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Composition operators on Hardy-Sobolev spaces and BMO-quasiconformal mappings

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Dedicated to 80th anniversary of Professor Vladimir Gutlyanskii

Abstract. In this paper we consider composition operators on Hardy-Sobolev spaces in connections with BMO-quasiconformal mappings. Using the duality of Hardy spaces and BMO-spaces we prove that BMO-quasiconformal mappings generate bounded composition operators from Hardy–Sobolev spaces to Sobolev spaces.

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1. Introduction

Composition operators on Sobolev spaces arise in the work by V. Maz'ya [38] in connection with the isoperimetric problem as operators generated by sub-areal mappings. In this pioneering work it was established a connection between geometrical properties of mappings and the corresponding Sobolev spaces. In the present paper we consider composition operators on Hardy–Sobolev spaces generated by BMO-quasiconformal mappings. The main result of the article states:

Let Hardy–Sobolev spaces $H^{1,n}_r(\Omega)$ are defined in Lipschitz bounded domains in $\Omega \subset \mathbb{R}^n$, Sobolev spaces $L^{1,n}(\widetilde{\Omega})$ are defined in bounded domains in $\widetilde{\Omega} \subset \mathbb{R}^n$ and $\varphi : \Omega \to \widetilde{\Omega}$ is a BMO-quasiconformal mapping. Then the inequality

$$\|f \circ \varphi^{-1} \mid L^{1,n}(\widetilde{\Omega})\| \leq \|Q \mid \mathrm{BMO}_z(\Omega)\|^{\frac{1}{n}} \|f \mid H^{1,n}_r(\Omega)\|,$$

where a measurable function $Q : \Omega \to \mathbb{R}$ be such that a quasiconformal distortion $K(\varphi) \leq Q$ a. e. in Ω, holds for any Lipschitz function $f \in \mathrm{Lip}(\Omega)$.

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BMO-quasiconformal mappings generalize the notion of quasiconformal mappings, because K-quasiconformal mappings are BMO-quasiconformal mappings with $Q := K \in \mathrm{BMO}(\Omega)$ [37]. Composition operators on Sobolev spaces in connections with quasiconformal mappings were considered in [54] in the frameworks of Reshetnyak's problem (1968). Note that this problem arises to quasiconformal mappings and Royden algebras [33, 43]. In [54] it was proved that a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$, where Ω, $\widetilde{\Omega}$ are domains in $\mathbb{R}^n$, generates by the composition rule $\varphi^*(f) = f \circ \varphi$ the bounded operator on Sobolev spaces

$$\varphi^* : L^{1,n}(\widetilde{\Omega}) \to L^{1,n}(\Omega),$$

if and only if φ is a quasiconformal mapping. In the case of Sobolev spaces $L^{1,p}(\widetilde{\Omega})$ and $L^{1,p}(\Omega)$, $p \neq n$, the analytic description was obtained in [52] using a notion of mappings of finite distortion introduced in [55]: a weakly differentiable mapping is called a mapping of finite distortion if $|D\varphi(x)| = 0$ a. e. on the set $Z = \{x \in \Omega : J(x, \varphi) = 0\}$. In [15] characterizations of composition operators in geometric terms for $n-1 < p < \infty$ were obtained.

The case of Sobolev spaces $L^{1,p}(\widetilde{\Omega})$ and $L^{1,q}(\Omega)$, $q < p$, is more complicated and in this case the composition operators theory is based on the countable-additive set functions, which are associated with norms of composition operators and were introduced in [50] (see also [56]). The main result of [50] gives analytic and capacitary characterizations of composition operators on Sobolev spaces (see, also [56]) in terms of mappings of finite distortion [23, 55]. Multipliers theory has been applied to the change of variable problem in Sobolev spaces in [40].

In the last decade the composition operators theory has been considered on some generalizations of Sobolev spaces, such as Besov spaces and Triebel–Lizorkin spaces, [22, 24, 25, 32, 44]. These types of composition operators have applications to the Calderón inverse conductivity problem [2]. Composition operators on Sobolev spaces over Banach function spaces (such as Orlicz, Lorentz, variable exponents etc.) have been considered in [26–30, 41, 42].

Remark that composition operators on Sobolev spaces have significant applications to the Sobolev embedding theory [14, 17] and to the spectral theory of elliptic operators, see, for example, [16, 19, 20]. In some cases the composition operators method allows one to obtain better estimates than the classical L. E. Payne and H. F. Weinberger estimates in convex domains [45].

The notion of Q-mappings was introduced in [34] (see also [35–37]). Recall that a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ of domains $\Omega, \widetilde{\Omega} \subset \mathbb{R}^n$ is called a Q-homeomorphism with a non-negative measurable function Q, if

$$M\left(\varphi\Gamma\right) \leqslant \int\limits_{\Omega} Q(x)\cdot\rho^{n}(x)dx$$

for every family Γ of rectifiable paths in Ω and every admissible function ρ for Γ.

The Q-mappings with a function Q belongs to the A_n-Muckenhoupt class are inverse to homeomorphisms generating bounded composition operators on the weighted Sobolev spaces [51] (see, also [53]). In the case $Q\in \mathrm{BMO}(\Omega)$ we have a class of BMO-quasiconformal mappings [37,46]. Note that BMO-quasiconformal mappings have significant applications in the Beltrami equation theory [5].

The aim of the present article is to study Q-mappings with $Q\in\mathrm{BMO}$ in connection with composition operators on Sobolev-type spaces. This leads us to consider composition operators on Hardy–Sobolev spaces.

The theory of Hardy spaces on the Euclidean space $\mathbb{R}^n$, arise in the work by E. M. Stein and G. Weiss in [49]. Later, C. Fefferman and E. M. Stein [4] systematically developed the real-variable theory for Hardy spaces $H^p(\mathbb{R}^n)$ with $p\in(0,1]$, which plays an important role in various fields of analysis (see, for example, [47]). Hardy spaces and BMO-spaces on domains of $\mathbb{R}^n$ were considered in [6, 7]. The current state of the art and references to applications of Hardy spaces on domains of $\mathbb{R}^n$ the reader will find in [13]. Composition operators on Hardy and Hardy–Sobolev spaces of analytic functions have been intensively studied for a long time and can be found, for example in [10, 48].

2. Hardy–Sobolev spaces

2.1. Sobolev spaces

Let E be a measurable subset of $\mathbb{R}^n$, $n\geq 2$. The Lebesgue space $L^p(E)$, $1\leq p<\infty$, is defined as a Banach space of p-summable functions $f: E\to\mathbb{R}$ equipped with the following norm:

$$\|f\mid L^p(E)\| = \Bigg(\int\limits_E |f(x)|^p\,dx\Bigg)^{\frac{1}{p}},\ \ 1\leq p<\infty.$$

If Ω is an open subset of $\mathbb{R}^n$, the Sobolev space $W^{1,p}(\Omega)$, $1\leq p<\infty$, is defined [39] as a Banach space of locally integrable weakly differentiable functions $f:\Omega\to\mathbb{R}$ equipped with the following norm:

$$\|f\mid W^{1,p}(\Omega)\| = \|f\mid L^p(\Omega)\| + \|\nabla f\mid L^p(\Omega)\|,$$

where ∇f is the weak gradient of the function f, i. e. $\nabla f = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$.

The homogeneous seminormed Sobolev space $L^{1,p}(\Omega)$, $1 \leq p < \infty$, is defined as a space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$\|f \mid L^{1,p}(\Omega)\| = \|\nabla f \mid L^p(\Omega)\|.$$

2.2. Hardy and Hardy–Sobolev spaces

Let us recall the classical definition of Hardy spaces $H^1(\mathbb{R}^n)$ [47]. Let Φ be a function belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \Phi(x)\, dx = 1$. For all $t > 0$, define $\Phi_t(x) = t^{-n}\Phi(x/t)$ and the vertical maximal function

$$\mathcal{M}f(x) = \sup_{t>0} |\Phi_t * f(x)|.$$

Let a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then f is said to be in $H^1(\mathbb{R}^n)$ if $\mathcal{M}f \in L^1(\mathbb{R}^n)$. The Hardy space $H^1(\mathbb{R}^n)$ is equipped with the norm

$$\|f \mid H^1(\mathbb{R}^n)\| := \|\mathcal{M}f \mid L^1(\mathbb{R}^n)\|.$$

There are several definitions of Hardy spaces [6, 7, 12] and Hardy–Sobolev spaces on domains $\Omega \subset \mathbb{R}^n$ (see, e.g. [1, 13]). Following [1] we define two type of Hardy spaces on Lipschitz domains in $\mathbb{R}^n$. The Hardy space $H^1_z(\Omega)$ is defined as a space of functions $f \in H^1(\mathbb{R}^n)$, such that $\operatorname{supp} f \subset \overline{\Omega}$. Endowed with the norm

$$\|f \mid H^1_z(\Omega)\| := \|f \mid H^1(\mathbb{R}^n)\|,$$

it is a Banach space.

The Hardy space $H^1_r(\Omega)$ is defined as a space of functions f which are restrictions to Ω of functions $F \in H^1(\mathbb{R}^n)$. If $f \in H^1_r(\Omega)$ then

$$\|f \mid H^1_r(\Omega)\| := \inf \|F \mid H^1(\mathbb{R}^n)\|,$$

where the infimum is taken over all functions $F \in H^1(\mathbb{R}^n)$ such that $F|_\Omega = f$. The space $H^1_r(\Omega)$ equipped with this norm is a Banach space. In [12], it was shown that $H^1_r(\Omega)$ can be define in terms of maximal function: $\|f \mid H^1_r(\Omega)\| = \|\mathcal{M}_\Omega f \mid L_1(\Omega)\|$,

$$\mathcal{M}_\Omega f(x) = \sup_{t \leq d(x,\partial\Omega)} |\Phi_t * f(x)|.$$

We define the Hardy–Sobolev space $HS^{1,p}_r(\Omega)$ $(HS^{1,p}_z)$, $1 \leq p < \infty$, as a space of weakly differentiable functions $f \in L^p(\Omega)$ such that $|\nabla f|^p \in H^1_r(\Omega)$ $(|\nabla f|^p \in H^1_z(\Omega))$ and equipped with the norms

$$\|f \mid HS^{1,p}_r(\Omega)\| := \|f \mid L^p(\Omega)\| + \||\nabla f|^p \mid H^1_r(\Omega)\|^{\frac{1}{p}},$$

$$\|f \mid HS_z^{1,p}(\Omega)\| := \|f \mid L^p(\Omega)\| + \||\nabla f|^p \mid H_z^1(\Omega)\|^{\frac{1}{p}}.$$

The homogeneous Hardy–Sobolev space $H_r^{1,p}(\Omega)$ ($H_z^{1,p}(\Omega)$), $1 \leq p < \infty$, we define as a space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorms:

$$\|f \mid H_r^{1,p}(\Omega)\| := \||\nabla f|^p \mid H_r^1(\Omega)\|^{\frac{1}{p}} \text{ and}$$

$$\|f \mid H_z^{1,p}(\Omega)\| := \||\nabla f|^p \mid H_z^1(\Omega)\|^{\frac{1}{p}}.$$

Let us prove that a function

$$\|\cdot\|_p : f \mapsto \||\nabla f|^p \mid H_r^1(\Omega)\|^{\frac{1}{p}}$$

is a seminorm (for the case of $H_z^1(\Omega)$ the proof is similar).

1. **Nonnegativity**:

$$\|f \mid H_r^{1,p}(\Omega)\| := \||\nabla f|^p \mid H_r^1(\Omega)\|^{\frac{1}{p}} \geq 0 \quad \text{for all } f \in H_r^{1,p}(\Omega).$$

2. **Absolute homogeneity**:

$$\begin{aligned}\|kf \mid H_r^{1,p}(\Omega)\| &:= \||k\nabla f|^p \mid H_r^1(\Omega)\|^{\frac{1}{p}} = \||k| \cdot |\nabla f|^p \mid H_r^1(\Omega)\|^{\frac{1}{p}} \\ &= |k|\||\nabla f|^p \mid H_r^1(\Omega)\|^{\frac{1}{p}} = |k|\|f \mid H_r^{1,p}(\Omega)\|\end{aligned}$$

for any $k \in \mathbb{R}$ and any $f \in H_r^{1,p}(\Omega)$.

3. **Triangle inequality**: Let functions $f, g \in H_r^{1,p}(\Omega)$. Then

$$\begin{aligned}&\|(f+g) \mid H_r^{1,p}(\Omega)\|^{\frac{1}{p}} = \||\nabla f + \nabla g|^p \mid H_r^1(\Omega)\|^{\frac{1}{p}} \\ &= \left(\int\limits_{\Omega} \sup_{t \leq d(x,\partial\Omega)} \left| \int\limits_{B(x,t)} |\nabla f(y) + \nabla g(y)|^p \Phi_t(x-t)\, dy \right| dx\right)^{\frac{1}{p}} \\ &\leq \left(\int\limits_{\Omega} \sup_{t \leq d(x,\partial\Omega)} \left| \int\limits_{B(x,t)} (|\nabla f(y)| + |\nabla g(y)|)^p \Phi_t(x-t)\, dy \right| dx\right)^{\frac{1}{p}}\end{aligned}$$

$$= \left(\int\limits_{\Omega} \sup_{t \leq d(x,\partial\Omega)} \left| \int\limits_{B(x,t)} \left((\Phi_t(x-t))^{\frac{1}{p}} |\nabla f(y)| + (\Phi_t(x-t))^{\frac{1}{p}} |\nabla g(y)|\right)^p dy \right| dx\right)^{\frac{1}{p}}.$$

Now, by using the Minkowski inequality, we have

$$\left(\int\limits_{\Omega}\sup_{t\leq d(x,\partial\Omega)}\left|\int\limits_{B(x,t)}\Big((\Phi_t(x-t))^{\frac{1}{p}}|\nabla f(y)|+(\Phi_t(x-t))^{\frac{1}{p}}|\nabla g(y)|\Big)^p\,dy\right|dx\right)^{\frac{1}{p}}$$

$$\leq\left(\int\limits_{\Omega}\sup_{t\leq d(x,\partial\Omega)}\left(\left|\int\limits_{B(x,t)}\Phi_t(x-t)|\nabla f(y)|^p\,dy\right|^{\frac{1}{p}}\right.\right.$$

$$\left.\left.+\left|\int\limits_{B(x,t)}\Phi_t(x-t)|\nabla g(y)|^p\,dy\right|^{\frac{1}{p}}\right)^p dx\right)^{\frac{1}{p}}$$

$$=\left(\int\limits_{\Omega}\left(\sup_{t\leq d(x,\partial\Omega)}\left|\int\limits_{B(x,t)}\Phi_t(x-t)|\nabla f(y)|^p\,dy\right|^{\frac{1}{p}}\right.\right.$$

$$\left.\left.+\sup_{t\leq d(x,\partial\Omega)}\left|\int\limits_{B(x,t)}\Phi_t(x-t)|\nabla g(y)|^p\,dy\right|^{\frac{1}{p}}\right)^p dx\right)^{\frac{1}{p}}.$$

Using the Minkowski inequality the second time, we obtain

$$\|(f+g)\mid H_r^{1,p}(\Omega)\|^{\frac{1}{p}}=\||\nabla f+\nabla g|^p\mid H_r^1(\Omega)\|^{\frac{1}{p}}$$

$$\leq\left(\int\limits_{\Omega}\left(\sup_{t\leq d(x,\partial\Omega)}\left|\int\limits_{B(x,t)}\Phi_t(x-t)|\nabla f(y)|^p\,dy\right|^{\frac{1}{p}}\right)^p dx\right)^{\frac{1}{p}}$$

$$+\left(\int\limits_{\Omega}\left(\sup_{t\leq d(x,\partial\Omega)}\left|\int\limits_{B(x,t)}\Phi_t(x-t)|\nabla g(y)|^p\,dy\right|^{\frac{1}{p}}\right)^p dx\right)^{\frac{1}{p}}=$$

$$= \left(\int\limits_{\Omega} \sup_{t \leq d(x,\partial\Omega)} \left| \int\limits_{B(x,t)} \Phi_t(x-t)|\nabla f(y)|^p \, dy \right| \, dx \right)^{\frac{1}{p}}$$

$$+ \left(\int\limits_{\Omega} \sup_{t \leq d(x,\partial\Omega)} \left| \int\limits_{B(x,t)} \Phi_t(x-t)|\nabla g(y)|^p \, dy \right| \, dx \right)^{\frac{1}{p}}$$

$$= \|\nabla f \mid H^1_r(\Omega)\|^{\frac{1}{p}} + \|\nabla g \mid H^1_r(\Omega)\|^{\frac{1}{p}}.$$

2.3. Duality of Hardy and BMO spaces

It is well-known, that dual to the Hardy space $H^1(\mathbb{R}^n)$ is a space $\mathrm{BMO}(\mathbb{R}^n)$, see, for example, [47]. Recall that a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$ is a function of bounded mean oscillation ($f \in \mathrm{BMO}(\mathbb{R}^n)$) [4] if

$$\|f \mid \mathrm{BMO}(\mathbb{R}^n)\| := \sup_B \frac{1}{B} \int_B |f(x) - f_B| \, dx < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ and $f_B = \frac{1}{B} \int_B f(x)dx$.

Since we consider the Hardy spaces defined on Lipschitz domains [1,7], we formulate the following version of duality (see [6, 8, 12].

Let Ω be a Lipschitz domain of $\mathbb{R}^n$. The space $\mathrm{BMO}_z(\Omega)$ is defined as being the space of all functions in $\mathrm{BMO}(\mathbb{R}^n)$ supported in $\overline{\Omega}$, equipped with the norm

$$\|f \mid \mathrm{BMO}_z(\Omega)\| := \|f \mid \mathrm{BMO}(\mathbb{R}^n)\|.$$

The dual of the space $H^1_r(\Omega)$ is the space $\mathrm{BMO}_z(\Omega)$.

The space $\mathrm{BMO}_r(\Omega)$ is defined as being the space of all restrictions to Ω of functions $\mathrm{BMO}(\mathbb{R}^n)$. It is equipped with the norm

$$\|f \mid \mathrm{BMO}_r(\Omega)\| := \inf \|F \mid \mathrm{BMO}(\mathbb{R}^n)\|,$$

where the infimum is taken over all functions $F \in \mathrm{BMO}(\mathbb{R}^n)$ such that $F|_\Omega = f$. In [9] it was shown, that $BMO_r(\Omega)$ can be described in another way, namely as a space of locally integrable function on Ω with

$$\|f \mid \mathrm{BMO}(\Omega)\| := \sup_Q \frac{1}{Q} \int_B |f(x) - f_Q| \, dx < \infty,$$

where the supremum is taken over all cubes $Q \subset \Omega$ with sides parallel to the axes. Then, the dual of the space $H^1_z(\Omega)$ is $\mathrm{BMO}_r(\Omega)$.

3. Q-quasiconformal mappings

3.1. Modulus and capacity

The theory of Q-quasiconformal mappings has been extensively developed in recent decades, see, for example, [37]. Let us give the basic definitions.

The linear integral is denoted by

$$\int_{\gamma} \rho\ ds = \sup \int_{\gamma'} \rho\ ds = \sup \int_{0}^{l(\gamma')} \rho(\gamma'(s))\ ds$$

where the supremum is taken over all closed parts γ' of γ and $l(\gamma')$ is the length of γ'. Let Γ be a family of curves in $\mathbb{R}^n$. Denote by $adm(\Gamma)$ the set of Borel functions (admissible functions) $\rho : \mathbb{R}^n \to [0, \infty]$ such that the inequality

$$\int_{\gamma} \rho\ ds \geqslant 1$$

holds for locally rectifiable curves $\gamma \in \Gamma$.

Let Γ be a family of curves in $\overline{\mathbb{R}^n}$, where $\overline{\mathbb{R}^n}$ is a one point compactification of the Euclidean space $\mathbb{R}^n$. The quantity

$$M(\Gamma) = \inf \int_{\mathbb{R}^n} \rho^n\ dx$$

is called the (conformal) module of the family of curves Γ [37]. The infimum is taken over all admissible functions $\rho \in adm(\Gamma)$.

Let Ω be a bounded domain in $\mathbb{R}^n$ and F_0, F_1 be disjoint non-empty compact sets in the closure of Ω. Let $M(\Gamma(F_0, F_1; \Omega))$ stand for the module of a family of curves which connect F_0 and F_1 in Ω. Then [37]

$$M(\Gamma(F_0, F_1; \Omega)) = \operatorname{cap}_n(F_0, F_1; \Omega)\,, \tag{3.1}$$

where $\operatorname{cap}_n(F_0, F_1; \Omega)$ is a conformal capacity of the condensor $(F_0, F_1; \Omega))$ [39].

Recall that a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ of domains $\Omega, \widetilde{\Omega} \subset \mathbb{R}^n$ is called a Q-homeomorphism [37], with a non-negative measurable function Q, if

$$M\left(\varphi\Gamma\right) \leqslant \int_{\Omega} Q(x) \cdot \rho^n(x) dx$$

for every family Γ of rectifiable paths in Ω and every admissible function ρ for Γ.

3.2. Mappings of finite distortion

Suppose a mapping $\varphi : \Omega \to \mathbb{R}^n$ belongs to the class $W^{1,1}_{\mathrm{loc}}(\Omega)$. Then the formal Jacobi matrix $D\varphi(x)$ and its determinant (Jacobian) $J(x,\varphi)$ are well defined at almost all points $x \in \Omega$. The norm $|D\varphi(x)|$ is the operator norm of $D\varphi(x)$, i. e., $|D\varphi(x)| = \max\{|D\varphi(x)\cdot h| : h \in \mathbb{R}^n, |h| = 1\}$. We also let $l(D\varphi(x)) = \min\{|D\varphi(x)\cdot h| : h \in \mathbb{R}^n, |h| = 1\}$.

Recall that a Sobolev mapping $\varphi : \Omega \to \mathbb{R}^n$ is the mapping of finite distortion if $D\varphi(x) = 0$ for almost all x from $Z = \{x \in \Omega : J(x,\varphi) = 0\}$ [55].

Let us define two p-distortion functions, $1 \leq p < \infty$, for Sobolev mappings of finite distortion $\varphi : \Omega \to \widetilde{\Omega}$.
The outer p-dilatation

$$K^O_p(x,\varphi) = \begin{cases} \frac{|D\varphi(x)|^p}{|J(x,\varphi)|}, & J(x,\varphi) \neq 0, \\ 0, & J(x,\varphi) = 0. \end{cases}$$

The inner p-dilatation

$$K^I_p(x,\varphi) = \begin{cases} \frac{|J(x,\varphi)|}{l(D\varphi(x))^p}, & J(x,\varphi) \neq 0, \\ 0, & J(x,\varphi) = 0. \end{cases}$$

Note that $K^I_n(x) \leq (K^O_n(x))^{n-1}$ and $K^O_n(x) \leq (K^I_n(x))^{n-1}$.

The maximal dilatation, or in short the dilatation, of φ at x is defined by

$$K_p(x) = K_p(x,\varphi) = \max(K^O_p(x,\varphi), K^I_p(x,\varphi)).$$

Let us recall the weak inverse theorem for Sobolev homeomorphisms [18] (see, also [11]).

Theorem 3.1. *Let $\varphi : \Omega \to \widetilde{\Omega}$, where Ω, $\widetilde{\Omega}$ are domains in $\mathbb{R}^n$, be a homeomorphism of finite distortion which belongs to the class $W^{1,p}_{\mathrm{loc}}(\Omega)$, $p \geq n-1$, and possesses the Luzin N-property (an image of a set of measure zero has measure zero). Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ be a mapping of finite distortion which belongs to the class $W^{1,1}_{\mathrm{loc}}(\widetilde{\Omega})$.*

Recall that homeomorphisms $\varphi : \Omega \to \widetilde{\Omega}$ of the class $W^{1,n}_{\mathrm{loc}}(\Omega)$ possess the Luzin N-property (an image of a set of measure zero has measure zero) [55].

4. BMO-quasiconformal mappings and composition operator

Given a function $Q : \Omega \to [1,\infty]$, a sense-preserving homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ is called to be Q-quasiconformal [34], if $\varphi \in W^{1,n}_{\mathrm{loc}}(\Omega)$ and

$K_n(x) \leq Q(x)$ for almost all $x \in \Omega$. If φ is Q-quasiconformal with $Q \in \mathrm{BMO}_r(\Omega)$, than φ is said to be a BMO-quasiconformal mapping. In [37], it was proven that every BMO-quasiconformal mapping is a Q-homeomorphism with some $Q \in \mathrm{BMO}_r$.

The first theorem represents a description of composition operators generated by BMO-quasiconformal homeomorphism.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz bounded domain, $\widetilde{\Omega} \subset \mathbb{R}^n$ be a bounded domain. Suppose there exists* BMO*-quasiconformal homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$. Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ generates by the composition rule $\left(\varphi^{-1}\right)^* = f \circ \varphi^{-1}$ a bounded composition operator*

$$\left(\varphi^{-1}\right)^* : H^{1,n}_z(\Omega) \cap \mathrm{Lip}(\Omega) \to L^{1,n}(\widetilde{\Omega}),$$

and the inequality

$$\|f \circ \varphi^{-1} \mid L^{1,n}(\widetilde{\Omega})\| \leq \|Q \mid \mathrm{BMO}_z(\Omega)\|^{\frac{1}{n}} \|f \mid H^{1,n}_z(\Omega)\|$$

holds for any Lipschitz function $f \in \mathrm{Lip}(\Omega)$.

Proof. Since $\varphi \in W^{1,n}_{\mathrm{loc}}(\Omega)$ then φ possesses the Luzin N-property, then the composition $f \circ \varphi^{-1}$ is well defined a. e. in $\widetilde{\Omega}$. Because $\varphi \in W^{1,n}_{\mathrm{loc}}(\Omega)$ and has a finite distortion, then $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ belongs to $W^{1,1}_{\mathrm{loc}}(\widetilde{\Omega})$ [18].

Now, let there be given a Lipschitz function $g \in H^{1,n}_z(\Omega)$. Then $g \circ \varphi^{-1}$ is weakly differentiable in $\widetilde{\Omega}$, and as long as φ has the Luzin N-property, the chain rule holds [23]. Hence

$$\|g \circ \varphi^{-1} \mid L^{1,n}(\widetilde{\Omega})\|^n = \int\limits_{\widetilde{\Omega}} |\nabla g \circ \varphi^{-1}(y)|^n \, dy$$

$$\leq \int_{\widetilde{\Omega}} |\nabla g|^n(\varphi^{-1}(y))|D\varphi^{-1}(y)|^n \, dy.$$

By the definition of BMO-quasiconformal mappings there exists measurable function $Q \in \mathrm{BMO}_r(\Omega)$, such that $K^I_n(x) \leq Q(x)$ for almost all $x \in \Omega$. Using the change of variables formula [3, 21], we obtain

$$\begin{aligned}\int_{\widetilde{\Omega}} &|\nabla g|^n(\varphi^{-1}(y))|D\varphi^{-1}(y)|^n dy \\ &= \int_{\Omega} |\nabla g|^n(x)|D\varphi^{-1}(\varphi(x))|^n |J(x,\varphi)| dx \\ &= \int_{\Omega} |\nabla g|^n(x) \frac{|J(x,\varphi)|}{l(D\varphi(x))^n} \, dx \leq \int_{\Omega} |\nabla g|^n(x) Q(x) \, dx.\end{aligned}$$

Now, by the duality of Hardy spaces H^1_z and BMO_r-spaces [6], we have

$$\int_\Omega |\nabla g|^n(x)Q(x)\ dx \leq \|Q \mid \mathrm{BMO}_r(\Omega)\| \cdot \|f \mid H^{1,n}_z(\Omega)\|^n.$$

Hence

$$\|f\circ\varphi^{-1} \mid L^{1,n}(\widetilde{\Omega})\| \leq \|Q \mid \mathrm{BMO}_r(\Omega)\|^{\frac{1}{n}} \|f \mid H^{1,n}_z(\Omega)\|$$

for any Lipschitz function $f \in H^{1,n}_z(\Omega)$. □

Let $\varphi : \Omega \to \widetilde{\Omega}$ be a homeomorphism. Then φ is called to be a BMO_p-quasiconformal mapping, if $\varphi \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and $K_p(x) \leq Q(x)$ for almost all $x \in \Omega$ and for some function $Q \in \mathrm{BMO}_r(\Omega)$.

In the case of BMO_p-quasiconformal mappings, we require the additional assumption of the Luzin N-property of a mapping φ in the case $n-1 \leq p < n$.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz bounded domain, $\widetilde{\Omega} \subset \mathbb{R}^n$ be a bounded domain. Suppose there exists BMO_p-quasiconformal homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$, $p \geq n-1$, which possesses the Luzin N-property, if $n-1 \leq p < n$. Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ generates by the composition rule $(\varphi^{-1})^* = f\circ\varphi^{-1}$ a bounded composition operator*

$$(\varphi^{-1})^* : H^{1,p}_z(\Omega) \cap \mathrm{Lip}(\Omega) \to L^{1,p}(\widetilde{\Omega}),$$

and the inequality

$$\|f\circ\varphi^{-1} \mid L^{1,p}(\widetilde{\Omega})\| \leq \|Q \mid \mathrm{BMO}_r(\Omega)\|^{\frac{1}{p}} \|f \mid H^{1,p}_z(\Omega)\|$$

holds for any Lipschitz function $f \in \mathrm{Lip}(\Omega)$.

Proof. Since φ possesses the Luzin N-property, then the composition $f\circ\varphi^{-1}$ is well defined a. e. in $\widetilde{\Omega}$. Because $\varphi \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $p \geq n-1$, has a finite distortion and possess the Luzin N-property, $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ belongs to $W^{1,1}_{\mathrm{loc}}(\widetilde{\Omega})$ [18].

Now, let there be given a Lipschitz function $g \in H^{1,p}_z(\Omega)$. Then $g\circ\varphi^{-1}$ is weakly differentiable in $\widetilde{\Omega}$, and as long as φ has the Luzin N-property, the chain rule holds [23]. Hence

$$\|g\circ\varphi^{-1} \mid L^{1,p}(\widetilde{\Omega})\|^p = \int\limits_{\widetilde{\Omega}} |\nabla g\circ\varphi^{-1}(y)|^p\ dy$$

$$\leq \int_{\widetilde{\Omega}} |\nabla g|^p(\varphi^{-1}(y))|D\varphi^{-1}(y)|^p\, dy.$$

By the definition of BMO-quasiconformal mappings there exists measurable function $Q \in \mathrm{BMO}_r(\Omega)$, such that $K_p^I(x) \leq Q(x)$ for almost all $x \in \Omega$. Using the change of variables formula [3, 21], we obtain

$$\int_{\widetilde{\Omega}} |\nabla g|^p(\varphi^{-1}(y))|D\varphi^{-1}(y)|^p dy \\ = \int_\Omega |\nabla g|^p(x)|D\varphi^{-1}(\varphi(x))|^p|J(x,\varphi)|dx \\ = \int_\Omega |\nabla g|^p(x)\frac{|J(x,\varphi)|}{l(D\varphi(x))^p}\, dx \leq \int_\Omega |\nabla g|^p(x)Q(x)\ dx.$$

Now, by the duality of Hardy spaces H_z^1 and BMO_r-spaces [6], we have

$$\int_\Omega |\nabla g|^p(x)Q(x)\ dx \leq \|Q \mid \mathrm{BMO}_r(\Omega)\| \cdot \|f \mid H_z^{1,p}(\Omega)\|^p.$$

Hence

$$\|f\circ\varphi^{-1} \mid L^{1,p}(\widetilde{\Omega})\| \leq \|Q \mid \mathrm{BMO}_r(\Omega)\|^{\frac{1}{p}} \|f \mid H_z^{1,p}(\Omega)\|$$

for any Lipschitz function $f \in H_z^{1,p}(\Omega)$. □

Using the duality between $H_r^1(\Omega)$ and $\mathrm{BMO}_z(\Omega)$ in the same manner we obtain the next two results:

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz bounded domain, $\widetilde{\Omega} \subset \mathbb{R}^n$ be a bounded domain. Suppose there exists* BMO*-quasiconformal homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ with $Q \in \mathrm{BMO}_z(\Omega)$. Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ generates by the composition rule $\left(\varphi^{-1}\right)^* = f\circ\varphi^{-1}$ a bounded composition operator*

$$\left(\varphi^{-1}\right)^* : H_r^{1,n}(\Omega) \cap \mathrm{Lip}(\Omega) \to L^{1,n}(\widetilde{\Omega}),$$

and the inequality

$$\|f\circ\varphi^{-1} \mid L^{1,n}(\widetilde{\Omega})\| \leq \|Q \mid \mathrm{BMO}_z(\Omega)\|^{\frac{1}{n}} \|f \mid H_r^{1,n}(\Omega)\|$$

holds for any Lipschitz function $f \in \mathrm{Lip}(\Omega)$.

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz bounded domain, $\widetilde{\Omega} \subset \mathbb{R}^n$ be a bounded domain. Suppose there exists BMO_p-quasiconformal homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$, $p \geq n-1$, with $Q \in \mathrm{BMO}_z(\Omega)$, which possesses the Luzin N-property, if $n-1 \leq p < n$. Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ generates by the composition rule $\left(\varphi^{-1}\right)^* = f\circ\varphi^{-1}$ a bounded composition operator*

$$\left(\varphi^{-1}\right)^* : H_r^{1,p}(\Omega) \cap \mathrm{Lip}(\Omega) \to L^{1,p}(\widetilde{\Omega}),$$

and the inequality

$$\|f \circ \varphi^{-1} \mid L^{1,p}(\widetilde{\Omega})\| \leq \|Q \mid \mathrm{BMO}_z(\Omega)\|^{\frac{1}{p}} \|f \mid H_r^{1,p}(\Omega)\|$$

holds for any Lipschitz function $f \in \mathrm{Lip}(\Omega)$.

We note the following regularity results also:

Theorem 4.5. *Given the mapping* $\varphi : \Omega \to \widetilde{\Omega}$.

1. *If the composition operator* $\varphi^* : H_r^{1,p}(\widetilde{\Omega}) \to L^{1,p}(\Omega)$ *is bounded, then* $\varphi \in L^{1,p}(\Omega)$.
2. *If the composition operator* $\varphi^* : H_r^{1,p}(\widetilde{\Omega}) \to H_r^{1,p}(\Omega)$ *is bounded, then* $\varphi \in H_r^{1,p}(\Omega)$.

Proof. We prove the theorem only for the first case. The second one is proved in a similar way.

Due to the boundedness of φ^*

$$\|f \circ \varphi \mid L^{1,p}(\Omega)\| \leq \|\varphi^*\| \|f \mid H_z^{1,p}(\widetilde{\Omega})\|.$$

Substitute the coordinate functions $f_j = y_j$, $j = 1, ..., n$, we obtain

$$\|f_j \mid H_r^{1,p}(\widetilde{\Omega})\| = \int_{\widetilde{\Omega}} \sup_{0<t\leq dist(x,\partial\widetilde{\Omega})} \left| \frac{1}{t^n} \int_{B(x,t)} \Phi(\frac{x-y}{t}) \right| dx$$
$$= \int_{\widetilde{\Omega}} \sup_{0<t\leq dist(x,\partial\widetilde{\Omega})} |1|\, dx = |\widetilde{\Omega}|.$$

Hence,

$$\|f_j \circ \varphi \mid L^{1,p}(\Omega)\| = \|\varphi_j \mid L^{1,p}(\Omega)\| \leq |\widetilde{\Omega}| \|\varphi^*\|.$$

□

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References

[1] Auscher, P., Russ, E., Tchamitchian, P. (2005). Hardy Sobolev spaces on strongly Lipschitz domains of $\mathbb{R}^n$. *Journal Funct. Anal., 218*, 54–109.

[2] Calderón, A.P. (1980). *On an inverse boundary value problem.* In: Seminar on Numerical Analysis and its Applications to Continuum Physics, Rio de Janeiro, 1980, 65–73.

[3] Federer, H. (1969). *Geometric measure theory.* Springer Verlag, Berlin.

[4] Fefferman, C., Stein, E.M. (1972). H^p spaces of several variables. *Acta Math., 129*, 137–193.

[5] Gutlyanskii, V., Ryazanov, V., Srebro, U., Yakubov, E. (2012). *The Beltrami Equation. A Geometric Approach.* Springer-Verlag, New York, NY.

[6] Chang, D.C. (1994). The dual of Hardy spaces on a bounded domain in $\mathbb{R}^n$. *Forum Math., 6*, 65–81.

[7] Chang, D.C., Krantz, S.G., Stein, E.M. (1993). H^p theory on a smooth domain in $\mathbb{R}^N$ and elliptic boundary value problems. *J. Funct. Anal., 114*, 286–347.

[8] Chang, D.-C., Dafni, G., Sadosky, C. (2005). *A Div-Curl Lemma in BMO on a Domain, Harmonic Analysis, Signal Processing, and Complexity.* Progress in Mathematics, vol 238. Birkhäuser, Boston, 55–65.

[9] Jones, P.W. (1980). Extension theorems for BMO. *Indiana Univ. Math. J., 29*, 41–66.

[10] Cowen, C., MaCluer, B. (1995). *Composition operators on spaces of analytic functions.* CRCPress, New York.

[11] Csörnyei, M., Hencl, S., Malý, J. (2010). Homeomorphisms in the Sobolev space $W^{1,n-1}$. *J. Reine Angew. Math., 644*, 221–235.

[12] Miyachi, A. (1990). H^p spaces over open subsets of $\mathbb{R}^n$. *Studia Math., 95*(3), 205–228.

[13] Chen, X., Jiang, R., Yang, D. (2016). Hardy and Hardy-Sobolev Spaces on Strongly Lipschitz Domains and Some Applications. *Anal. Geom. Metr. Spaces, 4*(1), 336–362.

[14] Gol'dshtein, V., Gurov, L. (1994). Applications of change of variables operators for exact embedding theorems. *Integral Equations Operator Theory, 19*, 1–24.

[15] Gol'dshtein, V., Gurov, L., Romanov, A. (1995). Homeomorphisms that induce monomorphisms of Sobolev spaces. *Israel J. Math., 91*, 31–60.

[16] Gol'dshtein, V., Pchelintsev, V., Ukhlov, A. (2018). On the First Eigenvalue of the Degenerate p-Laplace Operator in Non-convex Domains. *Integral Equations Operator Theory, 90* 43.

[17] Gol'dshtein, V., Ukhlov, A. (2009). Weighted Sobolev spaces and embedding theorems. *Trans. Amer. Math. Soc., 361*, 3829–3850.

[18] Gol'dshtein, V., Ukhlov, A. (2010). About homeomorphisms that induce composition operators on Sobolev spaces. *Complex Var. Elliptic Equ., 55*, 833–845.

[19] Gol'dshtein, V., Ukhlov, A. (2016). On the first Eigenvalues of Free Vibrating Membranes in Conformal Regular Domains. *Arch. Rational Mech. Anal., 221*(2), 893–915.

[20] Gol'dshtein, V., Ukhlov, A. (2017). The spectral estimates for the Neumann-Laplace operator in space domains. *Adv. in Math., 315*, 166–193.

[21] Hajlasz, P. (1993). Change of variables formula under minimal assumptions. *Colloq. Math., 64*, 93–101.

[22] Hencl, S., Koskela, P. (2013). Composition of quasiconformal mappings and functions in Triebel–Lizorkin spaces. *Math. Nachr., 286*, 669–678.

[23] Hencl, S., Koskela, P. (2014). *Lectures on Mappings of Finite Distortion.* Springer, Berlin/Heidelberg.

[24] Koch, H., Koskela, P., Saksman, E., Soto, T. (2014). Bounded compositions on scaling invariant Besov spaces. *J. Funct. Anal., 266*, 2765–2788.

[25] Koskela, P., Yang, D., Zhou, Y. (2011). Pointwise characterizations of Besov and Triebel-Lizorkin spaces and quasiconformal mappings. *Adv. Math., 226*, 3579–3621.

[26] Arora, S.C., Datt, G., Verma, S. (2009). Weighted composition operators on Orlicz–Sobolev spaces. *J. of the Australian Math. Soc., 83*(3), 327–334.

[27] Hencl, S., Kleprlik, L. (2012). Composition of q-quasiconformal mappings and function in Orlicz–Sobolev spaces. *Illinois J. Math., 56*(3), 931–955.

[28] Hencl, S., Kleprlik, L., Maly, J. (2014). Composition operator and Sobolev–Lorentz spaces $WL^{n,q}$. *Studia Mathematica, 221*, 197–208.

[29] Kleprlik, L. (2014). Composition operators on W^1X are necessarily induced by quasiconformal mappings. *Cent. Eur. J. Math., 12*(8), 1229–1238.

[30] Romanov, A.S. (2017). The composition operators in Sobolev spaces with variable exponent of summability. *Sib. Elektron. Mat. Izv., 14*, 794–806.

[31] Koskela, P., Saksman, E. (2008). Pointwise characterizations of Hardy-Sobolev functions. *Math. Res. Lett., 15*(4), 727–744.

[32] Koskela, P., Xiao, J., Ru-Ya Zhang, Yi., Zhou, Y. (2017). A quasiconformal composition problem for the Q-spaces. *J. Eur. Math. Soc. (JEMS), 19*, 1159–1187.

[33] Lewis, L.G. (1971). Quasiconformal mappings and Royden algebras in space. *Trans. Amer. Math. Soc., 158*, 481–492.

[34] Martio, O., Ryazanov, V., Srebro, U., Yakubov, E. (2001). To the theory of Q-homeomorphisms. *Dokl. Akad. Nauk Rossii, 381*, 20–22.

[35] Martio, O., Ryazanov, V., Srebro, U., Yakubov, E. (2004). Mappings with finite length distortion. *J. d'Anal. Math., 93*, 215–236.

[36] Martio, O., Ryazanov, V., Srebro, U., Yakubov, E. (2005). On Q-homeomorphisms. *Ann. Acad. Sci. Fenn. Math., 30*(1), 49–69.

[37] Martio, O., Ryazanov, V., Srebro, U., Yakubov, E. (2009). *Moduli in modern mapping theory.* Springer Monographs in Mathematics. Springer, New York.

[38] Maz'ya, V.G. (1969). Weak solutions of the Dirichlet and Neumann problems. *Trudy Moskov. Mat. Ob-va., 20*, 137–172.

[39] Maz'ya, V. (2010). *Sobolev spaces: with applications to elliptic partial differential equations.* Springer, Berlin/Heidelberg.

[40] Maz'ya, V., Shaposhbikovs, T.O. (1986). *Multipliers in Spaces of Differentiable Functions.* Leningrad Univ. Press.

[41] Menovshchikov, A.V. (2016). Composition operators in Orlicz-Sobolev spaces. *Siberian Math. J., 57*, 849–859.

[42] Menovshchikov, A.V. (2017). Regularity of the inverse of a homeomorphism of a Sobolev-Orlicz space. *Siberian Math. J., 58*, 649–662.

[43] Nakai, M. (1960). Algebraic criterion on quasiconformal equivalence of Riemann surfaces. *Nagoya Math. J., 16*, 157–184.

[44] Oliva, M., Prats, M. (2017). Sharp bounds for composition with quasiconformal mappings in Sobolev spaces. *J. Math. Anal. Appl., 451*, 1026–1044.

[45] Payne, L.E., Weinberger, H.F. (1960). An optimal Poincaré inequality for convex domains. *Arch. Rat. Mech. Anal., 5*, 286–292.

[46] Ryazanov, V., Srebro, U., Yakubov, E. (2001). BMO-quasiconformal mappings. *Journal d'Analyse Mathématique, 83*, 1–20.

[47] Stein, E.M. (1993). *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* Princeton Mathematical Series 43, Monographs in Harmonic AnalysisIII, Princeton University Press, Princeton, NJ.

[48] He, L., Cao, G.F., He, Z.H. (2015). Composition operators on Hardy-Sobolev spaces. *Indian J. Pure Appl. Math., 46*(3), 255–267.

[49] Stein, E.M., Weiss, G. (1960). On the theory of harmonic functions of several variables. The theory of H^p-spaces. *Acta Math., 103*, 25–62.

[50] Ukhlov, A. (1993). On mappings, which induce embeddings of Sobolev spaces. *Siberian Math. J., 34*, 185–192.

[51] Ukhlov, A., Vodop'yanov, S.K. (2008). Mappings associated with weighted Sobolev spaces. *Complex analysis and dynamical systems III, Contemp. Math., Amer. Math. Soc., Providence, RI, 455*, 369–382.

[52] Vodop'yanov, S.K. (1988). *Taylor Formula and Function Spaces.* Novosibirsk Univ. Press., Novosibirsk.

[53] Vodop'yanov, S.K. (2020). Composition Operators on Weighted Sobolev Spaces and the Theory of Q_p-Homeomorphisms. *Doklady Mathematics, 102*, 371–375.

[54] Vodop'yanov, S.K., Gol'dshtein, V.M. (1975). Structure isomorphisms of spaces W_n^1 and quasiconformal mappings. *Siberian Math. J., 16*, 224–246.

[55] Vodop'yanov, S.K., Gol'dshtein, V.M., Reshetnyak, Yu.G. (1979). On geometric properties of functions with generalized first derivatives. *Uspekhi Mat. Nauk, 34*, 17–65.

[56] Vodop'yanov, S.K., Ukhlov, A.D. (2002). Superposition operators in Sobolev spaces. *Russian Mathematics (Izvestiya VUZ), 46*(4), 11–33.

[57] Vodop'yanov, S.K., Ukhlov, A.D. (2004). Set Functions and Their Applications in the Theory of Lebesgue and Sobolev Spaces. I. *Siberian Adv. Math., 14*(4), 78–125.

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