \tag{2.5}$$

for each measurable set $E \subset D$ (see, e.g., (7.7) in [12]).

Set

$$L_f(x) := \limsup_{r\to 0} \frac{L_f(x,r)}{r}. \tag{2.6}$$

Taking into account Lemma 4.4 from [16], we claim that

Theorem 2.7. *Let D and D' be domains in smooth connected Riemannian manifolds $(\mathbb{M}, g)$ and $(\mathbb{M}', g')$ with geodesic distances d and d', respectively, and let a homeomorphism $f : D \to D'$ be K-quasiconformal in any geodesic ball $B(x_0, r) \subset U(x_0)$ with $r = \lambda\, d(x_0, \partial D)$, $0 < \lambda < 1$. Then f is K-quasiconformal in D, the function L_f is Borel measurable, $L_f \in L^n_{loc}(D)$, and the inequality*

$$C^{-2}\mu'_f(x) \le [L_f(x)]^n \le C^2[H_f(x)]^n \mu'_f(x) \tag{2.7}$$

holds for almost every $x \in D$.

Proof. It is well known that every connected Riemannian manifold $\mathbb{M}$ is a metric space (see Theorem 1.1). Using now Proposition 1.2, we claim that $\mathbb{M}$ is paracompact, that guarantees us the existence of a partition of unity (see, e.g., [6, Prop. 3.4.4]). Moreover, every open cover $\{U_i\}_{i\in I}$ in $\mathbb{M}$ has a countable, locally finite refinement $\{V_k\}$, $1 \le k \le N$, in $\mathbb{M}$ (see Lemma 1.3). Denote the coordinates on each chart by x_i^α, $\alpha = 1, \dots, n$.

Let now $\{\varphi_k\}$ be a partition of unity, which subordinate to this atlas, $\{\operatorname{supp}\varphi_k\} \subset \{V_k\}$. Using this partition we glue the charts of atlas $\{V_k\}$, $1 \le k \le N$, in $\mathbb{M}$. In view of the local finiteness of the atlas, consider the finite number of the maps V_{k_l}, $1 \le l \le N$. Let now

$U(x_0) = exp_{x_0}(W)$, $W \subset T_{x_0}M, U \subset D$, be an arbitrary normal neighborhood, whose closure $\overline{U}(x_0)$ is compact. Thus, in view of the local finiteness of the atlas $\{V_k\}$, this neighborhood $U(x_0)$ intersects only with finite number of $V_{k_1}, \dots, V_{k_l}$, $1 \le l \le N$. Let $B(x_0, r) \subset U(x_0)$ with $r = \lambda\ d(x_0, \partial D)$, $0 < \lambda < 1$ is an arbitrary geodesic ball, then f is K-quasiconformal in $B(x_0, r)$. This means that for any $x_k \in \{V_k\}$, $k = 1, \dots, N$, $\limsup\limits_{r \to 0} H_f(x_k, r) \le K < \infty$, under passing from chart to chart, and

$$H_f(x, r) = \begin{cases} \sum_{k=1}^{N} H_f(x_k, r) \cdot \varphi_k(x) & \text{for } x \in \{V_k\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

Clearly, in the normal neighborhood $U(x_0)$ we can rewrite equality (2.8) as $H_f(x, r) = \sum_{p=1}^{l} H_f(x_{k_p}, r) \cdot \varphi_{k_p}$.

Since $0 \le \varphi_k \le 1$ and $\sum_{k=1}^{N} \varphi_k = 1$, there exists an index j such that $\varphi_j(x) > 0$ and $H_f(x, r) \ge H_f(x_j, r) \cdot \varphi_j$, i.e., without loss of generality, one concludes that $H_f(x, r) = c \cdot H_f(x_j, r) \cdot \varphi_j$, where $c > 0$ is an arbitrary constant. Letting to $\limsup$ as $r \to 0$ in both sides, we obtain that $f : D \to D'$ is a K-quasiconformal mapping on D.

Prove now that the function L_f (see (2.6)) is Borel measurable and the inequality (2.7) holds for almost every $x \in D$. The Borel measurability of L_f follows from the fact that we can present a compact subset E of D as

$$E = \bigcup_i A_i = \{x \in E : L_f(x) < t\},$$

where $t > 0$ and the sets

$$A_i = \left\{ x \in E : d'(f(x), f(y)) \le \left(t - \frac{1}{i} \right) d(x, y) \right\}$$

are closed by continuity of f for all $0 < d(x, y) < \frac{d(E, \partial D)}{i}$ and $y \in E$.

Let $0 < r < d(x, \partial D)$. Then, by Lemma 1.4,

$$\frac{H^n(f(\overline{B}(x, r)))}{H^n(B(x, r))} \le \frac{C L_f^n(x, r)}{C^{-1} r^n} = C^2 \left(\frac{L_f(x, r)}{r} \right)^n,$$

i.e.,

$$C^{-2} \frac{H^n(f(\overline{B}(x, r)))}{H^n(B(x, r))} \le \left(\frac{L_f(x, r)}{r} \right)^n. \tag{2.9}$$

In view of (2.2) and Lemma 1.4, we have now

$$\left(\frac{L_f(x,r)}{r}\right)^n = \left(\frac{L_f(x,r)}{l_f(x,r)}\right)^n \left(\frac{l_f(x,r)}{r}\right)^n$$
$$\leq C^2 \left(\frac{L_f(x,r)}{l_f(x,r)}\right)^n \frac{H^n(f(\overline{B}(x,r)))}{H^n(B(x,r))}$$
$$= C^2 H_f^n(x,r)\frac{H^n(f(\overline{B}(x,r)))}{H^n(B(x,r))}. \tag{2.10}$$

Hence, taking into account the inequalities (2.9) and (2.10), the claim inequality (2.7) follows by letting r to zero (cf., e.g., [16, Lemma 4.4]).

It remains to prove that $L_f \in L^n_{loc}(D)$. Recall that if a homeomorphism $f : D \to D^{'}$ is a K-qc mapping, then f is a locally η-QS homeomorphism in D (see above Theorem 2.6). Thus, $[L_f(x)]^n \leq C_\eta \mu'_f(x)$ holds for any geodesic ball $B(x_0,r) \subset D$ (see (7.10) in [12]). Recall also that the Hausdorff measure $H^n(f(E)) < \infty$ for any continuous mapping f (see, [7, subsection 2.2.2] with Theorem 1.1). Thus, in light of the above and by inequality (2.5),

$$\int\limits_E [L_f(x)]^n \, dH^n \ \leq \ C_\eta \int\limits_E \mu_f(x)\, dH^n \leq C_\eta H^n(f(E)) < \infty,$$

and $L_f \in L^n_{loc}(D)$. □

Remark 2.8. [16, Rem. 3.1] If a homeomorphism f is η-quasisymmetric, then

$$H_f(x,r) \leq \eta(1). \tag{2.11}$$

Corollary 2.9. *Let D and D' be domains in smooth connected Riemannian manifolds $(\mathbb{M}, g)$ and $(\mathbb{M}', g')$ with geodesic distances d and d', respectively, and let a homeomorphism $f : D \to D'$ be K-quasiconformal in any geodesic ball $B(x_0,r) \subset U(x_0)$ with $r = \lambda\ d(x_0, \partial D)$, $0 < \ \lambda < \ 1$. Let also $\eta : [0,\infty) \to [0,\infty)$ be a homeomorphism. Then the inequality*

$$C^{-2}\mu'_f(x) \leq [L_f(x)]^n \leq C^2 \mu'_f(x)[\eta(1)]^n \tag{2.12}$$

holds for almost every $x \in B(x_0,r)$.

Proof. Since a homeomorphism f is K-quasiconformal in any geodesic ball $B(x_0,r) \subset U(x_0)$ with $r = \lambda\ d(x_0, \partial D)$, $0 < \ \lambda < \ 1$, then, by Theorem 2.7, f is K-quasiconformal in D, and, by Theorem 2.6, f is a locally η-QS homeomorphism in D. Taking into account Remark 2.8, inequality (2.7) and formula (2.3), we obtain the inequality (2.12). □

Corollary 2.10. *Let D and D' be domains in smooth connected Riemannian manifolds $(\mathbb{M}, g)$ and $(\mathbb{M}', g')$ with geodesic distances d and d', respectively, and let a homeomorphism $f : D \to D'$ be an η-QS mapping. Then* (2.12) *holds for almost every $x \in D$.*

Proof. Since a homeomorphism f is an η-QS mapping, then, by Theorem 2.6, f is a K-*qc* mapping. Thus, to obtain the desired inequality, we can apply now Theorem 2.7 and Remark 2.8. □

3. Boundary behavior of η-quasisymmetric homeomorphisms between Riemannian manifolds

In connection with a problem on quasiconformal extension, F. W. Gehring and O. Martio introduced in 1985 a class of space domains for which the quasiconformality of mappings is equivalent to the quasi-Möbius property (see [8]). It is so-called *quasiextremal distance domains*, briefly QED domains. This concepts is based on the modulus of a path family. So, they proved that a c-uniform domain is c_1-QED with $c_1 = c_1(c, n)$. Recall that Martio and Sarvas introduced the class of *uniform domains* in 1979. In particular, any uniform domain is contained in an A-QED class with some A depending only on n and the parameters of uniformity. For a planar simply connected domain, the properties of QED and uniformity are equivalent to the condition that ∂D is a quasicircle. An interesting investigation of the structure of A-QED domains was developed by S. Yang. He gave a complete description of 1-QED and 2-QED domains in $\overline{\mathbb{R}^n}$. In particular, a Jordan domain in $\overline{\mathbb{R}^n}$ is 2-QED if and only if it is a ball. Moreover, many other characterizations of uniform domains have been established. Uniform domains can be understood as a class of domains developed in the context of generalizing Riemann Mapping Theorem for quasiconformal mappings in $\mathbb{R}^n$ with $n \geq 3$, a question that still remains open. This class of domains has numerous geometric and function theoretic properties that make it useful for many fields of the modern Mathematical Analysis (see, for example, [23]).

Recently, the boundary behavior of mappings in metric spaces and on Riemannian manifolds is studied by many specialists in Function Theory. Some interesting research directions see, for example, in links from papers [1–4, 13, 14] and [20].

Recall now some necessary definitions.

Let $\Gamma = \{\gamma\}$ be a family of some curves on an n–dimension Riemannian manifold $(\mathbb{M}, g)$. We say that a measurable by Borel nonnegative function $\rho : \mathbb{M} \to [0, \infty]$ is *admissible* for Γ, if

$$\int\limits_{\gamma} \rho \, ds \geq 1$$

for every locally rectifiable curve $\gamma \in \Gamma$. Here ds corresponds to the natural parameter of the arc lengths on a curve γ, calculated with respect to the geodesic distance d.

The modulus of a family of curves Γ is defined by

$$M(\Gamma) := \inf_{\rho \, \in \, adm \, \Gamma} \int\limits_{\mathbb{M}} \rho^n dv_g.$$

For the sets A, B and C on the manifold $(\mathbb{M}, g)$ by the symbol $\Delta(A, B; C)$ we define the set of all curves $\gamma : [a, b] \to \mathbb{M}$, which join A and B in C, i.e. $\gamma(a) \in A$, $\gamma(b) \in B$ and $\gamma(t) \in C$ for all $t \in (a, b)$.

The following definitions can be found in [18].

We say that the boundary of the domain D is *weakly flat at the point* $x_0 \in \partial D$, if for every number $P > 0$ and every neighborhood U of x_0 there is its neighborhood $V \subset U$, such that

$$M(\Delta(E, F; D)) \geq P$$

for every continua E and F in D, crossing ∂U and ∂V.

The boundary of the domain D is *weakly flat*, if the corresponding property takes place at every point of the boundary.

We also say that the domain D is *uniform*, if there are the constants a and b such that every pair of points x_1 and x_2 in D can be joined by a rectifiable curve γ in D with $l(\gamma) \leq a \; d(x_1, x_2)$, $\min(s, l(\gamma) - s) \leq b \operatorname{dist}(\gamma(s), \partial D)$, where γ is parameterized by the length s.

The domain D is *locally connected at a point* $x_0 \in \partial D$, if for every neighborhood U of x_0 there is its neighborhood $V \subseteq U$, such that $V \cap D$ is connected (see, e.g., [18]).

Recall (see, e.g., [23]) that a set D in $\overline{\mathbb{R}^n}$ is *c-locally connected*, if there exists a constant $c \in (1, \infty)$ with the following property. For each $x_0 \in \mathbb{R}^n$ and $r > 0$,
(i) points in $D \cap \overline{B^n}(x_0, r)$ can be joined in $D \cap \overline{B^n}(x_0, cr)$ and
(ii) points in $D \setminus B^n(x_0, r)$ can be joined in $D \setminus B^n(x_0, r/c)$.

In particular, if D is c-locally connected and $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is K-quasiconformal, then $f(D)$ is c'-locally connected, where c' depends only on n, c, and K (see [18, p. 52]).

Following [23], we say that a domain D is of *quasiextremal distance*, abbr. QED domain (or, more precisely, A-QED domain), if

$$M(\Delta(E,F;\mathbb{M})) \leq A \cdot M(\Delta(E,F;D))$$

for some $A \in [1;\infty)$ and any continua E and F in D.

Combining Theorem 2.6 with the boundary behavior theorems on quasiconformal mappings (see, e.g., [19, 25]), and using the Lemma 1.4, we obtain the following results.

Theorem 3.1. *Suppose that D is A-QED and D' is c'-locally connected. If $f : D \to D^{'}$ is an η-QS homeomorphism, then f has a homeomorphic extension to $\overline{D}$.*

Corollary 3.2. *If D and D' are A-QED domains and $f : D \to D^{'}$ is an η-QS homeomorphism, then f has a homeomorphic extension to $\overline{D}$.*

The following result can be found in [4, Prop. 3].

Proposition 3.3. *Let a domain D be uniform at some point $x_0 \in \partial D$. Then D is an A-QED domain at the point x_0 for some $A \in [1,\infty)$.*

Combining Proposition 3.3 with Theorem 3.1 and Corollary 3.2 we obtain the following results.

Theorem 3.4. *Suppose that D is a uniform domain and D' is c'-locally connected. If $f : D \to D^{'}$ is an η-QS homeomorphism, then f has a homeomorphic extension to $\overline{D}$.*

Corollary 3.5. *If D and D' are uniform domains and $f : D \to D^{'}$ is an η-QS homeomorphism, then f has a homeomorphic extension to $\overline{D}$.*

Similarly [23], we say that a domain D is a *K-quasiball* if D is the image of the unit ball B^n by some K-quasiconformal mapping $f : \mathbb{M} \to \mathbb{M}'$.

Two-dimensional quasiballs are called quasidisks, and they play an important role in several fields. For a simply connected proper subdomain of $\mathbb{R}^2$, the properties of the K-quasidisk, c-uniform and A-QED are quantitatively equivalent.

Following the diagram in [23, p. 122], we have

Proposition 3.6. *Let D be a K-quasiball domain at some point $x_0 \in \partial D$. Then D is an uniform domain at the point x_0.*

Theorem 3.7. *Suppose that D is a K-quasiball domain and D' is c'-locally connected. If $f : D \to D^{'}$ is an η-QS homeomorphism, then f has a homeomorphic extension to $\overline{D}$.*

Corollary 3.8. *If D and D' are K-quasiball domains and $f : D \to D^{'}$ is an η-QS homeomorphism, then f has a homeomorphic extension to $\overline{D}$.*

The following results can be found in [4].

Proposition 3.9. [4, Cor. 1] *Uniform and QED domains on Riemannian manifolds $\mathbb{M}$ have weakly flat boundaries.*

Proposition 3.10. [4, Cor. 2] *Uniform and QED domains on Riemannian manifolds $\mathbb{M}$ are locally connected on the boundary.*

Combining Propositions 3.9 and 3.10 with Theorem 2.6, we obtain the following results.

Theorem 3.11. *If $f : D \to D'$ is an η-QS homeomorphism, ∂D is weakly flat, D' is locally connected on the boundary and $\overline{D'}$ is compact, then f admits a continuous extension $\overline{f} : \overline{D} \to \overline{D'}$.*

Theorem 3.12. *If $f : D \to D^{'}$ is an η-QS homeomorphism, and D and D' have weakly flat boundaries and compact closures, then f admits a homeomorphic extension $\overline{f} : \overline{D} \to \overline{D'}$.*

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