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Fourier transforms on weighted amalgam type spaces

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Dedicated to 80th anniversary of the Corresponding Member of the NAS of Ukraine V. Ya. Gutlyanskii

Abstract. We introduce weighted amalgam type spaces and analyze their relations with some known spaces. Integrability results for the Fourier transform of a function with the derivative from one of these spaces are proved. The obtained results are applied to the integrability of trigonometric series with the sequence of coefficients of bounded variation.

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1. Introduction

Integrability of the Fourier transform is one of the important problems of harmonic analysis; [26] or [30] are good sources to see how much depends on whether the Fourier transform is integrable or not. Taking [28] as a starting point, one can see that many of such results were and still are certain analogs of the results on the absolute convergence of Fourier series. Some time ago the results started to appear (see, e.g., [10, 12, 16]) that have their source in a different topic of periodic harmonic analysis, the so-called integrability of trigonometric series. The latter is a name, given with a slight abuse of terminology, to the part of the problem whether the given trigonometric series is the Fourier series of an integrable function in the case where it is known that the series converges almost everywhere. The strongest known conditions that ensures (along with certain other

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natural assumptions) the integrability of trigonometric series with the sequence of coefficients with bounded variation, can be found in [1, 27] and [7]. In their analogs, certain subspaces of the space of functions of bounded variation are involved. In the last years, most of such results are overviewed in the survey paper [20] and in the monographs [14] and [19].

Returning to trigonometric series, we have to mention that the spaces in [1] and [7] and the spaces in [27] (and certain related ones) are of different nature. Moreover, they are incomparable in the sense that no one of these spaces is embedded into the other. The construction in [1] and [7] has been transferred to the nonperiodic setting in [17]. Similarly, the space $A_{1,2}$ considered there is incomparable with the real Hardy space H^1 and its versions. Generalizing [1,7], and [17], we introduce a different type of function and sequence spaces. We say that a locally integrable function g defined on $\mathbf{R}_+$ belongs to $A^{u,w}_{p,r}$, $1 \le p \le \infty$, $1 < r < \infty$, if

$$\|g\|_{A^{u,w}_{p,r}} = \sum_{m=-\infty}^{\infty} \left\{ \sum_{j=1}^{\infty} w_j \left[\int_{j2^{-m}}^{(j+1)2^{-m}} |g(t)|^p u(t)^p \, dt \right]^{\frac{r}{p}} \right\}^{\frac{1}{r}} < \infty. \tag{1}$$

Here $\{w_j\}$ is a sequence of non-negative numbers and u is a non-negative locally integrable function, both will naturally be called weights. When $w_j = 1$ for all j or $u(t) \equiv 1$, we will just omit one or both of them in the notation. In case where both of them are such, we get the space $A_{p,r}$ with the norm

$$\|g\|_{A_{p,r}} = \sum_{m=-\infty}^{\infty} \left\{ \sum_{j=1}^{\infty} \left[\int_{j2^{-m}}^{(j+1)2^{-m}} |g(t)|^p \, dt \right]^{\frac{r}{p}} \right\}^{\frac{1}{r}} < \infty. \tag{2}$$

If $p = 1$, this space is of amalgam nature, since for $A_{1,r}$ each of the summands in m is the norm in the Wiener type amalgam space $W(L^1, \ell^r)$ for functions $2^m g(2^m t)$ on $[2^m, \infty)$ and zero otherwise, with ℓ^r, $1 \le r < \infty$, being the space of sequences $\{d_j\}$ endowed with the norm

$$\|\{d_j\}\|_{\ell^r} = \left(\sum_{j=1}^{\infty} |d_j|^r \right)^{\frac{1}{r}}$$

and the norm of a function $g : \mathbf{R}_+ \to \mathbf{C}$ from the amalgam space $W(L^1, \ell^r)$ is taken as

$$\|\{ \int_j^{j+1} |g(t)| \, dt \}\|_{\ell^r}.$$

In the general case, each summand is also of amalgam kind, that is, locally belongs to one space and then globally amalgamated by another space.

It is expected that like the known conditions for the integrability of the cosine and sine Fourier transforms are given in terms of belonging of the derivative of the considered function to $A_{1,r}$, the new ones will be given in terms of belonging of the derivative of the considered function to $A_{p,r}^{u,w}$. This is possible only if such a space is a subspace of L^1. Moving forward a little, we can see that this is the case for the spaces we are going to consider. More precisely, we will deal only with the power weights $w_j = j^\delta$ and $u(t) = t^\beta$. In this case we shall denote the corresponding space by $A_{p,r}^{\beta,\delta}$. For these spaces, a simple sufficient condition to be embedded into L^1 is $\beta = \frac{1}{p'}$. Indeed, this follows from

$$\|g\|_{A_{1,r}^{\beta,\alpha}} \ge C \sum_{m=-\infty}^{\infty} \Big[\int\limits_{2^{-m}}^{2^{-m+1}} |g(t)|^p t^{\beta p}\, dt \Big]^{\frac{1}{p}}.$$

By Hölder's inequality, the right-hand side can be estimated from below by $\|g\|_{L^1(\mathbf{R}_+)}$ if the integrals

$$\int\limits_{2^{-m}}^{2^{-m+1}} t^{-\beta p'}\, dt$$

are uniformly bounded. This is the case exactly if $\beta = \frac{1}{p'}$. Thus, we have proved that

$$A_{p,r}^{\frac{1}{p'},w} \subset L^1(\mathbf{R}_+). \tag{3}$$

In [17] (see also [14, Ch. 3]), the following results are proved. We study, for $\gamma = 0$ or 1, the Fourier transforms

$$\widehat{f}_\gamma(x) = \int\limits_0^\infty f(t) \cos(xt - \frac{\pi\gamma}{2})\, dt. \tag{4}$$

It is clear that $\widehat{f}_\gamma$ represents the cosine Fourier transform in the case $\gamma = 0$, while taking $\gamma = 1$ gives the sine Fourier transform.

Theorem 1. *Let f be locally absolutely continuous on $\mathbf{R}_+$ and vanishing at infinity, that is, $\lim\limits_{t\to\infty} f(t) = 0$, and $f' \in A_{1,r}$ with $1 < r \le 2$. Then for $x > 0$*

$$\widehat{f}_\gamma(x) = \frac{1}{x} f(\frac{\pi}{2x}) \sin\frac{\pi\gamma}{2} + \Gamma(x),$$

where $\gamma = 0$ or 1, and $\|\Gamma\|_{L^1(\mathbf{R}_+)} \lesssim \|f'\|_{A_{1,r}}$.

To be precise, in [17] Theorem 4 is given only for $r = 2$, however, for $1 < r \le 2$ the result follows immediately from the known fact that $\ell^{p_1} \subset \ell^{p_2}$ if $p_1 < p_2$.

Here and in what follows $\varphi \lesssim \psi$ means that $\varphi \le C\psi$ with C being an absolute constant while $\varphi \asymp \psi$ means that both $\varphi \lesssim \psi$ and $\psi \lesssim \varphi$.

In the present paper we prove similar results for the case of the weighted amalgam type spaces, with power weights. This is what the next Section 2 is about. Further, in Section 3, we study these spaces. One of the main conclusions of this study is as follows. There existed a scale O_r (see Section 3) of integrability spaces for the Fourier transforms of a function of bounded variation, with H^1 on the top. Similarly, the weighted amalgam type spaces form an independent scale of the same type, with $A_{1,2}$ on the top. Though $A_{1,2}$ and H^1 are incomparable, each contains subspaces from the both scales. These subspaces are intermixed among themselves. In the last Section 4, we apply the obtained results to the integrability of trigonometric series.

2. Main results

Before proving the results for certain $A_{p,r}^{\beta,\delta}$ spaces we immediately pose additional restrictions. Further to the already discussed $\beta = \frac{1}{p'}$, we will consider only special $\delta = r(\alpha - \frac{1}{p'})$. By this, a natural idea is to simplify notation and denote the corresponding space by $A_{p,r,\alpha}$. To be precise, we say that a locally integrable function g defined on $\mathbf{R}_+$ belongs to $A_{p,r,\alpha}$, $1 \le p \le \infty$, $1 < r < \infty$, $\alpha \ge 0$, if

$$\|g\|_{A_{p,r,\alpha}} = \sum_{m=-\infty}^{\infty} \left\{ \sum_{j=1}^{\infty} j^{r(\alpha-\frac{1}{p'})} \left[\int_{j2^{-m}}^{(j+1)2^{-m}} |g(t)|^p t^{\frac{p}{p'}}\, dt \right]^{\frac{r}{p}} \right\}^{\frac{1}{r}} < \infty. \quad (5)$$

Theorem 2. *Let f be a function on $\mathbf{R}_+$, locally absolutely continuous on $(0,\infty)$ and vanishing at infinity, that is, $\lim\limits_{t\to\infty} f(t) = 0$, and $f' \in A_{p,r,\alpha}$, with $p > 1$, $1 < r \le 2$ and $0 \le \alpha < 1 - \frac{1}{p}$. Then, for $x > 0$,*

$$\widehat{f_\gamma}(x) = \frac{1}{x} f(\frac{\pi}{2x}) \sin\frac{\pi\theta}{2} + F(x),$$

where $\theta = 0$ or 1, and

$$\int_0^\infty |F(x)|\, dx \lesssim \|f'\|_{A_{p,r,\alpha}}.$$

Proof of Theorem 1. In fact, prior to a certain point the proof will go along the same lines as that of the main result in [17]. For the reader's convenience and in order to make the presentation self-contained, we will not omit the details. That mention point is crucial: we apply modified and refined Young's inequality elaborated in [23] rather than the classical one as in [17]. Now, splitting the integral (4) and integrating by parts, we obtain

$$\widehat{f_\theta}(x) = -\frac{1}{x} f(\frac{\pi}{2x}) \sin\frac{\pi}{2}(1-\theta)$$
$$+ \int_0^{\frac{\pi}{2x}} f(t)\cos(xt - \frac{\pi\theta}{2})\,dt - \frac{1}{x}\int_{\frac{\pi}{2x}}^{\infty} f'(t)\sin(xt-\frac{\pi\theta}{2})\,dt.$$

Further,

$$\int_0^{\frac{\pi}{2x}} f(t)\cos(xt-\frac{\pi\theta}{2})\,dt$$
$$= \int_0^{\frac{\pi}{2x}} [f(t) - f(\frac{\pi}{2x})]\cos(xt - \frac{\pi\theta}{2})\,dt + \int_0^{\frac{\pi}{2x}} f(\frac{\pi}{2x})\cos(xt-\frac{\pi\theta}{2})\,dt$$
$$= -\int_0^{\frac{\pi}{2x}} \left[\int_t^{\frac{\pi}{2x}} f'(s)\,ds\right]\cos(xt-\frac{\pi\theta}{2})\,dt$$
$$+\frac{1}{x} f(\frac{\pi}{2x})\sin\frac{\pi}{2}(1-\theta) + \frac{1}{x} f(\frac{\pi}{2x})\sin\frac{\pi\gamma}{2}$$
$$=\frac{1}{x} f(\frac{\pi}{2x})\sin\frac{\pi\theta}{2} + \frac{1}{x} f(\frac{\pi}{2x})\sin\frac{\pi}{2}(1-\theta) + O\left(\int_0^{\frac{\pi}{2x}} s|f'(s)|\,ds\right).$$

Since

$$\int_0^\infty \int_0^{\frac{\pi}{2x}} s|f'(s)|\,ds\,dx = \frac{\pi}{2}\int_0^\infty |f'(s)|\,ds,$$

it follows from (3) that to prove the theorem it remains to estimate

$$\int_0^\infty \frac{1}{x}\left|\int_{\frac{\pi}{2x}}^\infty f'(t)\sin(xt-\frac{\pi\theta}{2})\,dt\right|\,dx.$$

We can study

$$\sum_{m=-\infty}^{\infty} \int_{2^m}^{2^{m+1}} \frac{1}{x} \left| \int_{2^{-m}}^{\infty} f'(t) \sin(xt - \frac{\pi\theta}{2})\, dt \right| dx \tag{6}$$

instead. Indeed,

$$\sum_{m=-\infty}^{\infty} \int_{2^m}^{2^{m+1}} \frac{1}{x} \left| \int_{2^{-m}}^{\frac{\pi}{2x}} f'(t) \sin(xt - \frac{\pi\theta}{2})\, dt \right| dx$$

$$\leq \sum_{m=-\infty}^{\infty} \int_{2^m}^{2^{m+1}} \frac{1}{x} \int_{\frac{1}{x}}^{\frac{\pi}{2x}} |f'(t)|\, dt\, dx \leq \int_0^{\infty} |f'(t)|\, dt,$$

and again (3) is applied.

We now intermit the proof of the theorem. In order to control the L^1 norm of the Fourier transform, no matter cosine or sine, by means of the $A_{p,r,\alpha}$ norm, the crucial role belongs to the bounds of a special sequence of integrals over the dyadic intervals $[2^m, 2^{m+1}]$, the summands in (6). Given an integrable function g, we define the sequence of functions

$$\widehat{G_m}(x) = \int_{2^{-m}}^{\infty} g(t) e^{-ixt}\, dt.$$

Obviously, this function is the Fourier transform of the function $G_m(t)$ which is $g(t)$ for $2^{-m} < t < \infty$ and zero otherwise.

The mentioned integrals are estimated in the next lemma. It will be proved along the same lines as Lemma 1 in [17]. There is a slight difference in applying the Hausdorff–Young inequality rather than the Parseval identity and essential difference in applying the weighted Young inequality instead of its usual unweighted version. For the sake of completeness, we do not omit the details of the proof of both the theorem and the lemma.

Lemma 1. *Let g be an integrable function on $\mathbf{R}_+$. Then for $m \in \mathbf{Z}$, $p > 1$, $1 < r \leq 2$ and $0 \leq \alpha < 1 - \frac{1}{p}$,*

$$\int_{2^m}^{2^{m+1}} \frac{|\widehat{G_m}(x)|}{x}\, dx \lesssim \left(\sum_{j=1}^{\infty} j^{r(\alpha - \frac{1}{p'})} \left[\int_{j2^{-m}}^{(j+1)2^{-m}} |g(t)|^p t^{\frac{p}{p'}}\, dt \right]^{\frac{r}{p}} \right)^{\frac{1}{r}}.$$

Proof of Lemma 1. We start with the following inequality:

$$\int\limits_{2^m}^{2^{m+1}} \frac{|\widehat{G_m}(x)|}{x}\,dx \lesssim \int\limits_{2^m}^{2^{m+1}} |\widehat{S_m}(x)\,\widehat{G_m}(x)|\,dx, \tag{7}$$

where $\widehat{S_m}(x) = \frac{\sin 2^{-m}x}{x}$. The latter can be considered as the Fourier transform, up to a constant, of the indicator function of the interval $[0, 2^{-m}]$. It follows from the formula (see (5) in [4, Ch.I, §4]; it is mentioned in Remark 12 in the cited literature of [4] that the formula goes back to Fourier)

$$\int_0^\infty \frac{\sin ax}{x}\cos yx\,dx = \begin{cases} \frac{\pi}{2}, & y < a; \\ \frac{\pi}{4}, & y = a; \\ 0, & y > a. \end{cases}$$

By Hölder's inequality, the right-hand side of (7) does not exceed

$$2^{\frac{m}{r}} \left(\int\limits_{2^m}^{2^{m+1}} |\widehat{S_m}(x)\,\widehat{G_m}(x)|^{r'}\,dx \right)^{\frac{1}{r'}}.$$

In fact, we no more need the integral over $[2^m, 2^{m+1}]$ (the factor $2^{\frac{m}{r}}$ we got from it was all we needed) and have to estimate

$$2^{\frac{m}{r}} \left(\int\limits_{\mathbf{R}} |\widehat{S_m}(x)\,\widehat{G_m}(x)|^{r'}\,dx \right)^{\frac{1}{r'}}. \tag{8}$$

Since G_m is integrable and S_m is in $L^{r'}$, $\widehat{S_m}(x)\widehat{G_m}(x)$ is the Fourier transform of their convolution; see Theorems 64 and 65 in [28, 3.13]. Applying the Hausdorff–Young inequality, and this is where the condition $1, r \le 2$ comes into play, we see that (8) is controlled with

$$2^{\frac{m}{r}} \left(\int\limits_{\mathbf{R}} |(S_m * G_m)(x)|^r\,dx \right)^{\frac{1}{r}}. \tag{9}$$

Further,

$$\widehat{G_m}(x) = \sum_{j=1}^\infty \int\limits_{j2^{-m}}^{(j+1)2^{-m}} g(t)e^{-ixt}\,dt = \sum_{j=1}^\infty \widehat{G_{m,j}}(x),$$

where

$$\widehat{G_{m,j}}(x) = \int\limits_{j2^{-m}}^{(j+1)2^{-m}} g(t)e^{-ixt}\,dt.$$

Correspondingly,

$$G_m(x) = \sum_{j=1}^{\infty} g_{m,j}(x),$$

with $g_{m,j}(x) = g(x)$ when $j2^{-m} \le x < (j+1)2^{-m}$ and zero otherwise. Representing (9) as

$$2^{\frac{m}{r}} \left(\int\limits_{\mathbf{R}} \left| \sum_{j=1}^{\infty} S_m * g_{m,j}(x) \right|^r dx \right)^{\frac{1}{r}},$$

let us analyze what the support of each summand

$$S_m * g_{m,j}(x) = \int\limits_{j2^{-m}}^{(j+1)2^{-m}} S_m(x-t)g(t)\,dt$$

is. Since we have $0 < x - t < 2^{-m}$, such a summand is supported within the interval $j2^{-m} \le x \le (j+2)2^{-m}$. Only two neighboring intervals may have an intersection of positive measure. Therefore, the value in (9) is dominated by

$$2^{\frac{m}{r}} \left(\sum_{\substack{j=1 \\ j\ is\ even}}^{\infty} \int\limits_{\mathbf{R}} |S_m * g_{m,j}(x)|^r\,dx \right)^{\frac{1}{r}}$$
$$+2^{\frac{m}{r}} \left(\sum_{\substack{j=1 \\ j\ is\ odd}}^{\infty} \int\limits_{\mathbf{R}} |S_m * g_{m,j}(x)|^r\,dx \right)^{\frac{1}{r}}.$$

The bound for each of the two values is the same and can be obtained by means of weighted Young's inequality for convolution (for non-weighted case, see, e.g., [26, Ch.V, §1]). We make use of Corollary 1 from [23]). Its variant for the L^r estimate of the convolution reads as follows. If $\varphi \in L^p_\alpha$ and $\psi \in L^{q,\infty}$ (the latter is the Lorentz space), then

$$\|\varphi * \psi\|_{L^r} \le \|\varphi\|_{L^p_\alpha} \|\psi\|_{L^{q,\infty}} \tag{10}$$

provided

(i) $1 < p \le r < \infty$ and $1 < q < \infty$;

(ii) $0 \le \alpha < 1 - \frac{1}{p}$;

(iii) $\frac{1}{r} = \frac{1}{q} + \frac{1}{p} + \alpha - 1$. Taking $\psi = S_m$ and $\varphi = g_{m,j}$, we obtain in each of the two cases the bound

$$2^{\frac{m}{r}} \left(\sum_{j=1}^{\infty} \|S_m\|_{L^{q,\infty}}^r \|g_{m,j}\|_{L^p_\alpha}^r \right)^{\frac{1}{r}}.$$

Since

$$\|S_m\|_{L^{q,\infty}}^r \asymp 2^{-m\frac{r}{q}},$$

we get the bound

$$2^{m(\frac{1}{r}-\frac{1}{q})} \left(\sum_{j=1}^{\infty} \left[\int_{j2^{-m}}^{(j+1)2^{-m}} |g(t)|^p t^{p\alpha}\, dt \right]^{\frac{r}{p}} \right)^{\frac{1}{r}}$$

$$\asymp 2^{m(\frac{1}{r}-\frac{1}{q}-\alpha)} \left(\sum_{j=1}^{\infty} j^{r\alpha} \left[\int_{j2^{-m}}^{(j+1)2^{-m}} |g(t)|^p\, dt \right]^{\frac{r}{p}} \right)^{\frac{1}{r}}.$$

By (iii), we have

$$2^{m(\frac{1}{r}-\frac{1}{q}-\alpha)} = 2^{m(\frac{1}{p}-1)}.$$

Taking this into account while summing up in m and fulfilling standard calculations, we complete the proof of the lemma. □

We are now in a position to complete the proof of the theorem merely by applying the proven Lemma 1 to each of the summands in (6). □

Remark 3. There are versions of weighted Young's inequality for power weights in [15] and in [6]. The assumptions in both involve additional parameter (space) but lead to exactly the same estimate as in the above lemma. We also mention that Young's inequality with general weights can be found in [25]. However, it is true under extremely complicated conditions, by no means practical, at least , in our setting. It is worth mentioning that the applied Young's inequality has been preceded in [22] and refined in [24]. What these give for our study is as follows: the applied inequality is sharp and none of the obtained versions can yield better estimates.

3. Properties of the $A_{p,r,\alpha}$ spaces

One of the basic properties is already established in the introduction. We mean (3), the embedding into L^1. The main result in the previous section is obtained under assumption that the derivative belongs to $A_{p,r,\alpha}$ and, correspondingly, to L^1. Since we deal with (locally) absolutely continuous functions, we deduce that such functions are of bounded variation, and the obtained results are within the scope of the Fourier transforms of functions with bounded variation (see [19]). We now continue with the comparison of $A_{p,r,\alpha}$ for different values of the parameters p, r and α. Further, we shall compare $A_{p,r,\alpha}$ with some other spaces related to the integrability problems for the Fourier transforms of functions of bounded variation.

Though the obtained results can be derived from certain embeddings, the restriction $1 < r \le 2$ apparently cannot. In fact, the relation of the case $r > 2$ to the problems of the integrability of the Fourier transforms is an open problem.

3.1. Comparison of $A_{p,r,\alpha}$ for different values of the parameters

As mentioned, for fixed p and α, the space is becomes smaller when r changes from 2 towards 1. Since always $\alpha - \frac{1}{p'} < 0$, the larger is α the wider is the space, with the smallest one for $\alpha = 0$. Let us now figure out what happens for different p. For simplicity, we will consider only the case $r = 2$. Let $p_1 < p_2$. For m fixed, we apply Hölder's inequality to each of the m-th summands with the exponential $P = \frac{p_2}{p_1}$. Its conjugate $P' = \frac{p_2}{p_2 - p_1}$. Thus,

$$\int_{j2^{-m}}^{(j+1)2^{-m}} |g(t)|^{p_1} t^{\frac{p_1}{p_1'}}\, dt$$

$$\le \left(\int_{j2^{-m}}^{(j+1)2^{-m}} |g(t)|^{p_2} t^{\frac{p_2}{p_2'}}\, dt \right)^{\frac{1}{p_2}} \left(\int_{j2^{-m}}^{(j+1)2^{-m}} t^{-\frac{p_1}{p_1'}P'} t^{-\frac{p_1}{p_2'}P'}\, dt \right)^{\frac{1}{P'}}$$

$$\lesssim \left(\int_{j2^{-m}}^{(j+1)2^{-m}} |g(t)|^{p_2} t^{\frac{p_2}{p_2'}}\, dt \right)^{\frac{1}{p_2}} \left(\int_{j2^{-m}}^{(j+1)2^{-m}} \frac{dt}{t} \right)^{\frac{1}{P'}}$$

$$\lesssim j^{\frac{p_1-p_2}{p_2}} \left(\int\limits_{j2^{-m}}^{(j+1)2^{-m}} |g(t)|^{p_2} t^{\frac{p_2}{p_2'}} \, dt \right)^{\frac{1}{p_2}}.$$

Since

$$j^{2(\alpha-\frac{p_1-1}{p_1})} j^{2\frac{p_1-p_2}{p_1p_2}} = j^{2(\alpha-\frac{1}{p_2'})},$$

we obtain

$$A_{p_2,2,\alpha} \subset A_{p_1,2,\alpha}. \tag{11}$$

Therefore, for $\alpha > 0$ and $p > 1$, we have

$$A_{p,2,\alpha} \subset A_{1,2,\alpha} \subset A_{1,2}. \tag{12}$$

The space on the right is the widest in its range, which makes the results in [17] less restrictive than the obtained in this work. It is interesting to compare the proofs of both as well. To obtain the weighted amalgam type condition for the integrability of the Fourier transform, in both [17] and in the present work, we apply the classical Young inequality and its weighted extension, respectively. The latter holds only for $p > 1$, therefore the proof of the obtained result does not reduce to the older one in this way. However, this can be done in a different manner. In the definition of the space $A_{p,r}^{\frac{1}{p'},r(\alpha+\frac{1}{p'})}$, we can try to let p tend to 1. For this, $\alpha = 0$ should be taken. In this way, both subscripts tend to zero, and limiting space becomes $A_{1,r}$. This is the same as the embedding above.

3.2. Comparison with the D. Borwein spaces

There is a scale of subspaces of L^1 proved to be convenient in the problems of integrability of the Fourier transform of a function of bounded variation.

Definition 4. For $1 < p < \infty$, the space O_p is the space of functions g with finite norm

$$\|g\|_{O_p} = \int_0^\infty \left(\frac{1}{x} \int_x^{2x} |g(t)|^p \, dt \right)^{\frac{1}{p}} dx. \tag{13}$$

All these spaces and their sequence analogs first appeared in the paper by D. Borwein [5], but became – for sequences – widely known after the paper by G. A. Fomin [9]. More details can be found in [19, Ch. 1].

Further, for $p = \infty$ we define the corresponding space as follows.

Definition 5. The space O_∞ is the space of functions g with finite norm

$$\|g\|_{O_\infty} = \int_0^\infty \operatorname*{ess\,sup}_{x\le t\le 2x} |g(t)|\,dx. \tag{14}$$

This space is connected with the names of D. K. Faddeev, Beurling, Telyakovskii; for more details see [3].

Let us compare O_p with $A_{p,2,\alpha}$, $\alpha \ge 0$. We have

$$\begin{aligned}\int_0^\infty \left(\frac{1}{x}\int_x^{2x} |g(t)|^p\,dt\right)^{\frac{1}{p}} dx &= \sum_{m=-\infty}^{\infty} \int\limits_{2^m}^{2^{m+1}} \left(\frac{1}{x}\int_x^{2x} |g(t)|^p\,dt\right)^{\frac{1}{p}} dx \\ &\le \sum_{m=-\infty}^{\infty} \int\limits_{2^m}^{2^{m+1}} \left(\frac{1}{x}\int_{2^m}^{2^{m+2}} |g(t)|^p\,dt\right)^{\frac{1}{p}} dx \\ &\lesssim \sum_{m=-\infty}^{\infty} \left(\int_{2^m}^{2^{m+1}} |g(t)|^p t^{p-1}\,dt\right)^{\frac{1}{p}}.\end{aligned}$$

This leads to the embedding

$$A_{p,2,0} \subset O_p. \tag{15}$$

On the other hand, there holds

$$O_r \subset A_{1,r}, \tag{16}$$

for $1 < r \le \infty$. Indeed, applying Hölder's inequality for $1 < r < \infty$, we obtain

$$\int\limits_{j2^m}^{(j+1)2^m} |g(t)|\,dt \le 2^{\frac{m}{r'}} \left(\int\limits_{j2^m}^{(j+1)2^m} |g(t)|^r\,dt\right)^{\frac{1}{r}}.$$

Summing up in j and m, we see that the norm in $A_{1,r}$ is dominated by

$$\sum_{m=-\infty}^{\infty}\left(2^{\frac{mp}{r'}}\int_{2^m}^{\infty}|g(t)|^r\,dt\right)^{\frac{1}{r}}$$

$$=\sum_{m=-\infty}^{\infty}\int_{2^m}^{2^{m+1}}\frac{dx}{2^m}\left(2^{\frac{mp}{r'}}\int_{2^m}^{\infty}|g(t)|^r\,dt\right)^{\frac{1}{r}}$$

$$=\sum_{m=-\infty}^{\infty}\int_{2^m}^{2^{m+1}}\left(\frac{1}{2^m}\int_{2^m}^{\infty}|g(t)|^r\,dt\right)^{\frac{1}{r}}$$

$$\le 2^{\frac{1}{r}}\sum_{m=-\infty}^{\infty}\int_{2^m}^{2^{m+1}}\left(\frac{1}{x}\int_{2^m}^{\infty}|g(t)|^r\,dt\right)^{\frac{1}{r}}dx$$

$$\le 2^{\frac{1}{r}}\sum_{m=-\infty}^{\infty}\int_{2^m}^{2^{m+1}}\left(\frac{1}{x}\int_{\frac{x}{2}}^{\infty}|g(t)|^r\,dt\right)^{\frac{1}{r}}dx$$

$$=2\sum_{m=-\infty}^{\infty}\int_{2^{m-1}}^{2^m}\left(\frac{1}{x}\int_{x}^{\infty}|g(t)|^r\,dt\right)^{\frac{1}{r}}dx$$

$$=2\int_0^{\infty}\left(\frac{1}{x}\int_{x}^{\infty}|g(t)|^r\,dt\right)^{\frac{1}{r}}dx.$$

For $r=\infty$, we arrive at the required estimate immediately.

4. Integrability of trigonometric series

We mention that the elaborated in [16] Fourier transform approach to obtaining the results on the integrability of trigonometric series allows one to derive the latter from the results on the Fourier transform of a function of bounded variation (for a more general approach, see the recent paper [21]). For instance, in [17], we reproduced in this way the results from [1] and [7], almost readily and even in a slightly more general form. In [18], it was not a reproduction but completely new results on the integrability of trigonometric series, with no earlier analogs, since such were the prototype results on the Fourier transform. Similarly, in this work we derive also completely new results on the integrability of trigonometric series.

We begin with the definition of the discrete weighted amalgam type spaces. We shall not give them in full generality as for function but restrict ourselves only to the analogs of $A_{p,r,\alpha}$. We say that a sequence $a = \{a_n\}$ belongs to $a_{p,r,\alpha}$, $1 \le p \le \infty$, $1 < r < \infty$, $\alpha \ge 0$, if

$$\|a\|_{p,r,\alpha} = \sum_{m=-\infty}^{\infty} \left\{ \sum_{j=1}^{\infty} j^{r(\alpha - \frac{1}{p'})} \left[\sum_{n=j2^{-m}}^{(j+1)2^{-m}-1} |a_n|^p n^{\frac{p}{p'}} \right]^{\frac{r}{p}} \right\}^{\frac{1}{r}} < \infty. \quad (17)$$

We will deal with the sequences of the coefficients of the cosine series

$$a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx \quad (18)$$

and the sine series

$$\sum_{n=1}^{\infty} b_n \sin nx. \quad (19)$$

The problem is to find assumptions on the sequences of coefficients $\{a_n\}$, $\{b_n\}$ under which the series is the Fourier series of an integrable function. One of the basic assumptions is that the sequence $\{a_n\}$ or $\{b_n\}$ is of bounded variation, written $\{a_n\} \in bv$ or $\{b_n\} \in bv$, that is, satisfies the condition (for $\{a_n\}$; similarly for $\{b_n\}$)

$$\sum_{n=1}^{\infty} |\Delta a_n| < \infty, \quad (20)$$

where $\Delta a_n = a_n - a_{n+1}$ and similarly for Δb_n.

Since the passage from the estimates for the Fourier transforms of a function of bounded variation f to the corresponding trigonometric series, with the coefficients being the values of f at the lattice points, is described many times (see, e.g., [16] or [17]), we omit most of the details and the proof except the facts needed to state the result. Roughly speaking, the results for the Fourier transforms of a function of bounded variation can be rewritten for the trigonometric series with the coefficients being the values of the transformed function at the lattice points just by replacing integrals by sums.

First, given series (18) or (19) with the null sequence of coefficients being in an appropriate sequence space, set for $x \in [n, n+1]$

$$A(x) = a_n + (n-x)\Delta a_n, \qquad a_0 = 0,$$
$$B(x) = b_n + (n-x)\Delta b_n.$$

So, we construct a corresponding function by means of linear interpolation of the sequence of coefficients. Secondly, the result due to Trigub [29, Th. 4] (see also [30]; an earlier version, for functions with compact support, is due to Belinsky [2]) allows us to pass from estimating trigonometric series (18) and (19) to estimating the Fourier transform of $A(t)$ and $B(t)$, respectively.

Theorem 6. *If the coefficients* $\{a_n\}$ *in* (18) *and* $\{b_n\}$ *in* (19) *tend to* 0 *as* $n \to \infty$, *and the sequences* $\{\Delta a_n\}$ *and* $\{\Delta b_n\}$ *are in* $a_{p,r,\alpha}$, *with* $p > 1$, $1 < r \le 2$ *and* $0 \le \alpha < 1 - \frac{1}{p}$, *then* (18) *represents an integrable function on* $[0, \pi]$, *and*

$$\sum_{n=1}^{\infty} b_n \sin nx = \frac{1}{x} B(\frac{\pi}{2x}) + F(x), \tag{21}$$

where $\int\limits_0^{\pi} |F(x)|\, dx \lesssim \|\{\Delta b_n\}\|_{a_{p,r,\alpha}}$.

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References

[1] Aubertin, B., Fournier, J.J.F. (1993). Integrability theorems for trigonometric series. *Studia Math., 107*, 33–59.

[2] Belinsky, E. S. (1975). On asymptotic behavior of integral norms of trigonometric polynomials. In: *Metric Questions of the Theory of Functions and Mappings, 6.* Nauk. Dumka, Kiev, 15–24 (in Russian).

[3] Belinsky, E., Liflyand, E., Trigub, R. (1997). The Banach algebra A^* and its properties. *J. Fourier Anal. Appl., 3*, 103–129.

[4] Bochner, S. (1959). *Lectures on Fourier integrals.* Princeton Univ. Press, Princeton, N.J.

[5] Borwein, D. (1965). Linear functionals connected with strong Cesáro summability. *J. London Math. Soc., 40*, 628–634.

[6] Bui, H.-Q. (1994). Weighted Young's Inequality and Convolution Theorems on Weighted Besov Spaces. *Math. Nachr., 170*, 25–37.

[7] Buntinas, M., Tanović-Miller, N. (1991). New integrability and L^1-convergence classes for even trigonometric series II. Appr. Theory, J. Szabados and K. Tandori (eds.). *Colloq. Math. Soc. János Bolyai, 58.* North-Holland, Amstedam, 103–125.

[8] Feichtinger, H. (1992). Wiener amalgams over Euclidean spaces and some of their applications. In: *Function spaces* (Edwardsville, IL, 1990). Lect. Notes Pure Appl. Math., 136. Dekker, New York, 123–137.

[9] Fomin, G.A. (1978). A Class of Trigonometric Series. *Matem. Zametki, 23*, 213–222; transl. in *Math. Notes, 23*, 117–123.

[10] Fridli, S. (2001). Hardy Spaces Generated by an Integrability Condition. *J. Approx. Theory, 113*, 91–109.

[11] Fournier, J.J.F., Stewart, J. (1985). Amalgams of L^p and ℓ^q. *Bull. Amer. Math. Soc. (N.S.), 13*, 1–21.

[12] Giang, D.V., Móricz, F. (1994). Multipliers of Fourier transforms and series on L^1. *Arch. Math., 62*, 230–238.

[13] Heil, C. (2003). *An Introduction to Weighted Wiener Amalgams.* Wavelets and their Applications (Chennai, 2002), M. Krishna, R. Radha and S. Thangavelu, eds., Allied Publishers, New Delhi, 183–216.

[14] Iosevich, A., Liflyand, E. (2014). *Decay of the Fourier transform: analytic and geometric aspects.* Birkhäuser.

[15] Kerman, R.A. (1983). Convolution theorems with weights. *Trans. Amer. Math. Soc., 280*, 207–219.

[16] Liflyand, E. (1993). Fourier transforms of functions from certain classes. *Anal. Math., 19*, 151–168.

[17] Liflyand, E. (2013). Fourier transforms on an amalgam type space. *Monatsh. Math., 172*, 345–355.

[18] Liflyand, E. (2018). The Fourier transform of a function of bounded variation: symmetry and asymmetry. *J. Fourier Anal. Appl., 24*, 525–544.

[19] Liflyand, E. (2019). *Functions of Bounded Variation and their Fourier Transforms*, Birkhäuser.

[20] Liflyand, E., Samko, S., Trigub, R. (2012). The Wiener algebra of absolutely convergent Fourier integrals: an overview. *Anal. Math. Physics, 2*, 1–68.

[21] Liflyand, E., Trigub, R. (2021). Wiener algebras and trigonometric series in a coordinated fashion. *Constructive Approx., 54*, arXiv:1910.02777, 2019.

[22] Nursultanov E., Tikhonov, S. (2011). Convolution inequalities in Lorentz spaces. *J. Fourier Anal. Appl., 17*, 486–505.

[23] Nursultanov, E., Tikhonov, S. (2015). Weighted norm inequalities for convolution and Riesz potential. *Potential Anal., 42*, 435–456.

[24] Nursultanov, E., Tikhonov, S., Tleukhanova, N. (2018). Norm convolution inequalities in Lebesgue spaces. *Rev. Mat. Iberoam., 34*, 811–838.

[25] Rakotondratsimba, Y. (2002). Weighted Young Inequalities for Convolutions. *Southeast Asian Bull. Math., 26*, 77–99.

[26] Stein, E.M., Weiss, G.(1971). *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton Univ. Press, Princeton, N.J.

[27] Telyakovskii, S.A. (1964). Integrability conditions for trigonometric series and their applications to the study of linear summation methods of Fourier series. *Izv. Akad. Nauk SSSR, Ser. Matem., 28*, 1209–1236 (in Russian).

[28] Titchmarsh, E.C. (1937). *Introduction to the Theory of Fourier Integrals*. Clarendon Press, Oxford.

[29] Trigub, R.M. (1989). Multipliers of Fourier series and approximation of functions by polynomials in spaces C and L. *Dokl. Akad. Nauk SSSR, 306*, 292–296; transl. in (1989). *Soviet Math. Dokl., 39*, 494–498.

[30] Trigub, R.M., Belinsky, E.S. (2004). *Fourier Analysis and Appoximation of Functions*, Kluwer.

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