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A representation of the Weierstrass integral via the Poisson integrals

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Abstract. In our investigation, we have presented the second order linear partial differential equation in polar coordinates. Considering this differential equation on the unit disk, we have obtained the one-dimensional heat equation. It is well-known that the heat equation can be solved taking into account the boundary condition for the general solution on the unit circle. In our paper, the boundary value problem is solved by the well-known method called the separation of variables. As a result, the general solution to the boundary value problem is presented in terms of the Fourier series. After that, the expressions for the Fourier coefficients are used with the aim to transform the Fourier series expansion for the general solution to the boundary value problem into the so-called Weierstrass integral that is represented via the so-called Weierstrass kernel. A representation of the Weierstrass kernel via the infinite geometric series is derived by a way to parameterize a complicated function via a simplified function. The derivation of the corresponding parametrization is based on two well-known integrals. As a result, a complicated function of the natural argument is represented in the form of the double integral that contains a simplified function of the same natural argument. So, the double-integral representation of the Weierstrass kernel is derived. To obtain this result, the integral representation of the so-called Dirac delta function is taken into account. The found expression for the Weierstrass kernel is substituted into the expression for the Weierstrass integral. As a result, it was found that the Weierstrass integral can be considered to be the double-integral that contains the Poisson integral and the conjugate Poisson integral.

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1. Introduction

Let us consider the second order linear partial differential equation

$$\rho \frac{\partial U}{\partial \rho} + \frac{\partial^2 U}{\partial x^2} = 0 \tag{1.1}$$

in two independent variables ρ and x. Considering the differential equation (1.1) on the unit disk $(0 < \rho < 1)$, we obtain the one-dimensional heat equation. In the presence of the boundary condition

$$\lim_{\rho \to 1} U(\rho, x) = f(x), \tag{1.2}$$

it is well-known that the one-dimensional heat equation (1.1) can be solved by a way to parameterize a complicated operator function via a simplified operator function [1]. In the presence of the periodicity property $f(x + 2\pi) = f(x)$, the one-dimensional heat equation (1.1) with the boundary condition (1.2) can be solved by separation of variables. So, the periodic function $f(x)$ and the general solution $U(\rho, x)$ to the one-dimensional heat equation (1.1) can be expanded in the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{+\infty} (a_k \cos kx + b_k \sin kx) \tag{1.3}$$

and

$$U(\rho, x) = \frac{a_0}{2} + \sum_{k=1}^{+\infty} \rho^{k^2} (a_k \cos kx + b_k \sin kx) \tag{1.4}$$

that contain the Fourier coefficients

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x')\, dx', \tag{1.5}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \cos kx' dx' \tag{1.6}$$

and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \sin kx' dx'. \tag{1.7}$$

The Fourier series expansion (1.4) is a formal solution to the one-dimensional heat equation (1.1). To obtain a closed-form solution to this equation, we need to substitute the Fourier coefficients (1.5), (1.6) and (1.7) into the Fourier series expansion (1.4). Introducing the notation $U(\rho, x) \equiv W(\rho; f; x)$, the integral [2]

$$W(\rho; f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \left\{ \frac{1}{2} + \sum_{k=1}^{+\infty} \rho^{k^2} \cos k(x - x') \right\} dx' \qquad (1.8)$$

can be obtained. It is significant to note that the quantity (1.8) is called the Weierstrass integral [3] of the function f. Taking into account the new notation, the boundary condition (1.2) can be replaced by the boundary condition

$$\lim_{\rho \to 1} W(\rho; f; x) = f(x). \qquad (1.9)$$

In approximation theory, the summation of the Fourier series between curly brackets of the formula (1.8) is a problem of a great significance. In the current research, we suggested a way to compute the sum of the above-mentioned Fourier series. It is important to note that this way is based on a way to parameterize a complicated function via a simplified function. As a result, the double-integral representation of the Weierstrass integral is obtained. The obtained result can be useful for future explorations in the field of approximation theory.

2. Weierstrass kernel

A way to parameterize a complicated function via a simplified function means that a general term of a Fourier series can be replaced by its integral representation. This can enable us to represent an unknown Fourier series via some well-known series. In our investigation, the general term of the Fourier series between curly brackets of the formula (1.8) will be considered to be a double integral.

In the formula (1.8), the Fourier series between curly brackets is called the Weierstrass kernel. A double-integral representation of the Weierstrass kernel can be derived from the parametrization

$$\rho^{k^2} \cos k(x - x') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} e^{\pm i\xi\eta} \rho^{k(1+|\eta|)} \cos\left[k(x + \xi - x') - \xi\right] d\eta. \qquad (2.1)$$

The parametrization (2.1) can be useful to compute the infinite sum

$$\sum_{k=1}^{+\infty} \rho^{k^2} \cos k\left(x-x'\right)$$
$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} e^{\pm i\xi\eta} \sum_{k=1}^{+\infty} \rho^{k(1+|\eta|)} \cos\left[k\left(x+\xi-x'\right)-\xi\right] d\eta. \quad (2.2)$$

The further calculations can be simplified by the identity

$$\int_{-\infty}^{+\infty} \delta\left(\xi\right) \frac{\cos\xi}{2} d\xi = \frac{1}{2} \quad (2.3)$$

that contains the so-called Dirac delta function [4]

$$\delta\left(\xi\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{\pm i\xi\eta} d\eta. \quad (2.4)$$

Substituting the integral representation (2.4) of the Dirac delta function into the identity (2.3), we can derive the identity

$$\frac{1}{2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} e^{\pm i\xi\eta} \frac{\cos\xi}{2} d\eta. \quad (2.5)$$

The sum of the identities (2.2) and (2.5) is the Weierstrass kernel

$$\frac{1}{2} + \sum_{k=1}^{+\infty} \rho^{k^2} \cos k\left(x-x'\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} e^{\pm i\xi\eta} \left(\frac{\cos\xi}{2}\right.$$
$$\left. + \sum_{k=1}^{+\infty} \rho^{k(1+|\eta|)} \cos\left[k\left(x+\xi-x'\right)-\xi\right]\right) d\eta. \quad (2.6)$$

In the right side of the identity (2.6), we have the absolutely convergent series, because $0 < \rho^{1+|\eta|} \leq \rho < 1$. A well-known sum of an infinite geometric series (see [5], p. 48 or see [6], p. 573) enables us to compute the infinite sum

$$\frac{\cos\xi}{2} + \sum_{k=1}^{+\infty} \rho^{k(1+|\eta|)} \cos\left[k\left(x+\xi-x'\right)-\xi\right]$$
$$= \frac{\cos\xi}{2} \frac{1-\rho^{2(1+|\eta|)}}{1-2\rho^{1+|\eta|}\cos\left(x+\xi-x'\right)+\rho^{2(1+|\eta|)}}$$
$$+ \frac{\rho^{1+|\eta|}\sin\left(x+\xi-x'\right)}{1-2\rho^{1+|\eta|}\cos\left(x+\xi-x'\right)+\rho^{2(1+|\eta|)}} \sin\xi. \quad (2.7)$$

We need to substitute the absolutely convergent series (2.7) into the right side of the identity (2.6). This enables us to derive the double-integral representation of the Weierstrass kernel

$$\frac{1}{2}+\sum_{k=1}^{+\infty}\rho^{k^2}\cos k\left(x-x'\right)$$
$$=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}d\xi\int\limits_{-\infty}^{+\infty}e^{\pm i\xi\eta}\left(\frac{\rho^{1+|\eta|}\sin\left(x+\xi-x'\right)\sin\xi}{1-2\rho^{1+|\eta|}\cos\left(x+\xi-x'\right)+\rho^{2(1+|\eta|)}}\right.$$
$$\left.+\frac{\cos\xi}{2}\frac{1-\rho^{2(1+|\eta|)}}{1-2\rho^{1+|\eta|}\cos\left(x+\xi-x'\right)+\rho^{2(1+|\eta|)}}\right)d\eta. \tag{2.8}$$

3. Weierstrass integral

Let us now substitute the Weierstrass kernel (2.8) into the expression (1.8) for the Weierstrass integral. Then we can derive the double-integral representation

$$W\left(\rho;f;x\right)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}d\xi\int\limits_{-\infty}^{+\infty}e^{\pm i\xi\eta}\left\{P\left(\rho^{1+|\eta|};f;x+\xi\right)\cos\xi\right.$$
$$\left.+\ \bar{P}\left(\rho^{1+|\eta|};f;x+\xi\right)\sin\xi\right\}d\eta \tag{3.1}$$

that contains the Poisson integral [7–11]

$$P\left(\rho^{1+|\eta|};f;x+\xi\right)$$
$$=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{1-\rho^{2(1+|\eta|)}}{1-2\rho^{1+|\eta|}\cos\left(x+\xi-x'\right)+\rho^{2(1+|\eta|)}}f\left(x'\right)dx' \tag{3.2}$$

and the conjugate Poisson integral [12, 13]

$$\bar{P}\left(\rho^{1+|\eta|};f;x+\xi\right)$$
$$=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}\frac{\rho^{1+|\eta|}\sin\left(x+\xi-x'\right)}{1-2\rho^{1+|\eta|}\cos\left(x+\xi-x'\right)+\rho^{2(1+|\eta|)}}f\left(x'\right)dx'. \tag{3.3}$$

In the case of the Weierstrass integral (3.1), we have the boundary condition (1.9). In the case of the integrals (3.2) and (3.3), the analogous boundary conditions can also be derived. Let us consider the Poisson integral (3.2). Taking into account the Fourier series expansion

$$\frac{1}{2}\frac{1-\rho^{2(1+|\eta|)}}{1-2\rho^{1+|\eta|}\cos\left(x+\xi-x'\right)+\rho^{2(1+|\eta|)}}$$
$$=\frac{1}{2}+\sum_{k=1}^{+\infty}\rho^{k(1+|\eta|)}\cos k\left(x+\xi-x'\right), \tag{3.4}$$

we can transform the integral (3.2) into the integral

$$P\left(\rho^{1+|\eta|};f;x+\xi\right)$$
$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(x'\right)\left\{\frac{1}{2}+\sum_{k=1}^{+\infty}\rho^{k(1+|\eta|)}\cos k\left(x+\xi-x'\right)\right\}dx'. \tag{3.5}$$

The Fourier coefficients (1.5), (1.6) and (1.7) can be useful to transform the integral (3.5) into the Fourier series expansion

$$P\left(\rho^{1+|\eta|};f;x+\xi\right)=\frac{a_0}{2}$$
$$+\sum_{k=1}^{+\infty}\rho^{k(1+|\eta|)}\left\{a_k\cos k\left(x+\xi\right)+b_k\sin k\left(x+\xi\right)\right\}. \tag{3.6}$$

Let us analyze the Fourier series expansion (3.6) in the asymptotical case $\rho\to 1$. Taking the Fourier series representation (1.3) of the periodic function $f(x)$ into account, we can derive the boundary condition

$$\lim_{\rho\to 1}P\left(\rho^{1+|\eta|};f;x+\xi\right)=f\left(x+\xi\right). \tag{3.7}$$

Let us now consider the conjugate Poisson integral (3.3). Introducing the new integration variable $t=x'-x-\xi$, the integral (3.3) can be transformed into the integral

$$\bar{P}\left(\rho^{1+|\eta|};f;x+\xi\right)$$
$$=-\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\rho^{1+|\eta|}\sin t}{1-2\rho^{1+|\eta|}\cos t+\rho^{2(1+|\eta|)}}f\left(t+x+\xi\right)dt. \tag{3.8}$$

Let us now analyze the conjugate Poisson integral (3.8) in the asymptotical case $\rho\to 1$. Then we can derive the boundary condition

$$\lim_{\rho\to 1}\bar{P}\left(\rho^{1+|\eta|};f;x+\xi\right)=\bar{f}\left(x+\xi\right) \tag{3.9}$$

that contains the function

$$\bar{f}(x) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t+x)\cot\frac{t}{2}dt. \tag{3.10}$$

The double-integral representation (3.1) of the Weierstrass integral can also be analyzed in the asymptotical case $\rho \to 1$. The boundary conditions (1.9), (3.7) and (3.9) must be taken into account with the aim to derive the identity

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} e^{\pm i\xi\eta}\left\{f(x+\xi)\cos\xi + \bar{f}(x+\xi)\sin\xi\right\}d\eta. \tag{3.11}$$

The identities (3.1) and (3.11) can be used with the aim to calculate the difference

$$\begin{aligned} &f(x) - W(\rho; f; x) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} e^{\pm i\xi\eta}\left(\left[f(x+\xi) - P\left(\rho^{1+|\eta|}; f; x+\xi\right)\right]\cos\xi\right. \\ &\quad \left. + \left[\bar{f}(x+\xi) - \bar{P}\left(\rho^{1+|\eta|}; f; x+\xi\right)\right]\sin\xi\right)d\eta. \end{aligned} \tag{3.12}$$

The double-integral representation (3.1) of the Weierstrass integral $W(\rho; f; x)$ and the double-integral representation (3.12) of the difference $f(x) - W(\rho; f; x)$ are the main results of our investigation.

Conclusion

To calculate a sum of the Fourier series between curly brackets of the formula (1.8), we derived the parametrization (2.1) that enables one to represent the complicated function $\rho^{k^2}\cos k\left(x - x'\right)$ of k via the simplified function $\rho^{k(1+|\eta|)}\cos\left[k\left(x+\xi-x'\right)-\xi\right]$ of k. As a result, the integral kernel (2.6) of the Weierstrass integral (1.8) is represented via the absolutely convergent series (2.7). In fact, a sum of the Fourier series (2.7) can be calculated taking into account a well-known sum of an infinite geometric series. So, the double-integral representation (2.8) of the Weierstrass kernel is derived. The identity (2.8) enables one to derive the identity (3.1) that contains the Weierstrass integral (1.8), the Poisson integral (3.2) and the conjugate Poisson integral (3.8). In addition to the boundary condition (1.9) for the Weierstrass integral (1.8), the boundary condition (3.7) for the Poisson integral (3.2) and the boundary condition

(3.9) for the conjugate Poisson integral (3.8) can also be derived. It is also important to note that the obtained results can be useful to derive the identity (3.12) that contains the differences

$$f(x) - W(\rho; f; x),$$

$$f(x+\xi) - P\left(\rho^{1+|\eta|}; f; x+\xi\right)$$

and

$$\bar{f}(x+\xi) - \bar{P}\left(\rho^{1+|\eta|}; f; x+\xi\right).$$

All the results obtained in this investigation can be useful for future explorations in the field of approximation theory.

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