Український математичний вісник
Том 18 **(2021)**, № 2, 196 – 208
https://doi.org/10.37069/1810-3200-2021-18-2-4

An alternative capacity in metric measure spaces

Olli Martio

Dedicated to Vladimir Gutlyanskii on his 80^{th} birthday

Abstract. A new condenser capacity $\mathrm{Cap}_\mathrm{p}^\mathrm{M}(E,G)$ is introduced as an alternative to the classical Dirichlet capacity in a metric measure space X. For $p>1$ it coincides with the M_p–modulus of the curve family $\Gamma(E,G)$ joining ∂G to an arbitrary set $E\subset G$ and for $p=1$ it lies between $AM_1(\Gamma(E,G))$ and $M_1(\Gamma(E,G))$. Moreover, the $\mathrm{Cap}_\mathrm{p}^\mathrm{M}(E,G)$–capacity has good measure theoretic regularity properties with respect to the set E. The $\mathrm{Cap}_\mathrm{p}^\mathrm{M}(E,G)$–capacity uses Lipschitz functions and their upper gradients. The doubling property of the measure μ and Poincaré inequalities in X are not needed.

2010 MSC. 31C15, 31B15.

Key words and phrases. Capacity, M– and AM–modulus.

1. Introduction

The Dirichlet p–capacity $\mathrm{cap}_p(E,G)$ of a condenser (E,G), developed by G. Choquet, is the most commonly used capacity in analysis. The modulus of a curve family offers an alternative approach to capacity. In a metric measure space X curve families play a more central role than in $\mathbb{R}^n$ since the Fubini theorem is not available in X. For example, in X the modulus method is used to construct so called Newtonian spaces which have many properties common to the first order Sobolev spaces in $\mathbb{R}^n$. The constructions require that the metric space X is so called good metric space, i.e. the measure μ in X is doubling and X supports a Poincaré inequality in addition to various topological properties, see [2], [7] and [13].

The purpose of this paper is to introduce an alternative acapacity, the $\mathrm{Cap}_\mathrm{p}^\mathrm{M}$–capacity, which is directly connected to the M_p–modulus for $p>1$

Received 12.06.2021

and uses neither the doubling property nor the Poincaré inequalities. For $p = 1$ the $\mathrm{Cap}_\mathrm{p}^\mathrm{M}$–capacity lies between the AM_1– and M_1–modulus.

The $\mathrm{Cap}_\mathrm{p}^\mathrm{M}$–capacity, $p > 1$, offers a more straightforward approach to the classical Dirichlet capacity cap_p and their equivalence is considered in Section 4.

2. M_p– and AM_p–modulus

Let (X, d) be a metric space equipped by a Borel regular measure μ which is finite on compact sets. We also assume that X is proper, i.e. bounded closed sets are compact. From this it follows that X is complete.

A continuous mapping $\gamma\colon [a, b] \to X$ is called a *curve*. We say that a curve γ is a *path* if it has a finite and non–zero total length; in this case we parametrize γ by its arclength. The *locus* of γ is defined as $\gamma([0, \ell])$ and denoted by $\langle\gamma\rangle$ and the length of γ by $\ell(\gamma)$.

Let Γ be a family of paths in X. A non–negative Borel function ρ is *M–admissible*, or simply admissible, for Γ if

$$\int_\gamma \rho\, ds \geq 1$$

for every $\gamma \in \Gamma$. For $p \geq 1$ the *M_p–modulus* of Γ is defined as

$$M_p(\Gamma) = \inf \int_X \rho^p\, d\mu$$

where the infimum is taken over all admissible functions ρ.

A sequence of non–negative Borel functions ρ_i, $i = 1, 2, \dots$, is *AM–admissible*, or simply admissible, for Γ if

$$\liminf_{i\to\infty} \int_\gamma \rho_i\, ds \geq 1 \tag{2.1}$$

for every $\gamma \in \Gamma$. The *approximation modulus* Γ is defined as

$$AM_p(\Gamma) = \inf_{(\rho_i)} \Big\{ \liminf_{i\to\infty} \int_X \rho_i^p\, d\mu \Big\} \tag{2.2}$$

where the infimum is taken over all AM–admissible sequences (ρ_i) for Γ.

Since the space X is proper, instead of admissible Borel functions it is possible to use lower semicontinuous non–negative functions as admissible for the M_p– and AM_p–modulus, see e.g. [5, Proposition 7.14].

For the following lemma, we refer to [8], [12] and [9] for the properties of the AM_p–modulus and to [2], [6] and [1] for those of the M_p-modulus, $p \geq 1$.

Lemma 2.1. *The AM_p– and M_p–modulus are outer measures in the set of all paths in X, i.e.*
(a) $AM_p(\emptyset) = 0$
(b) $\Gamma_1 \subset \Gamma_2 \implies AM_p(\Gamma_1) \leq AM_p(\Gamma_2)$.
(c) $\Gamma = \bigcup_{j=1}^{\infty} \Gamma_j \implies AM_p(\Gamma) \leq \sum_{i=j}^{\infty} AM_p(\Gamma_j)$.
(d) $AM_1(\Gamma) \leq M_1(\Gamma)$ *and* $AM_p(\Gamma) = M_p(\Gamma)$, $p > 1$, *for every path family* Γ.
The properties (a)–(c) also hold for the M_p–modulus.

We employ following notation for path families associated with an arbitrary set $E \subset X$ and an open bounded set $G \supset E$:

$$\Gamma(E, G) = \Gamma(E) \cap \Gamma(X \setminus G) \text{ and } \Gamma(E) = \{\gamma : \gamma \text{ meets } E\}.$$

3. $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$–capacity

In this section we assume that X is a proper metric space with a Borel regular measure μ and introduce a new capacity for the condenser (E, G) where E is an arbitrary subset of a bounded open set G in X. Since a metric space usually has plenty of Lipschitz functions but need not contain many curves such a capacity is not possible without an assumption that guarantees plenitude of curves and we use the quasiconvexity property of X, i.e. there is $c < \infty$ such that for all $x, y \in X$, $x \neq y$, there exists a path γ joining x to y whose length satisfies $\ell(\gamma) \leq c\, d(x, y)$. Note that we do not need the quasiconvexity property for G but for X.

A complete doubling p–Poincaré space X is quasiconvex, see [2, Chapter 4], but the converse is not true as simple examples show.

We mostly work with Lipschitz functions in X. For such a function u a non-negative Borel function g is an upper gradient of u in X if for every path γ in X

$$|u(\gamma(\ell)) - u(\gamma(0))| \leq \int_\gamma g\, ds,$$

see [2, Chapters 1–2] for the properties of functions and their upper gradients. The lower pointwise dilatation

$$|\nabla u(x)| = \liminf_{r \to 0} \sup_{y \in B(x,r)} \frac{|u(y) - u(x)|}{r}$$

is an upper gradient of u, see [2, Proposition 1.14]. In $\mathbb{R}^n$, $|\nabla u(x)|$ is a unique minimal upper gradient for a Lipschitz function u, see [2, Examples A1].

Let G be a fixed bounded open set in X and E an arbitrary subset G. An increasing sequence (u_i) of non–negative Lipschitz functions in X is called *admissible*, $(u_i) \in Ad(E,G)$, for the condenser (E,G) if $u_i = 0$ in $X \setminus G$ and

$$\liminf_{i\to\infty} u_i(x) \geq 1$$

for $x \in E$ For $p \geq 1$ we define

$$\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E,G) = \inf\Big\{ \liminf_{i\to\infty} \int_G g_i^p \, d\mu : (u_i) \in Ad(E,G) \text{ and } g_i \text{ is an upper gradient of } u_i \Big\}.$$

It is obvious that the $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$–capacity is monotone, i.e.

$$E_1 \subset E_2 \subset G \Longrightarrow \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E_1,G) \leq \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E_2,G). \tag{3.1}$$

In the rest of this section we assume that X is proper and quasiconvex and $G \subset X$ is a bounded open set.

Theorem 3.1. *If E is an arbitrary subset of G, then for $p \geq 1$*

$$AM_p(\Gamma(E,G)) \leq \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E,G) \leq M_p(\Gamma(E;G)). \tag{3.2}$$

For $p > 1$ the above inequalities are equalities and, in particular,

$$\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E,G) = M_p(\Gamma(E;G)). \tag{3.3}$$

Proof. The first inequality $\leq$ in (3.2) is classical. For completeness we recall the proof. Let (u_i) be an $Ad(E,G)$–admissible sequence and g_i an upper gradient of u_i. Now (g_i) is an AM–admissible sequence for $\Gamma(E,G)$ because for each path $\gamma \in \Gamma(E,G)$ with $\gamma(\ell) \in E$

$$\liminf_{i\to\infty} \int_\gamma g_i \, ds \geq \liminf_{i\to\infty} u_i(\gamma(\ell)) \geq 1.$$

Hence

$$AM_p(\Gamma(E,G)) \leq \liminf_{i\to\infty} \int_G g_i^p \, d\mu$$

and since this holds for all sequences $(u_i) \in Ad(E,G)$ and all upper gradients g_i of u_i, the left side of (3.2) follows.

For the second inequality in (3.2) we use a modification of the method in [2, Lemmata 5.25 and 5.26]. Let $\tilde\rho$ be a lower semicontinuous M–admissible function for $\Gamma(E,G)$. We may assume that $\tilde\rho = 0$ in $X \setminus G$ and

$$\int_G \tilde\rho^p \, d\mu < \infty.$$

Let $\tau > 0$ and set $\rho = \tilde{\rho} + \tau$ in X. Now ρ is lower semicontinuous in X and since X is proper there is an increasing sequence of continuous functions $\rho_i : X \to [0, \infty)$ such that

$$\lim_{i\to\infty} \rho_i(x) = \rho(x)$$

for every $x \in X$. We may assume that $\rho_i \geq \tau$ in X.

For each i define

$$u_i(x) = \inf\Big\{ \int_\gamma \rho_i \, ds : \gamma \text{ joins } X \setminus G \text{ to } x \Big\}$$

for $x \in G$ and $u_i(x) = 0$ for $x \in X \setminus G$. Note that each path γ which meets $X \setminus G$ and $x \in G$ has a subpath meeting $X \setminus G$ at $\gamma(0)$ only. Hence in the definition of $u_i(x)$, $x \in G$, we can consider only paths γ which lie in G except at $\gamma(0)$. The sequence (u_i) is increasing and we show that each u_i is an $C_i\, c$–Lipschitz function where $C_i = \sup\{\rho_i(x) : x \in X\}$ and c is the quasigeodesic constant of X. Consider first the case where x, $y \in G$. By symmetry we may assume $u_i(y) \geq u_i(x)$. Let $\varepsilon > 0$ and choose a path γ_{xy} joining x to y with $\ell(\gamma_{xy}) \leq c\, d(x, y)$. By the definition of $u_i(x)$ there is a path γ_x from $X \setminus G$ to x such that

$$u_i(x) > \int_{\gamma_x} \rho_i \, ds - \varepsilon.$$

Joining the paths γ_x and γ_{xy} together we obtain a path γ from $X \setminus G$ to y and now

$$u_i(y) - u_i(x) \leq \int_\gamma \rho_i \, ds - \int_{\gamma_x} \rho_i \, ds + \varepsilon \leq \int_{\gamma_{xy}} \rho_i \, ds + \varepsilon \leq C_i\, c\, d(x, y) + \varepsilon$$

and letting $\varepsilon \to 0$ we obtain the required Lipschitz bound for u_i. If $x \in X \setminus G$ and $y \in G$, then $u_i(x) = 0$ and choosing γ_{xy} as before we have

$$u_i(y) - u_i(x) = u_i(y) \leq \int_{\gamma_{xy}} \rho_i \, ds \leq C_i\, c\, d(x, y).$$

For y, $x \in X \setminus G$ the inequality is trivial.

The function ρ_i is an upper gradient of u_i. If γ is a path joining y and x which lie in G, then by symmetry we can assume that $u_i(y) \geq u_i(x)$ and for $\varepsilon > 0$ we can choose a path γ_x joining $X \setminus G$ to x such that

$$u_i(x) > \int_{\gamma_x} \rho_i \, ds - \varepsilon$$

and joining the paths γ_x and γ together we obtain the path $\tilde{\gamma}$ joining $X \setminus G$ to y. Thus

$$u_i(y) - u_i(x) \le \int_{\tilde{\gamma}} \rho_i \, ds - \int_{\gamma_x} \rho_i \, ds + \varepsilon = \int_{\gamma} \rho_i \, ds + \varepsilon$$

and letting $\varepsilon \to 0$ we obtain the required inequality. If $y \in X \setminus G$ and $x \in G$ and γ is a path joining y to x, then

$$u_i(y) - u_i(x) = u_i(y) \le \int_{\gamma} \rho_i \, ds.$$

The case x, $y \in X \setminus G$ is again trivial.

Next let $\varepsilon > 0$ and

$$E_\varepsilon = \bigl\{ x \in E : \lim_{i \to \infty} u_i(x) < 1 - \varepsilon \bigr\}.$$

Fix $x \in E_\varepsilon$. Then there is a sequence of paths γ_i from $X \setminus G$ to x such that for each i

$$1 - \varepsilon > u_i(x) \ge \int_{\gamma_i} \rho_i \, ds - 2^{-i} \varepsilon. \tag{3.4}$$

Now for $j \ge i$

$$u_i(x) \le u_j(x) \le \int_{\gamma_j} \rho_j \, ds.$$

Let

$$L = \liminf_{i \to \infty} \ell(\gamma_i)$$

and reparameterize the paths γ_i as $\tilde{\gamma}_i(t) = \gamma_i(t \ell(\gamma_i))$, $t \in [0, 1]$. Since we may assume that each γ_i meets $X \setminus G$ at $\gamma_i(0)$ only,

$$\ell(\gamma_i) \le \int_{\gamma_i} \frac{\rho_i}{\tau} \, ds \le \frac{1 - \varepsilon}{\tau} < \frac{1}{\tau}$$

and so the curves $\tilde{\gamma}_i$ are $\ell(\gamma_i)$–Lipschitz and uniformly $1/\tau$–Lipschitz and thus an equicontinuous family of mappings from $[0, 1]$ to the compact space $\overline{G}$. By the Ascoli theorem there is a subsequence of $(\tilde{\gamma}_i)$, denoted again by $(\tilde{\gamma}_i)$, which converges uniformly to a $1/\tau$–Lipschitz curve $\tilde{\gamma} : [0, 1] \to \overline{G}$. Clearly $\tilde{\gamma}(0) \in X \setminus G$ and $\tilde{\gamma}(1) = x$ and by the continuity of ρ_j in $\overline{G}$

$$\lim_{i \to \infty} \rho_j(\tilde{\gamma}_i(t)) = \rho_j(\tilde{\gamma}(t))$$

for each j and $t \in [0, 1]$.

Next let γ be the reparametrization of $\tilde{\gamma}$ by arch length and note that

$$\ell(\gamma) \le \liminf_{i \to \infty} \ell(\gamma_i) = L.$$

Denote by

$$s_i(t) = \ell(\tilde{\gamma}_i|[0,t]),\ t \in [0,1]$$

the length function of $\tilde{\gamma}_i$ and by

$$s(t) = \ell(\tilde{\gamma}|[0,t]),\ t \in [0,1]$$

the length function of $\tilde{\gamma}$. Now $s_i^{'}(t) = \ell(\gamma_i)$ for $t \in (0,1)$ and at the point $t_0 \in (0,1)$ of the differentiability of s we have for $t_0 < t_1 \leq 1$

$$\frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{\ell(\tilde{\gamma}|[t_0,t_1])}{t_1 - t_0} \leq \liminf_{i\to\infty} \frac{\ell(\tilde{\gamma}_i|[t_0,t_1])}{t_1 - t_0}$$

$$= \frac{1}{t_1 - t_0} \liminf_{i\to\infty} \int_{t_0}^{t_1} s_i^{'}(t)\,dt = \liminf_{i\to\infty} \frac{\ell(\gamma_i)(t_1 - t_0)}{t_1 - t_0} = L$$

and hence $s^{'}(t) \leq L$ for a.e. $t \in [0,1]$. For $j \geq i$ we have

$$\int_{\tilde{\gamma}_j} \rho_j\,ds \geq \int_{\tilde{\gamma}_j} \rho_i\,ds = \int_0^1 \rho_i(\tilde{\gamma}_j(t))s_j^{'}(t)\,dt = \ell(\gamma_j)\int_0^1 \rho_i(\tilde{\gamma}_j(t))\,dt. \quad (3.5)$$

Note that the function s is absolutely continuous because $\tilde{\gamma}$ is a Lipschitz curve. Now (3.4), (3.5) and the continuity of ρ_i yield for each i

$$1 - \varepsilon \geq \liminf_{j\to\infty} \int_{\tilde{\gamma}_j} \rho_j\,ds \geq L\int_0^1 \rho_i(\tilde{\gamma}(t))\,dt$$

$$\geq \int_0^1 \rho_i(\tilde{\gamma}(t))s^{'}(t)\,dt = \int_\gamma \rho_i\,ds.$$

This leads to contradiction since by the Lebesgue increasing convergence theorem for every path $\gamma \in \Gamma(E,G)$

$$\lim_{i\to\infty} \int_\gamma \rho_i\,ds = \int_\gamma \rho\,ds \geq \int_\gamma \tilde{\rho}\,ds \geq 1$$

and thus

$$\lim_{i\to\infty} u_i(x) \geq 1 - \varepsilon$$

for each $x \in E$.

Now $(u_i/(1-\varepsilon))$ is an admissible sequence for $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E,G)$ and by the Lebesgue bounded convergence theorem

$$\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E,G) \leq \liminf_{i\to\infty} \int_G \Big(\frac{\rho_i}{1-\varepsilon}\Big)^p d\mu = \int_G \Big(\frac{\tilde{\rho}+\tau}{1-\varepsilon}\Big)^p d\mu$$

and since $\tilde{\rho}$ is an arbitrary M–admissible function for $\Gamma(E,G)$ letting $\varepsilon \to 0$ and $\tau \to 0$ we complete the proof for the right inequality of (3.2).

Because $AM_p(\Gamma(E,G)) = M_p(\Gamma(E,G))$ for $p > 1$, the equality case in (3.2) is clear. □

Lemma 3.1. *If $E \subset G$ is an arbitrary set and $p \geq 1$, then there is a Borel set $E' \supset E$ such that*

$$\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E', G) = \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E, G).$$

Proof. If $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E, G) = \infty$ we can choose $E' = G$ and then the monotonicity of $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$ implies

$$\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(G, G) \geq \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E, G) = \infty.$$

Suppose that $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E, G) < \infty$ and for each $j \in \mathbb{N}$ choose an $Ad(E, G)$ admissible sequence (u_i^j) such that

$$\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E, G) \geq \liminf_{i \to \infty} \int_G (g_i^j)^p \, d\mu - 1/j \tag{3.6}$$

where g_i^j is an upper gradient of u_i^j.

Now the set

$$F^j = \left\{ x \in G : \liminf_{i \to \infty} u_i^j(x) \geq 1 \right\}$$

is a Borel set and $F^j \supset E$. The set $E' = \cap_j F^j$ is a Borel set and contains E and thus $Cap_p(E', G) \geq Cap_p(E, G)$. Using (3.6) we obtain the converse inequality

$$\begin{aligned} Cap_p(E', G) &\leq \liminf_{j \to \infty} \liminf_{i \to \infty} \int_G (g_i^j)^p \, d\mu \\ &\leq \liminf_{j \to \infty} (\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E, G) + 1/j) = \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E, G). \end{aligned}$$

□

From Theorem 3.1 and Lemma 3.1 we obtain:

Corollary 3.1. *If $p > 1$ then for each set $E \subset G$ there is a Borel set $E' \supset E$ such that $M_p(\Gamma(E, G)) = M_p(\Gamma(E', G))$.*

Since for $p > 1$, $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E, G) = M_p(\Gamma(E, G))$ for all sets $E \subset G$, the $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E, G)$–capacity inherits all the properties of the M_p–modulus in a quasiconvex space X. For the properties of the Choquet capacity see Section 4.

Theorem 3.2. *The $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$–capacity, $p > 1$, has the following properties:*
(a) $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$ is subadditive. i.e. if $E_i \subset G$, $i = 1, 2, \ldots$, then

$$\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(\bigcup_i E_i, G) \leq \sum_i \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E_i, G).$$

(b) If $K_1 \supset K_2 \supset \dots$ are compact sets in G, then

$$\lim_{i\to\infty} \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(K_i, G) = \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(\bigcap_i K_i, G).$$

(c) $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$ is a Choquet capacity, i.e. for a Suslin set $E \subset G$,

$$\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E, G) = \sup \big\{ \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(K, G) : K \subset E \text{ compact} \big\}.$$

Proof. The subadditivity of the M_p–modulus is well known, see Lemma 2.1, and hence Theorem 3.1 implies (a). For (b) let $K = \cap_i K_i$ and note that by the monotonicity

$$\lim_{i\to\infty} \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(K_i, G) \geq \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(K, G).$$

For the reverse inequality let $\varepsilon > 0$ and choose a sequence $(u_i) \in Ad(K, G)$ such that

$$\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(K, G) \geq \liminf_{i\to\infty} \int_G g_i^p \, d\mu - \varepsilon. \tag{3.7}$$

The function $u = \lim_i u_i$ is lower semicontinuous in G as a limit of an increasing sequence of continuous functions u_i. Thus the set $U = \{x \in G : u(x) > 1 - \varepsilon\}$ is open and contains K. Now there is i_0 such that $K_i \subset U$ for $i \geq i_0$ and thus $(u_i/(1-\varepsilon)) \in Ad(K_i, G]$ for $i \geq i_0$. By (3.7)

$$\lim_{i\to\infty} \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(K_i, G) \leq \liminf_{i\to\infty} \int_G \big(\frac{g_i}{1-\varepsilon}\big)^p \, d\mu \leq \frac{\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(K, G) + \varepsilon}{(1-\varepsilon)^p}$$

and letting $\varepsilon \to 0$ we obtain (b).

The map $E \mapsto \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E, G)$ is monotone and satisfies (a) and (b). Hence by the Choquet capacitibility theorem, see [4], it satisfies (c). □

4. Dirichlet capacity

In this section we compare the $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$–capacity to the classical Dirichlet capacity cap_p and we first recall its definition and basic properties due to G. Choquet. Originally this capacity used C_0^∞-functions in $\mathbb{R}^n$ and their gradients but the upper gradients for Lipschitz functions work as well in a metric measure space X, see [2, Section 6.3].

We again assume that X is a proper quasigeodesic space and $G \subset X$ is a fixed bounded open set.

Let K be a compact subset of G and $Ad_C(K, G)$ the family of all Lipschitz functions such that $u \geq 1$ in K and $u = 0$ in $X \setminus G$. Define

$$\operatorname{cap}_p(K,G) = \inf\Big\{ \int_G g^p\, d\mu :\, u \in Ad_C(K,G),$$
$$g \text{ an upper gradient of } u\Big\},$$

Obviously the infimum does not change if restricted to test functions satisfying $0 \le u \le 1$. The condition that a test function $u \in Ad_C(K,G)$ satisfies $u = 0$ in $X \setminus G$ can be replaced, due to the continuity of u, by the requirement that u has compact support in G.

If $U \subset G$ is open, then we set

$$\operatorname{cap}_p(U,G) = \sup\{\operatorname{cap}_p(K,G) :\, K \subset U \text{ compact}\}$$

and for an arbitrary set $E \subset G$

$$\operatorname{cap}_p(E,G) = \inf\{\operatorname{cap}_p(U,G) :\, U \text{ open}, E \subset U \subset G\}.$$

Now there are two definitions for $\operatorname{cap}_p(E,G)$ when E is compact but since the competitors are continuous the both definitions give the same value.

The cap_p–capacity, $p \ge 1$, has the following properties:
(i) monotonicity: $E_1 \subset E_2 \Longrightarrow \operatorname{cap}(E_1,G) \le \operatorname{cap}_p(E_2,G)$.
(ii) subadditivity: $E_1 \subset E_2 \subset \ldots \Longrightarrow \lim_i \operatorname{cap}_p(E_i,G) = \operatorname{cap}_p(\bigcup_i E_i)$.
(iii) $K_1 \supset K_2 \supset \ldots$ compact $\Longrightarrow \lim_i \operatorname{cap}_p(K_i,G) = \operatorname{cap}_p(\bigcap_i K_i,G)$.
By the Choquet capacibility theorem for all Suslin sets $E \subset G$

$$\operatorname{cap}_p(E,G) = \sup\big\{\operatorname{cap}_p(K,G) :\, K \subset E \text{ compact}\big\}.$$

For the Choquet theory see [4] and [3] and for the Dirichlet capacity in X, [2, Section 6.3].

We frequently use the following lemma:

Lemma 4.1. *For $p \ge 1$ and $K \subset G$ compact the equality*

$$\operatorname{cap}_p(K,G) = M_p(\Gamma(K,G)) = AM_p(\Gamma(K,G)) = \mathrm{Cap}_\mathrm{p}^\mathrm{M}(K,G) \tag{4.1}$$

holds.

Proof. For $p = 1$ by [11, Lemma 3.3]

$$\operatorname{cap}_1(K,G) = M_1(\Gamma(K,G)) = AM_1(\Gamma(K,G))$$

and thus by Theorem 3.1 equality holds in (4.1). For $p > 1$, (4.1) is well known, see e.g. [2, Chapter 5]. and [1]. Note that for $p > 1$, $M_p(\Gamma) = AM_p(\Gamma)$ for every path family Γ in X, see [11, Lemma 3.3]. □

Lemma 4.2. *If $E \subset G$ is an arbitrary set and $p \geq 1$, then*

$$\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E, G) \geq \mathrm{cap}_p(E, G). \tag{4.2}$$

Proof. Choose a Borel set $E' \supset E$ such that $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E, G) = \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E', G)$ and now by Lemma 4.1 and the Choquet capacibility theorem

$$\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E, G) = \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E', G) \geq \sup_{\{K \subset E' \text{ compact}\}} \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(K, G)$$

$$= \sup_{\{K \subset E' \text{ compact}\}} \mathrm{cap}_p(K, G) = \mathrm{cap}_p(E', G) \geq \mathrm{cap}_p(E, G).$$

□

Lemma 4.3. *Suppose that E is an arbitrary subset of G. For $p > 1$*

$$AM_p(\Gamma(E, G)) = M_p(\Gamma(E, G)) \leq \mathrm{cap}_p(E, G) \tag{4.3}$$

and for $p = 1$

$$AM_1(\Gamma(E, G)) \leq \mathrm{cap}_1(E, G). \tag{4.4}$$

Proof. The first equality in (4.3) is due to the fact that $AM_p = M_p$ for $p > 1$. For the inequality in (4.3) let $U \subset G$ be an open set with $U \supset E$ and choose compact sets $K_1 \subset K_2 \subset ... \subset U$ such that $\cup_i K_i = U$. Now $\Gamma(K_i, G) \subset \Gamma(K_{i+1}, G)$ and

$$\bigcup_i \Gamma(K_i, G) = \Gamma(U, G)$$

and since $p > 1$ we have $M_p(\Gamma(U, G)) = \lim_{i \to \infty} M_p(\Gamma(K_i, G))$, see [1], and then from (4.1) it follows

$$M_p(\Gamma(U, G)) = \lim_{i \to \infty} M_p(\Gamma(K_i, G)) = \lim_{i \to \infty} \mathrm{cap}_p(K_i, G) = \mathrm{cap}_p(U, G).$$

Since this holds for all open sets $U \supset E$

$$M_p(\Gamma(E, G)) \leq \inf_{G \supset U \supset E} M_p(\Gamma(U, G))$$

$$= \inf_{G \supset U \supset E} \mathrm{cap}_p(\Gamma(U, G)) = \mathrm{cap}_p(E, G).$$

For (4.4) we can proceed as above but now by [8, Lemma 3.11] we have

$$\mathrm{cap}_1(U, G) = \lim_{i \to \infty} \mathrm{cap}_1(K_i, G) = \lim_{i \to \infty} M_1(\Gamma(K_i, G)) \geq AM_1(\Gamma(U, G))$$

and hence for every open set $U \supset E$

$$\mathrm{cap}_1(U, G) \geq AM_1(\Gamma(U, G)) \geq AM_1(\Gamma(E, G)).$$

□

The following summarizes the situation for Suslin sets for $p = 1$.

Lemma 4.4. *If $E \subset G$ is a Suslin set, then*

$$\operatorname{cap}_1(E,G) = AM_1(\Gamma(E,G)) \le \operatorname{Cap}_1^{\mathrm{M}}(E,G) \le M_1(\Gamma(E,G)). \qquad (4.5)$$

Proof. To prove the first equality in (4.5) it suffices to show, by (4.4), that

$$\operatorname{cap}_1(E,G) \le AM_1(\Gamma(E,G)). \qquad (4.6)$$

Since E is a Suslin set, the Choquet capacibility theorem yields

$$\operatorname{cap}_1(E,G) = \sup \left\{ \operatorname{cap}_1(K,G) : \, K \subset E \text{ compact} \right\}$$

and for each compact set $K \subset E$

$$\operatorname{cap}_1(K,G) = AM_1(\Gamma(K,G)) \le AM_1(\Gamma(E,G))$$

and (4.6) follows. The rest follows from Theorem 3.1 and Lemma 4.2. □

The following theorem summarizes the situation for $p > 1$ and $p = 1$, respectively.

Theorem 4.1. *If X is a proper quasigeodesic metric space, $G \subset X$ a bounded open set and $E \subset G$ an arbitrary set, then for $p > 1$*

$$AM_p(\Gamma(E,G)) = \operatorname{cap}_p(E,G) = \operatorname{Cap}_{\mathrm{p}}^{\mathrm{M}}(E,G) = M_p(\Gamma(E,G)) \qquad (4.7)$$

and for $p = 1$

$$AM_1(\Gamma(E,G)) \le \operatorname{cap}_1(E,G) \le \operatorname{Cap}_1^{\mathrm{M}}(E,G) \le M_1(\Gamma(E,G)). \qquad (4.8)$$

Proof. Since $AM_p(\Gamma(E,G)) = M_p(\Gamma(E,G))$ for $p > 1$ the proof for (4.7) follows from Theorem 3.1, Lemmata 4.2 and 4.3. The inequalities in (4.8) follow from Theorem 3.1, (4.4) and (4.2). □

References

[1] Ambrosio, L., Di Marino, S., Savaré, G. (2015). On the duality between p-modulus and probability measures. *J. Eur. Math. Soc., 17*, 1817–1853.

[2] Björn, A., Björn, J. (2011). *Nonlinear potential theory on metric spaces.* EMS Tracts Math., vol. 17. European Mathematical Society (EMS), Zurich.

[3] Brelot, M. (1967). *Lectures on potential theory.* Notes by K. N. Gowrisankaran and M. K. Venkatesha Murthy, second edition, revised and enlarged with the help of S. Ramaswamy, Tata Institute of Fundamental Research Lectures on Mathematics, N 19 Tata Institute of Fundamental Research, Bombay.

[4] Choquet, G. (1953–54). Theory of capacities. *Ann. Inst. Fourier, 5*, 131–295.

[5] Folland, G.B. (1999). *Real analysis.* 2nd ed., Wiley, New York.

[6] Fuglede, B. (1957). Extremal length and functional completion. *Acta Math., 98*, 171–219.

[7] Heinonen, J., Koskela, P., Shanmugalingam, N., Tyson, J. (2015). *Sobolev spaces on metric measure spaces.* Cambridge University Press.

[8] Honzlová–Exnerová, V., Kalenda, O., Malý, J., Martio, O. Plans on measures and *AM*–modulus. *J. Funct. Anal.*, preprint arXiv 1904.04527 (to appear).

[9] Honzlová–Exnerová, V., Malý, J., Martio, O. (2017). Modulus in Banach function spaces. *Ark. Mat., 55*(1), 105–130.

[10] Honzlová–Exnerová, V., Malý, J., Martio, O. (2018). Functions of bounded variation and the *AM*–modulus in $\mathbb{R}^n$. *Nonlinear Anal., 177*, 553–571.

[11] Honzlová–Exnerová, V., Kalenda, O., Malý, J., Martio, O. *AM*–modulus and Hausdorff measure of codimension one in metric measure spaces. *Math. Nachr.* (to appear).

[12] Martio, O. (2016). Functions of bounded variation and curves in metric measure spaces. *Adv. Calc. Var., 9*(4), 305–322.

[13] Shanmugalingam, N. (2000). Newtonian spaces: An extension of Sobolev spaces to metric measure spaces. *Rev. Mat. Iberoam., 16*, 243–279.

Contact information

Olli Martio

Department of Mathematics and Statistics,
FI-00014 University of Helsinki, Finland
E-Mail: olli.martio@helsinki.fi