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Asymptotic soliton-like solutions to the Benjamin–Bona–Mahony equation with variable coefficients and a strong singularity

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Abstract. The paper deals with constructing an asymptotic one-phase soliton-like solution to the Benjamin–Bona–Mahony equation with variable coefficients and a strong singularity making use of the non-linear WKB technique. The influence of the small-parameter value on the structure and the qualitative properties of the asymptotic solution, as well as the accuracy with which the solution satisfies the considerable equation, have been analyzed. It was demonstrated that due to the strong singularity, it is possible to write explicitly not only the main term of the asymptotics but at least its first-order term.

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1. Introduction

The paper deals with the Benjamin–Bona–Mahony equation (shortly, the BBM equation) with variable coefficients

$$a(x,t,\varepsilon)u_t + b(x,t,\varepsilon)u_x + c(x,t,\varepsilon)uu_x = \varepsilon^{2n}u_{xxt}, \tag{1.1}$$

that is a generalization of the BBM equation [1–4] of the following form

$$u_t + u_x + uu_x - u_{xxt} = 0. \tag{1.2}$$

Equation (1.2) describes the propagation of long waves with small amplitude on the liquid surface and is called the regularized Korteweg-de Vries equation [5, 6].

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The coefficients $a(x,t,\varepsilon)$, $b(x,t,\varepsilon)$, $c(x,t,\varepsilon)$ of equation (1.1) are infinitely differentiable quantities of the variables $(x,t) \in \mathbf{R} \times [0;T]$, as well as a small parameter ε, and can be represented as asymptotic (according to Poincaré) series

$$a(x,t,\varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^k a_k(x,t),$$

$$b(x,t,\varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^k b_k(x,t),$$

$$c(x,t,\varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^k c_k(x,t). \tag{1.3}$$

The major coefficients of series (1.3) are assumed to satisfy the condition $a_0(x,t)\, b_0(x,t)\, c_0(x,t) \neq 0$ for all $(x,t) \in \mathbf{R} \times [0;T]$.

The paper deals with equation (1.1) when the natural number n in the power exponent of singularity exceeds unity, i.e. the case of strong singularity is considered. The aim of the paper is to construct an asymptotic one-phase soliton-like solution to equation (1.1) and analyze the influence of the power exponent in the singularity on the qualitative properties of the asymptotic solution, as well as the structure of the algorithm for its construction.

In this paper, it was established that the power exponent of small parameter in (1.1) affects the structure of equations for terms in the asymptotic expansion, the algorithm of its construction, and the accuracy with which the found approximate solution satisfies the original equation. This is demonstrated in detail below. Here, we only note that due to the strong singularity in (1.1), the differential equations for the regular and singular parts of the asymptotic expansion change, which leads to some new qualitative properties of the singular part of the asymptotics. In particular, the consequence of such changes in the structure of the mentioned differential equations is a possibility to explicitly determine not only the main term of the singular part of the asymptotics but at least the first-order term.

In addition, the strong singularity effect also appears in the definition of the function that determines the so-called discontinuity curve for the asymptotic one-phase soliton-like solution, since the orthogonality condition is used not only through the main term, as is in the case $n = 1$.

In the paper, we used the nonlinear WKB technique [7–9] and an approach for constructing the asymptotic one-phase soliton-like solution with variable coefficients in form (1.1) with $n = 1$, for which asymptotic one-phase soliton-like solutions [10] and asymptotic soliton-like Σ-

solutions [11] were constructed, and the Cauchy problem [12] for equation (1.1) was considered.

2. Preliminary notes and definitions

Recall some notations and definitions.

By $S = S(\mathbf{R})$, let us denote a space of quickly decreasing functions, i.e. functions that are infinitely differentiable on the set $\mathbf{R}$ and for which the condition

$$\sup_{x\in\mathbf{R}} \left| x^m \frac{d^n}{dx^n} u(x) \right| < +\infty$$

is fulfilled for any integers $m, n \geq 0$.

By $G_1 = G_1(\mathbf{R}\times[0;T]\times\mathbf{R})$, let us denote a linear space of infinitely differentiable functions $f = f(x,t,\tau)$, $(x,t,\tau) \in \mathbf{R}\times[0;T]\times\mathbf{R}$ for which the following conditions are fulfilled [6, 8]:

1^0) the relationship

$$\lim_{\tau\to+\infty} \tau^n \frac{\partial^p}{\partial x^p} \frac{\partial^q}{\partial t^q} \frac{\partial^r}{\partial \tau^r} f(x,t,\tau) = 0, \quad (x,t) \in K, \tag{2.1}$$

is true for any non-negative integers n, p, q, r uniformly with respect to (x,t) on any compact set $K \subset \mathbf{R}\times[0;T]$;

2^0) there exists an infinitely differentiable function $f^-(x,t)$ such that the equality

$$\lim_{\tau\to-\infty} \tau^n \frac{\partial^p}{\partial x^p} \frac{\partial^q}{\partial t^q} \frac{\partial^r}{\partial \tau^r} \big(f(x,t,\tau) - f^-(x,t)\big) = 0, \quad (x,t) \in K,$$

is true.

Let $G_1^0 = G_1^0(\mathbf{R}\times[0;T]\times\mathbf{R}) \subset G_1$ be a space of functions $f = f(x,t,\tau) \in G_1$, $(x,t,\tau) \in \mathbf{R}\times[0;T]\times\mathbf{R}$, such that the condition

$$\lim_{\tau\to-\infty} f(x,t,\tau) = 0$$

is fulfilled uniformly with respect to the variables (x,t) on any compact $K \subset \mathbf{R}\times[0;T]$.

Below, we apply the concept of the asymptotic one-phase soliton-like solution of equation (1.1). It is based on the following definition.

Definition [6, 8]. *A function $u = u(x,t,\varepsilon)$, where ε is a small parameter, is called the asymptotic one-phase soliton-like solution if, for any $N \geq 0$, it can be written as the asymptotic series*

$$u(x,t,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j \left[u_j(x,t) + V_j(x,t,\tau)\right] + O(\varepsilon^{N+1}), \ \ \tau = \frac{x-\varphi(t)}{\varepsilon^n}, \tag{2.2}$$

where $\varphi(t) \in C^{\infty}([0;T])$ *is a scalar real function; the coefficients* $u_j(x,t)$, $j = \overline{0,N}$, *are infinitely differentiable (at the points* $t = 0$ *and* $t = T$, *the left and right derivatives, respectivly, are considered), and* $V_0(x,t,\tau) \in G_1^0$, $V_j(x,t,\tau) \in G_1$, $j = \overline{1\,,N}$.

The value $x - \varphi(t)$ is called *the phase* of the one-phase soliton-like function $u(x,t,\varepsilon)$, and the set $\Gamma = \{(x,t) : x = \varphi(t), t \in [0;T]\}$ is called *the discontinuity curve* of the function.

3. General scheme for constructing asymptotic one-phase soliton-like solutions

Let us move on constructing asymptotic one-phase soliton-like solutions of equation (1.1). The solution is searched [10] as an asymptotic series (2.2) for which we use the following notation:

$$u(x,t,\varepsilon) = Y_N(x,t,\tau,\varepsilon) + O(\varepsilon^{N+1}), \tag{3.1}$$

where

$$Y_N(x,t,\tau,\varepsilon) = U_N(x,t,\varepsilon) + V_N(x,t,\tau,\varepsilon),$$

$$U_N(x,t,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j u_j(x,t), \quad V_N(x,t,\tau,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j V_j(x,t,\tau).$$

For asymptotic series (2.2), it is necessary to define the coefficients of the regular part $U_N(x,t,\varepsilon)$ and the singular part $V_N(x,t,\tau,\varepsilon)$. Equations for the regular and singular parts are found from (1.1) in accordance with property (2.1). The functions $u_j(x,t)$, $j = \overline{0\,,N}$, should only be smooth, but the functions $V_j(x,t,\tau)$, $j = \overline{0\,,N}$, must belong to some functional spaces.

Using the property $V_0(x,t,\tau) \in G_1^0$, $V_j(x,t,\tau) \in G_1$ $j = \overline{1\,,N}$, and substituting series (2.2) into equation (1.1), we obtain the partial differential equations: first for the regular part $U_N(x,t,\varepsilon)$ and then for the singular part $V_N(x,t,\tau,\varepsilon)$ of the asymptotics. Separating these relations, we find equations for the coefficients of asymptotic series (2.2). In particular, the coefficients of the regular part $u_j(x,t)$, $j = \overline{0,N}$, are determined from the system

$$a_0(x,t)\frac{\partial u_0}{\partial t} + b_0(x,t)\frac{\partial u_0}{\partial x} + c_0(x,t)u_0\frac{\partial u_0}{\partial x} = 0, \tag{3.2}$$

$$a_0(x,t)\frac{\partial u_j}{\partial t} + b_0(x,t)\frac{\partial u_j}{\partial x} + c_0(x,t)\left(u_j\frac{\partial u_0}{\partial x} + u_0\frac{\partial u_j}{\partial x}\right)$$
$$= f_j(x,t,u_0,u_1,\ldots,u_{j-1}), \quad j = \overline{1,N}, \tag{3.3}$$

where the functions $f_j(x,t,u_0,u_1,\ldots,u_{j-1})$, $j = \overline{1,N}$, are calculated recursively.

It should be remarked that the solutions of equations (3.2), (3.3) can be found by means of the method of characteristics [13]. Therefore in further we assume that the regular part is known.

Regarding the finding of the terms of the singular part, we note that this problem is much more complicated than the problem of determining the terms of the regular part, and it is solved in several steps [14]. This is implemented as follows: first, the terms of the singular part of the asymptotics are determined on the discontinuity curve Γ, which is considered a priori known. Then, from the orthogonality condition [15], a differential equation is found for the function $\varphi = \varphi(t)$, which defines the discontinuity curve. Finally, the main term $V_0(x,t,\tau)$ is constructed explicitly, and the prolongation of the functions $V_j(x,t,\tau)$, $j = \overline{1,N}$, from the discontinuity curve Γ to some its neighborhood is carried out in such a way that the terms of the singular part of the constructed solution belong to the space G_1. Due to the last condition, the constructed solution is a certain deformation of the soliton solution of the BBM equation with constant coefficients.

4. The main term of asymptotic solution

The general form of the main term of the regular part of asymptotics (2.2) can be implicitly determined via an arbitrary function of a complete system of first integrals of quasilinear differential equations that look like

$$\frac{dt}{a_0(x,t)} = \frac{dx}{b_0(x,t) + u_0 c_0(x,t)} = \frac{du_0}{0}.$$

Since the regular part of asymptotics (2.2) plays only the role of background function, we will focus attention on defining the terms composing its singular part, especially since the qualitative properties of the asymptotic one-phase soliton-like solutions, which are some deformations of soliton solutions, reveal themselves owing just to the main term in the singular part of asymptotics (2.2). Preliminarily, it should be noted that although the function $\varphi = \varphi(t)$ is used to determine the terms in the singular part of asymptotics (2.2), starting from the main one, this function does not affect the accuracy [10] with which the main term of asymptotics (2.2), i.e. the sum of the main terms in the regular and singular parts, satisfies (1.1).

The function $v_0(t,\tau) = V_0(x,t,\tau)\big|_{x=\varphi(t)}$ is found as a solution of the following differential equation:

$$\varphi'(t)\frac{\partial^3 v_0}{\partial \tau^3} + \big(b_0(\varphi,t) - a_0(\varphi,t)\varphi'(t) + c_0(\varphi,t)u_0(\varphi,t)\big)\,\frac{\partial v_0}{\partial \tau}$$

$$+c_0(\varphi,t)\,v_0\frac{\partial v_0}{\partial \tau} = 0. \tag{4.1}$$

Equation (4.1) is derived from the relationship for the singular part $V_N(x,t,\tau)$ taking into account the asymptotic series with respect to a small parameter for the coefficients of equation (1.1) in the neighborhood of the discontinuity curve Γ.

The solution of equation (4.1) in the space G_1^0 is written in the form

$$v_0(t,\tau) = \frac{3A(\varphi,t)}{c_0(\varphi,t)}\,\cosh^{-2}\left(\frac{1}{2}\sqrt{\frac{A(\varphi,t)}{\varphi'(t)}}(\tau + C_0(t))\right), \tag{4.2}$$

where we use the notation $A(\varphi,t) = \varphi' a_0(\varphi,t) - b_0(\varphi,t) - c_0(\varphi,t)u_0(\varphi,t)$, $\varphi = \varphi(t)$, and $C_0(t)$ is a constant of integration (t is a parameter). Here, we suppose that the condition

$$\varphi'\,A(\varphi,t) > 0 \tag{4.3}$$

is satisfied.

Taking into account the property $V_0(x,t,\tau) \in G_1^0$, the main term of asymptotic expansion (2.2) can be written by the formula $V_0(x,t,\tau) = v_0(t,\tau)$. The following statement establishes the accuracy with which the function $V_0(x,t,\tau)$ satisfies equation (1.1).

Theorem 4.1. *Let the following conditions take place:*

1^0) *the functions* $a_0(x,t)$, $b_0(x,t)$, $c_0(x,t) \in C^\infty(\mathbf{R}\times[0;T])$, *and the inequality* $a_0(x,t)\,b_0(x,t)\,c_0(x,t) \neq 0$, $(x,t) \in \mathbf{R}\times[0;T]$ *holds;*

2^0) *the function* $\varphi(t) \in C^\infty([0;T])$ *satisfies inequality* (4.3).

Then the function

$$Y_0(x,t,\varepsilon) = u_0(x,t) + V_0(x,t,\tau), \tag{4.4}$$

where $u_0(x,t)$ *is a solution of equation* (3.2), *and the function* $V_0(x,t,\tau) = v_0(t,\tau)$ *is defined by formula* (4.2), *is a main term of the one-phase soliton-like solution to equation* (1.1) *and satisfies the equation with an accuracy* $O\left(\varepsilon^{-(n-1)}\right)$ *on the set* $\mathbf{R}\times[0;T]$.

Moreover, function (4.4) *satisfies equation* (1.1) *with an accuracy* $O(\varepsilon)$ *as* $\tau \to \pm\infty$.

The statement of theorem 4.1 follows from the construction algorithm and is similar to the proof of Theorem 1 in [10]. Therefore, we do not present the proof here.

Remark 4.1. For $n > 1$, the accuracy with which function (4.4) satisfies equation (1.1) on the set $\mathbf{R} \times [0;T]$ decreases with the increasing number n, while the accuracy with which function (4.4) satisfies equation (1.1) as $\tau \to \pm\infty$ does not depend on the number n, i.e. the accuracy remains unchanged. For $n = 1$, theorem 4.1 is a particular case of Theorem 1 in [10].

Remark 4.2. Formula (4.2) actually determines only the form (structure) of the main term in the singular part of asymptotics (2.2), since the function contains an arbitrary function $\varphi = \varphi(t)$ which has to belong to the space $C^{(1)}([0;T])$ and satisfy condition (4.3). It follows from the proof of theorem 4.1.

5. Higher terms of asymptotic expansion

Let us proceed to the determination of the main term of the asymptotic soliton-like solution (2.2) of equation (1.1) using the procedure mentioned on p. 3. For this purpose, we write down the equations for higher terms of the singular part of the asymptotics on the discontinuity curve.

The functions $v_j = v_j(t,\tau) = V_j(x,t,\tau)\big|_{x=\varphi(t)}$, $j = \overline{1,N}$, satisfy the following differential equations:

$$\varphi'(t)\frac{\partial^3 v_j}{\partial \tau^3} + \big(b_0(\varphi,t) - a_0(\varphi,t)\varphi'(t) + c_0(\varphi,t)u_0(\varphi,t)\big)\,\frac{\partial v_j}{\partial \tau}$$

$$+c_0(\varphi,t)\frac{\partial}{\partial \tau}\,(v_0 v_j) = \mathcal{F}_j(t,\tau), \qquad (5.1)$$

where the functions

$$\mathcal{F}_j(t,\tau) = F_j(t, V_0(x,t,\tau),\ldots,V_{j-1}(x,t,\tau),u_0(x,t),\ldots,u_j(x,t))\big|_{x=\varphi(t)}$$

are determined after finding $u_0(x,t)$, $u_1(x,t)$, …, $u_j(x,t)$, $V_0(x,t,\tau)$, $V_1(x,t,\tau)$, …, $V_{j-1}(x,t,\tau)$, $j = \overline{1,N}$, recursively.

Let us find the conditions for the existence of solutions to equations (5.1) in the space G_1 and obtain theorems for the justification of found asymptotics. We will also study the influence of the power exponent n in equation (1.1)) on the kind of the equations for the singular part of the asymptotics. For this purpose, we write down the functions $\mathcal{F}_1(t,\tau)$ and

$\mathcal{F}_2(t,\tau)$ in equations (5.1) for $n=2,3,\dots$ in the explicit form. For these n-values, we have

$$\mathcal{F}_1(t,\tau)=\big(\varphi' a_1(\varphi,t)-b_1(\varphi,t)-c_0(\varphi,t)u_1(\varphi,t)$$

$$-c_1(\varphi,t)\left(u_0(\varphi,t)+v_0\right)\big)\frac{\partial v_0}{\partial\tau}. \tag{5.2}$$

It is easy to see that the function $\mathcal{F}_1(t,\tau)$ for the case $n=2,3,\dots$ has a simpler form than in the case $n=1$ (see formula (18) in [10] for comparison), while in the case $n=1$, the function $\mathcal{F}_2(t,\tau)$ has a cumbersome form. For $n=2$, the function is somewhat simplified and is represented by the formula

$$\mathcal{F}_2(t,\tau)=-a_0(\varphi,t)\frac{\partial v_0}{\partial t} \tag{5.3}$$

$$+\left(\varphi' a_{0x}(\varphi,t)-b_{0x}(\varphi,t)-\frac{\partial}{\partial x}\left(c_0(\varphi,t)u_0(\varphi,t)\right)\right)\tau\frac{\partial v_0}{\partial\tau}$$

$$+\left(-b_2(\varphi,t)-c_2(\varphi,t)u_0(\varphi,t)-c_1(\varphi,t)u_1(\varphi,t)-c_0(\varphi,t)u_2(\varphi,t)\right)\frac{\partial v_0}{\partial\tau}$$

$$+\big(\varphi' a_1(\varphi,t)-b_1(\varphi,t)-c_1(\varphi,t)u_0(\varphi,t)-c_0(\varphi,t)(u_1(\varphi,t)+v_1)\big)\frac{\partial v_1}{\partial\tau}$$

$$-\left(\tau c_{0x}(\varphi,t)+c_2(\varphi,t)\right)v_0\frac{\partial v_0}{\partial\tau}-c_1(\varphi,t)\frac{\partial}{\partial\tau}\left(v_0v_1\right)$$

$$-c_0(\varphi,t)u_{0x}(\varphi,t)v_0-\frac{\partial^3 v_0}{\partial\tau^2\partial t}.$$

In the case $n>2$, the function is greatly simplified, being represented by the formula

$$\mathcal{F}_2(t,\tau)=-\left(b_2(\varphi,t)+c_2(\varphi,t)u_0(\varphi,t)+c_1(\varphi,t)u_1(\varphi,t)\right. \tag{5.4}$$

$$\left.+c_0(\varphi,t)u_2(\varphi,t)+c_2(\varphi,t)v_0\right)\frac{\partial v_0}{\partial\tau}-c_1(\varphi,t)\frac{\partial}{\partial\tau}\left(v_0v_1\right)$$

$$+\big(\varphi' a_1(\varphi,t)-b_1(\varphi,t)-c_1(\varphi,t)u_0(\varphi,t)-c_0(\varphi,t)\left(u_1(\varphi,t)+v_1\right)\big)\frac{\partial v_1}{\partial\tau}.$$

Thus, an increase in the singularity power exponent in equation (1.1) leads to a certain simplification of the equations for the first terms in the singular part of the asymptotics.

5.1. Orthogonality condition

Let us now consider the conditions under which equations (5.1) have solutions in the space G_1. In this case, it becomes possible to obtain an asymptotic solution to equation (1.1), which is close (in a certain sense) to the soliton solution. In [10,15], the following properties were established:

1) under the conditions $\mathcal{F}_j(t,\tau) \in G_1^0$, $j = \overline{1,N}$, there exists a solution to equation (5.1) in the space G_1 if and only if the orthogonality condition

$$\int_{-\infty}^{+\infty} \mathcal{F}_j(t,\tau) v_0(t,\tau) d\tau = 0, \quad j = \overline{1,N}, \tag{5.5}$$

is fulfilled;

2) then the solution looks like

$$v_j(t,\tau) = \nu_j(t)\eta_j(t,\tau) + \psi_j(t,\tau), \tag{5.6}$$

where we use the notation

$$\nu_j(t) = (-a_0(\varphi,t)\varphi'(t) + b_0(\varphi,t) + c_0(\varphi,t)u_0(\varphi,t))^{-1} \lim_{\tau\to-\infty} \Phi_j(t,\tau);$$

the function $\eta_j(t,\tau) \in G_1$ satisfies the condition

$$\lim_{\tau\to-\infty} \eta_j(t,\tau) = 1;$$

$\psi_j(t,\tau)$ is a function from the space G_1^0; $c_j(t)$, $j = \overline{1,N}$, are integration constants;

$$\Phi_j(t,\tau) = \int_{-\infty}^{\tau} \mathcal{F}_j(t,\xi) d\xi + E_j(t); \tag{5.7}$$

$E_j(t)$, $j = \overline{1,N}$, are integration constants (t is a parameter) which are chosen according to the condition

$$\lim_{\tau\to+\infty} \Phi_j(t,\tau) = 0, \quad j = \overline{1,N}.$$

Under conditions $\mathcal{F}_j(t,\tau) \in G_1^0$, $j = \overline{1,N}$, and the orthogonality condition (5.5), the functions $v_j(t,\tau)$, $j = \overline{1,N}$, belong to the space G_1^0 if and only if the relation [12]

$$\lim_{\tau\to-\infty} \Phi_j(t,\tau) = 0, \quad j = \overline{1,N}, \tag{5.8}$$

is true. In (5.6), the coefficients $\nu_j(t)$ are equal to zero.

The orthogonality condition (5.5) should be checked at each stage of determination of the next term in the singular part of the asymptotics. This condition is also used to derive the differential equation for the function $\varphi = \varphi(t)$.

Formula (5.2) implies the property $\mathcal{F}_1(t,\tau) \in G_1^0$. So, the orthogonality condition (5.5) is true for $j = 1$ and therefore equation (5.1) for $v_1(t,\tau)$ has a solution in the space G_1. In the particular case, when

$$a(x,t,\varepsilon) = a(x,t), \quad b(x,t,\varepsilon) = b(x,t), \quad c(x,t,\varepsilon) = c(x,t), \tag{5.9}$$

the function $\mathcal{F}_1(t,\tau)$ has a simple form,

$$\mathcal{F}_1(t,\tau) = -c_0(\varphi(t),t)u_1(\varphi(t),t)v_{0\tau}(t,\tau).$$

As a consequence, in this case, we come to the following statement.

Lemma 5.1. *If the coefficients of equation* (1.1) *satisfies condition* (5.9), *the inequality* $a(x,t)\,b(x,t)\,c(x,t) \neq 0$ *holds for all* $(x,t) \in \mathbf{R}\times[0;T]$, *and the background function* $U_1(x,t,\varepsilon) = 0$, *then, for all* $(x,t) \in \mathbf{R}\times[0;T]$, *the function* $V_0(x,t,\tau) = v_0(t,\tau)$, *where* $v_0(t,\tau)$ *is written by formula* (4.2), *satisfies equation* (1.1) *with an accuracy* $O\left(\varepsilon^{-(n-2)}\right)$.

The proof of this statement is performed in a standard way and essentially uses the property that the function $v_1(t,\tau) = 0$ satisfies the corresponding equation of system (5.1) because $\mathcal{F}_1(t,\tau) = 0$.

The meaning of this lemma is that under sufficiently general conditions, the accuracy of the constructed main term in the asymptotic soliton-like solution can be higher than that established in Theorem 4.1.

Consider the next terms in the singular part of the asymptotics on the curve Γ. The general solution of each inhomogeneous equation in (5.1) can be found recursively, for example, by the method of arbitrary constant variation. The solution is written by the formula [10]

$$\begin{gathered} v_j(t,\tau) = \left(\int_{\tau_0}^{\tau} \Phi_j(t,\xi)v_{0\tau}(t,\xi)\,d\xi + c_{j1}(t)\right) v_{0\tau}(t,\tau)\int_{\tau_0}^{\tau} v_{0\tau}^{-2}(t,\xi)\,d\xi \\ -\left(\int_{\tau_0}^{\tau} \Phi_j(t,\xi)v_{0\tau}(t,\xi)\int_{\tau_0}^{\xi} v_{0\tau}^{-2}(t,\eta)\,d\eta\,d\xi + c_{j2}(t)\right) v_{0\tau}(t,\tau), \end{gathered} \tag{5.10}$$

where $c_{j1}(t)$, $c_{j2}(t)$, $j = \overline{1,N}$, are constants of integration.

Due to the simple expression for $\mathcal{F}_1(t,\tau)$ in (5.2), the function $v_1(t,\tau)$ can be explicitly found for any n. Preliminarily, it should be noted that the orthogonality condition (5.5) for the function $\mathcal{F}_1(t,\tau)$ is satisfied for all sufficiently smooth functions $\varphi = \varphi(t)$.

By means of direct integration, from formula (5.7) for $\Phi_1(t,\tau)$, we have

$$\Phi_1(t,\tau) = \big[\varphi'(t)a_1(\varphi,t) - b_1(\varphi,t) - c_0(\varphi,t)u_1(\varphi,t) - c_1(\varphi,t)u_0(\varphi,t)\big] \times$$
$$\times v_0(t,\tau) - \frac{1}{2}\, c_1(\varphi,t)v_0^2(t,\tau), \quad \varphi = \varphi(t).$$

From the property $v_0(t,\tau) \in G_1^0$, it follows that the function $\Phi_1(t,\tau)$ satisfies condition (5.8) for all smooth functions $\varphi = \varphi(t)$, and thus the function $v_1(t,\tau)$, as a solution to equation (5.1), belongs to the space G_1^0. This function can be written explicitly. Really, from (5.10), we have

$$v_1(t,\tau) = \left(\frac{D_1}{4\alpha^2} - \frac{D_2}{\alpha^2}\right)\cosh^{-2}(\alpha t) \tag{5.11}$$
$$-\left(\frac{D_1}{4\alpha} + \frac{D_2}{3\alpha}\cosh^{-6}(\alpha t)\right)\tau\,\cosh^{-3}(\alpha\tau)\sinh(\alpha\tau),$$

where

$$D_1 = 3\,\frac{A(\varphi,t)}{c_0(\varphi,t)}\,\big[\varphi'(t)a_1(\varphi,t) - b_1(\varphi,t)$$
$$-c_0(\varphi,t)u_1(\varphi,t) - c_1(\varphi,t)u_1(\varphi,t)\big],$$
$$D_2 = -\frac{9}{2}\,\frac{A^2(\varphi,t)c_1(\varphi,t)}{c_0^2(\varphi,t)}, \quad \alpha = \frac{1}{2}\sqrt{\frac{A(\varphi,t)}{\varphi'(t)}}, \quad \varphi = \varphi(t),$$

i.e. the function $v_1(t,\tau)$ is quickly decreasing with respect to the variable τ.

Analogously, from (5.10), one can explicitly find the function $v_2(t,\tau)$, as well as the other functions, but the corresponding formulas are cumbersome.

The fulfillment of conditions (5.5) and (5.8) for $j = 2, 3, \ldots$, depends on the value of the singularity power exponent n in equation (1.1). For $n > 2$, using formula (5.4), it is easy to make sure that due to the properties $v_0(t,\tau), v_1(t,\tau) \in G_1^0$, the corresponding function $\Phi_2(t,\tau)$ satisfies condition (5.8) for all smooth functions $\varphi = \varphi(t)$. Therefore, the function $v_2(t,\tau)$, as a solution to equation (5.1), is an element of the space G_1^0 and can be found according to (5.10).

On the contrary, for $n = 2$, the function $\Phi_2(t,\tau)$ of form (5.7), (5.3) may not satisfy condition (5.5) for all smooth functions $\varphi = \varphi(t)$ in the general case. Thus, the condition can be used to find the differential equation for the function $\varphi = \varphi(t)$. In particular, under these circumstances, in the case $n = 2$, from the orthogonality condition (5.8) for $j = 2$, it is possible to derive a differential equation for the function $\varphi = \varphi(t)$ in the following form

$$\left[A_1(\varphi,t)\varphi'^2(t) + A_2(\varphi,t)\varphi'(t) + A_3(\varphi,t)\right]\varphi''(t) + A_4(\varphi,t)\varphi'^4(t)$$
$$+A_5(\varphi,t)\varphi'^3(t) + A_6(\varphi,t)\varphi'^2(t) + A_7(\varphi,t)\varphi'(t) = 0, \qquad (5.12)$$

where the coefficients $A_k(\varphi,t)$, $k = \overline{1,7}$, are written as follows:

$$A_1 = 24\, a_0^2\, c_0,\ A_2 = -8\, a_0\, c_0\, \alpha,\ A_3 = -\, c_0\, \alpha^2,$$

$$A_4 = -40\, c_{0x}\, a_0^2 + 30 a_0\, a_{0x}\, c_0,$$

$$A_5 = \left[60 c_{0x}\, \alpha + 20 a_{0t}\, c_0 - 24 a_0\, c_{0t} - 30 c_0\, \alpha_x + 20 c_0^2\, u_{0x}\right] a_0 - 15 a_{0x}\, c_0\, \alpha,$$

$$A_6 = -20 a_0\, c_0\, \alpha_t - 5 a_{0t}\, c_0 \alpha + 15 c_0\, \alpha\, \alpha_x + 28 a_0\, c_{0t}\, \alpha - 20 c_0^2 u_{0x}\, \alpha - 20 c_{0x}\, \alpha^2,$$

$$A_7 = 5\, c_0\, \alpha\, \alpha_t - 20\, c_{0t}\, \alpha^2.$$

Here, $\alpha = b_0 + c_0 u_0$, $a_0 = a_0(\varphi,t)$, $b_0 = b_0(\varphi,t)$, $c_0 = c_0(\varphi,t)$, and $u_0 = u_0(\varphi,t)$.

Regarding the solution of the nonlinear equation (5.12), we note the following:

1) the Cauchy problem for equation (5.12) has a (local) solution under sufficiently general conditions for the functions $a_0(x,t)$, $b_0(x,t)$, and $c_0(x,t)$ [10];

2) the interval of existence of the solution to equation (5.12) depends on the main coefficients of problem (1.1) and the main term in the regular part of asymptotics (2.2), as well as on the initial conditions for the function $\varphi = \varphi(t)$;

3) in effect, this interval defines a value set of the variables (x,t) for which the constructed asymptotic solution exists. If the coefficients of equation (1.1) are determined for all its arguments, of considerable interest is the case when the nonlinear equation (5.12) has a global solution, i.e. a solution defined for all $t \in \mathbf{R}$. An example of such a solution was constructed in [16] for the Korteweg-de Vries equation with variable coefficients and a singular perturbation of the first order.

We also note that for $n > 2$, since the orthogonality condition (5.5) at $j = 2$ is satisfied for all smooth functions, then, in order to find the differential equation for the function $x = \varphi(t)$, it is necessary to analyze the orthogonality conditions for the equations for the next terms in the singular part of the asymptotics, i.e. for $j > 2$. This can be done quite easily in the case of certain coefficients of equation (1.1).

5.2. Prolongation of functions $v_j(t,\tau)$ from the discontinuity curve

Now we will find the functions $V_j(x,t,\tau)$, $j=\overline{1,N}$, using the prolongation of $v_j(t,\tau)$, $j=\overline{1,N}$, from the curve Γ into some its neighborhood. Above, according to the property $v_0(t,\tau)\in G_1^0$, the prolongation of the function $v_0(t,\tau)$ is defined by means of the formula $V_0(x,t,\tau)=v_0(t,\tau)$. Similarly, if $v_j(t,\tau)\in G_1^0$, then put $V_j(x,t,\tau)=v_j(t,\tau)$. In particular, the function $V_1(x,t,\tau)$ for all $n\geq 2$ is defined by the formula $V_1(x,t,\tau)=v_1(t,\tau)$.

Note that in the case $v_j(t,\tau)\in G_1^0$, $j=\overline{1,N}$, i.e. if the functions $\mathcal{F}_j(t,\tau)$, $j=\overline{1,N}$, satisfy condition (5.8), then the asymptotic solution to equation (1.1) is written as

$$Y_N(x,t,\varepsilon)=\sum_{j=0}^{N}\varepsilon^j\left[u_j(x,t)+v_j(t,\tau)\right]. \tag{5.13}$$

Theorem 5.1. *Let $n=2$ and the following conditions be fulfilled:*

1^0) *the functions $a_k(x,t)$, $b_k(x,t)$, $c_k(x,t)\in C^\infty(\mathbf{R}\times[0;T])$, $k=\overline{0,N}$, and the inequality $a_0(x,t)\,b_0(x,t)\,c_0(x,t)\neq 0$, $(x,t)\in\mathbf{R}\times[0;T]$, is satisfied;*

2^0) *equation* (5.12) *has a solution $\varphi(t)\in C^\infty([0;T])$, for which inequality* (4.3) *is true;*

3^0) *the function $\mathcal{F}_j(t,\tau)\in G_1^0$, $j=\overline{2,N}$, and conditions* (5.5), (5.8) *hold.*

Then the asymptotic one-phase soliton-like solution to equation (1.1) *is written as* (5.13) *and satisfies equation* (1.1) *with an accuracy $O(\varepsilon^{N-1})$ on the set $\mathbf{R}\times[0;T]$. Moreover, for $\tau\to\pm\infty$, function* (5.13) *satisfies equation* (1.1) *with an accuracy $O(\varepsilon^{N+1})$, $N\in\mathbf{N}$.*

For the case $n>2$, the following statement is true.

Theorem 5.2. *Let $n>2$ and the following conditions be fulfilled:*

1^0) *the condition 1^0 of theorem 5.1 takes place;*

2^0) *the function $\varphi(t)\in C^\infty([0;T])$ and satisfies inequality* (4.3)*;*

3^0) *the functions $\mathcal{F}_j(t,\tau)\in G_1^0$, $j=\overline{2,N}$, and conditions* (5.5), (5.8) *are true.*

Then the asymptotic one-phase soliton-like solution to equation (1.1) *is written as* (5.13) *and the function satisfies equation* (1.1) *with an accuracy $O(\varepsilon^{N-n+1})$ on the set $\mathbf{R}\times[0;T]$. Moreover, for $\tau\to\pm\infty$, function* (5.13) *satisfies equation* (1.1) *with an accuracy $O(\varepsilon^{N+1})$, $N\in\mathbf{N}$.*

The proofs of theorems 5.1 and 5.2 are carried out in the standard way and are similar to the proof of Theorem 1 from [10] to the accuracy with which the constructed asymptotic solution satisfies equation (1.1). If the property $v_j(t,\tau) \in G_1^0$ is not fulfilled, then taking into account formula (5.6), the prolongation of the functions $v_j(t,\tau)$ from the curve Γ into some its neighborhood is realized by the form

$$V_j(x,t,\tau) = u_j^-(x,t)\eta(t,\tau) + \psi_j(t,\tau), \tag{5.14}$$

where $u_j^-(x,t)$ is a solution of the Cauchy problem

$$\Lambda u_j^-(x,t) = f_j^-(x,t), \qquad u_j^-(x,t)\Big|_{\Gamma} = \nu_j(t). \tag{5.15}$$

Here, the differential operator Λ is written by the formula

$$\Lambda = a_0(x,t)\frac{\partial}{\partial t} + b_0(x,t)\frac{\partial}{\partial x} + c_0(x,t)u_0(x,t)\frac{\partial}{\partial x} + c_0(x,t)u_{0x}(x,t).$$

The equation in (5.15) is obtained from (1.1) by substituting (5.14) in (1.1) and calculating the limit at $\tau \to -\infty$.

Since, taking into account inequality (4.3) for all $t \in [0;T]$, the curve Γ is transversal to the characteristics of the operator Λ, the Cauchy problem (5.15) is well defined. Therefore, in the domain $\Omega_\mu(\Gamma) = \{(x,t) \in \mathbf{R} \times [0;T] : |x - \varphi(t)| < \mu\}$, for sufficiently small values of μ, the problem has a unique solution $u_j^-(x,t) \in C^\infty(\Omega_\mu(\Gamma))$, which exists under general conditions on the coefficients of the operator Λ.

Moreover, the Cauchy problem (5.15) can have a solution defined not only in some neighborhood of the curve Γ, but also in an unbounded (with respect to the variable x) domain. For example, the function $u_j^-(x,t)$, $j = \overline{2,N}$, can be defined in the domain $\{(x,t) : x - \varphi(t) < \mu,\ t \in [0;T]\}$ and then the asymptotic solution to equation (1.1) is written as

$$Y_N(x,t,\varepsilon) = \begin{cases} \sum\limits_{j=0}^{N} \varepsilon^j \left[u_j(x,t) + V_j(x,t,\tau)\right], & (x,t) \in \Omega_\mu(\Gamma), \\ \sum\limits_{j=0}^{N} \varepsilon^j u_j(x,t) + \sum\limits_{j=2}^{N} \varepsilon^j u_j^-(x,t), & (x,t) \in D^-, \\ \sum\limits_{j=0}^{N} \varepsilon^j u_j(x,t), & (x,t) \in D^+, \end{cases} \tag{5.16}$$

where

$$D^- = \{(x,t) \in \mathbf{R} \times [0;T] : x - \varphi(t) \le -\mu\},$$
$$D^+ = \{(x,t) \in \mathbf{R} \times [0;T] : x - \varphi(t) \ge \mu\},$$

Summarizing the considerations set out above, we obtain the following statement.

Theorem 5.3. *Let $n = 2$ in equation* (1.1) *and the following conditions be fulfilled:*

1^0) *the conditions* 1^0, 2^0 *of theorem 5.1 take place;*

2^0) *the functions* $\mathcal{F}_j(t,\tau) \in G_1^0$, $j = \overline{2,N}$, *and the orthogonality conditions* (5.5) *are true for them;*

3^0) *the Cauchy problem* (5.15) *has a solution in the set* D^-.

Then the asymptotic one-phase soliton-like solution to equation (1.1) *is written in form* (5.16) *and satisfies the equation with an accuracy* $O(\varepsilon^{N-1})$ *on the set* $\mathbf{R} \times [0;T]$. *Moreover, for* $\tau \to \pm\infty$, *function* (5.16) *satisfies equation* (1.1) *with an accuracy* $O(\varepsilon^N)$, $N \in \mathbf{N}$.

Theorem 5.4. *Let* $n > 2$ *in equation* (1.1); *both conditions* 1^0, 2^0 *of theorem 5.2 and condition* 3^0 *of theorem 5.3 be true; and the functions* $\mathcal{F}_j(t,\tau) \in G_1^0$, $j = \overline{2,N}$, *satisfy condition* (5.5).

Then the asymptotic one-phase soliton-like solution to equation (1.1) *is written in form (5.16) and satisfies the equation with an accuracy* $O(\varepsilon^{N-n+1})$ *on the set* $\mathbf{R} \times [0;T]$. *Moreover, for* $\tau \to \pm\infty$, *function* (5.16) *satisfies equation* (1.1) *with an accuracy* $O(\varepsilon^{N-n+2})$, $N \in \mathbf{N}$.

Conclusion

In this paper, an asymptotic single-phase soliton-like solution of the Benjamin-Bona-Mahony equation with variable coefficients and a strong singularity, i.e. when the power exponent $2n$ of the small parameter is greater than 2, has been constructed using the nonlinear WKB method. An algorithm of constructing the solution has been described and the influence of the power exponent of the small parameter on the structure of the asymptotic solution, as well as on the algorithm of its construction, the qualitative properties of the asymptotic solution, and the accuracy with which the constructed approximate solution satisfies the initial equation, has been analyzed.

It was also shown that due to the strong singularity in (1.1), the right-hand sides of the differential equations for determining the terms in the regular and singular parts of the asymptotics change, which leads to some new qualitative properties of the asymptotics. The consequence of such changes is the ability to find explicitly not only the main term of the singular part of the asymptotics but also at least its first-order term. The obtained results are consistent with the results published earlier [10] for the case of quadratic singularity, i.e. for $n = 1$.

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