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On the asymptotics of the principal moments of inertia of a convex body in the isotropic state

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Abstract. We recall Bourgain's conjecture on the upper bound for the main moments of inertia of a multidimensional body in the isotropic state. We also recall the currently known bounds obtained by Bourgain and Klartag. Then we evaluate the moments of inertia of the multidimensional ball and of the multidimensional cube, and state a conclusive comment corroborating the Bourgain's conjecture.

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1. Introduction

A homogeneous convex body $K \subset \mathbb{R}^n$ (of density 1) is said to be *in the isotropic state*, or, equivalently, *in the isotropic position*, if its center of inertia is located at the origin of Cartesian coordinates $x = (x_1, \dots, x_n)$ of the space $\mathbb{R}^n$, the body has unit volume (mass), and for any $i, j = 1, \dots, n$

$$\int_K x_i x_j dv = \delta_{ij} L_K^2 \, ,$$

where L_K does not dependent on i, j. The quantity L_K is called *isotropy constant* or *isotropic constant* of a body.

Any convex body can be brought into an isotropic state by a composition of translations and contractions–expansions.

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In the isotropic state of the body its tensor of inertia is not only diagonal but, moreover, all its eigenvalues (values of the main moments of inertia) coincide:

$$I_1 = \cdots = I_n = (n-1)L_K^2\,. \tag{1.1}$$

It was proved by J. Bourgain in [2] that the isotropic constant L_K of any convex body $K \subset \mathbb{R}^n$ in the isotropic state admits a uniform upper bound $L_K < Cn^{1/4}\log n$. This upper bound was improved by B. Klartag in [4] to $L_K < Cn^{1/4}$. According to V. Milman [6], the following question stated by J. Bourgain remains open: does L_K admit a *universal* upper bound $L_K < C$, where the constant C is uniform for all convex bodies K in all dimensions n? (This problem is also known as *slicing problem* and *the hyperplane conjecture*, see [3] for more details.)

Below we evaluate the quantities L_K for the ball and for the cube in $\mathbb{R}^n$.

2. Principal moments of inertia of the n-dimensional ball.

The section in [5] devoted to mechanics contains an exercise suggesting to calculate the moments of inertia of certain three-dimensional bodies. A useful hint is suggested for the case of the ball. The hint is equally valid not only for the case of the three-dimensional ball, but for the ball in the space $\mathbb{R}^n$ of any dimension n. We present below the corresponding calculation.

Recall that the volume v_n of the unit n-dimensional ball B^n (the ball of unit radius) in the n-dimensional Euclidean space $\mathbb{R}^n$ is expressed by the formula

$$v_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} = \frac{\pi^{\frac{n}{2}}}{\frac{n}{2}\Gamma(\frac{n}{2})}\,.$$

It follows from this formula that for $n \gg 1$ the quantity v_n admits the following asymptotics: $v_n \simeq \frac{1}{\sqrt{\pi n}}\left(\sqrt{\frac{2\pi e}{n}}\right)^n$. Since the ball b^n of unit volume in $\mathbb{R}^n$ has radius $r_n = v_n^{-\frac{1}{n}}$, we get $r_n \simeq \sqrt{\frac{n}{2\pi e}}$ for $n \gg 1$.

We calculate now the principal moments of inertia $I_1 = \cdots = I_n$ of the ball $K = b^n$ of unit volume. Following the indication of the book [5] we note that since

$$I_k = \int_{b^n} (x_1^2 + \cdots + \widehat{x_k^2} + \cdots + x_n^2)dv\,,$$

where $k = 1, \ldots, n$ and the term x_k^2 is omitted, we have

$$I_1 + \cdots + I_n = (n-1)\int_{b^n} r^2 dv\,,$$

where $r^2 = x_1^2 + \cdots + x_n^2$.

Thus,

$$I_1 = \cdots = I_n = \frac{n-1}{n} \int_{b^n} r^2 dv \,.$$

Passing to polar coordinates, we have

$$\int_{b^n} r^2 dv = \sigma_{n-1} \int_0^{r_n} r^{n+1} dr = \frac{1}{n+2}\, \sigma_{n-1} r_n^{n+2} = \frac{n}{n+2}\, r_n^2 \,,$$

where σ_{n-1} is the area ($(n-1)$-measure) of the unit sphere in $\mathbb{R}^n$, and r_n is the radius of the ball b^n of unit volume in $\mathbb{R}^n$.

Taking into account the asymptotics $r_n = v_n^{-\frac{1}{n}} \simeq \sqrt{\frac{n}{2\pi e}}$ for $n \gg 1$, obtained above, we conclude that $I_1 = \cdots = I_n \simeq r_n^2 \simeq \frac{n}{2\pi e}$ for $n \to \infty$. By equation (1.1) we get the following asymptotic value of the isotropy constant of a ball b^n of unit volume:

$$L_{b^n}^2 \simeq \frac{1}{2\pi e} \approx \frac{1}{17.0795} \text{ as } n \to \infty \,.$$

3. Principal moments of inertia of the n-dimensional cube.

We calculate now the principal moments of inertia $I_1 = \cdots = I_n$ and the isotropy constant of the n-dimensional unit cube $K = I^n = [-\frac{1}{2}, \frac{1}{2}]^n$.

Note that since

$$I_k = \int_{I^n} (x_1^2 + \cdots + \widehat{x_k^2} + \cdots + x_n^2) dv \,,$$

where $k = 1, \ldots, n$ and the term x_k^2 is omitted, we have

$$I_1 + \cdots + I_n = (n-1) \int_{I^n} r^2 dv \,,$$

where $r^2 = x_1^2 + \cdots + x_n^2$.

Thus,

$$I_1 = \cdots = I_n = \frac{n-1}{n} \int_{I^n} r^2 dv =: \frac{n-1}{n}\, M_n \,.$$

We have

$$\begin{aligned} \int_{I^n} r^2 dv &= \int_{I^n} (x_1^2 + \cdots + x_n^2) dv \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{I^{n-1}(x_n)} (x_1^2 + \cdots + x_n^2) dv_{n-1} \right) dx_n \,. \end{aligned}$$

Using the notation M_n introduced above, we can write

$$M_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{I^{n-1}(x_n)} (x_1^2 + \cdots + x_n^2) dv_{n-1} \right) dx_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} (M_{n-1} + x_n^2) dx_n \,.$$

Thus,

$$M_n = M_{n-1} + \frac{1}{12} \,.$$

Since $M_1 = \frac{1}{12}$, we get $M_n = n\frac{1}{12}$.

Combining equation $L_{I^n}^2 = \frac{1}{n-1} I_1$ (see (1.1)) and $I_1 = \frac{n-1}{n} M_n$, we get

$$L_{I^n}^2 = \frac{1}{12} \,.$$

4. Comment

Comparing the asymptotic values of isotropic constants of the ball of unit volume and of the unit cube, we observe that they are relatively close to each other despite the fact that the cube is not too spherical, especially in high dimensions when the distance between its parallel faces remains equal to one, while the diameter of the unit cube in $\mathbb{R}^n$ is $\sqrt{n}$.

Note the following phenomenon of concentration of measure valid for the space $\mathbb{R}^n$ of large dimension n. Most of the volume of a multidimensional ball is concentrated in a small neighborhood of its boundary sphere. There is almost nothing inside.

Note also that the average radius of the unit cube in $\mathbb{R}^n$, is the same as the radius of the ball of unit volume[1]. All but asymptotically negligible part of the mass of the cube is concentrated in the vicinity of the sphere of this radius. One can speculate that this phenomenon of concentration of measure corroborates the Bourgain's conjecture on the uniform boundedness of the isotropy constants L_K.

References

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[1] In general, fix the origin of the polar coordinate system at an inner point of a convex body $K \subset \mathbb{R}^n$. Denote by $r(\theta)$ the radius vector of the boundary point of the body in the direction $\theta \in S^{n-1}$. Assuming that the sphere S^{n-1} is endowed with the uniform probability measure $d\sigma$, the following relation holds: $\int_{S^{n-1}} r^n(\theta) d\sigma = V(K)/v_n$, where $V(K)$ is the volume of the body K and v_n is the volume of the ball of unit radius in the space $\mathbb{R}^n$ [1].

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