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On Ahlfors–Beurling Operator

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Dedicated to 80th anniversary of Professor Vladimir Gutlyanskii

Abstract. We investigate regularity properties of solutions of Beltrami equation expressed in terms of moduli of continuity. In particular, we prove that a class of Calderon–Zygmund operators, including Ahlfors–Beurling operator, preserves certain type of modulus of continuity of compactly supported functions. We also prove a purely topological result which easily gives injectivity of normal solutions of Beltrami equation.

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1. Introduction

We investigate questions related to Beltrami equation $f_{\overline{z}} = \mu f_z$, where $\|\mu\|_\infty < 1$. This equation is of fundamental importance in the theory of quasiconformal mappings in the plane, see [1, 5]. Note that no regularity is assumed on the Beltrami coefficient μ and therefore one looks for a solution in the local Sobolev space $W^{1,p}_{\rm loc}(\mathbb{C})$ for a suitably chosen $p > 2$. One of the questions we investigate is: what are conditions on μ that ensure that solution f is continuously differentiable? A partial result in that direction is given in Section 3, using a result on Calderon–Zygmund singular integral operators obtained in Section 2. In section 4 we address a problem present in demonstrating that a solution f of Beltrami equation is quasiconformal. We show that injectivity of f can be derived by purely topological arguments. In the last section we discuss weakly closed forms, in simple situations they can be used for solving Beltrami equation by explicit method of curvilinear integration.

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The notation we use is standard, the Riemann sphere is denoted by $\overline{\mathbb{C}}$ and the unit disc by $\mathbb{D}$. The boundary of a set S is denoted by bS. Surface measure is denoted by $d\sigma$. Also, for $a \in \mathbb{R}^n$ and $0 < R_1 < R_2 < +\infty$, we have a spherical ring $A(a; R_1, R_2) = \{x \in \mathbb{R}^n : R_1 < |x - a| < R_2\}$.

In this paper χ denotes a mollifier, namely a nonnegative C^∞ function on $\mathbb{C}$ or $\mathbb{R}^n$ supported in the unit disc (unit ball) with integral equal to 1. We set $\chi_\varepsilon(x) = \varepsilon^{-n}\chi(x/\varepsilon)$, $\varepsilon > 0$.

1.1. Additional comments

Concerning the subject shortly described above it seems appropriate first to add a few comments suggested by the reviewer. We can consider Theorem 1 and Propositions 2 and 4 as the main results of the paper. This theorem is applied to the Ahlfors–Beurling operator (on functions with compact support), which plays a crucial role in quasiconformal theory. The classical applications of this operator have been related mainly with the Hölder continuous functions and with homeomorphisms having distributional derivatives. The important deep generalizations of this theory have been given by the Polish and Ukrainian schools, see for example [1, 2, 5]. Theorem 1 extends these results to more general moduli of continuity. Its applications to quasiconformal theory given by Propositions 2 and 4 involve solving some injectivity problem.

2. Calderon-Zygmund operators and moduli of continuity

It is known that Ahlfors–Beurling operator S, formally defined by

$$(Sf)(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta)}{(z-\zeta)^2}\, d\xi\, d\eta \tag{1}$$

preserves Hölder continuity of order α of compactly supported functions if $0 < \alpha < 1$. An analogous one-dimensional result is Privalov's theorem which essentially states that Hilbert's operator preserves α – Hölder continuity. Both results are valid for Calderon–Zygmund singular integral operators acting on periodic functions, see [3]. It is our aim here to generalize these results to moduli of continuity more general than $\omega(t) = t^\alpha$, $0 < \alpha < 1$, and to deal with non periodic functions.

We investigate a singular integral operator $T = T_K$ with a kernel

$$K(x) = K_\Omega(x) = \Omega(x^\star)|x|^{-n}, \qquad x \in \mathbb{R}^n, \quad x \neq 0, \tag{2}$$

where $x^\star = x/|x|$ and Ω is a C^1 function on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. A basic requirement is that Ω has the following cancellation property:

$$\int_{\mathbb{S}^{n-1}} \Omega(\xi) d\sigma(\xi) = 0. \tag{3}$$

The operator T is defined by the following formula:

$$Tf(x) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} K_\varepsilon(y)f(x-y)dy, \tag{4}$$

where K_ε denotes a truncated kernel:

$$K_\varepsilon(x) = \begin{cases} K(x) & |x| \geq \varepsilon \\ 0 & |x| < \varepsilon \end{cases}. \tag{5}$$

It is a deep result of Calderon–Zygmund theory that the limit in (4) exists both in L^p norm and almost everywhere if $f \in L^p(\mathbb{R}^n)$, $1 < p < +\infty$; moreover T is bounded on L^p: $\|Tf\|_p \leq C_p\|f\|_p$ for $1 < p < \infty$. Let us note that T is not bounded neither on L^1 nor on L^∞.

We are interested in action of T on certain classes of continuous functions. Let us choose a majorant ω, i.e. a continuous increasing and concave function $\omega(t)$, $t \geq 0$, such that $\omega(0) = 0$ and $\omega(\lambda t) \leq C_\lambda\omega(t)$, $\lambda > 1$. It is said that $f : \mathbb{R}^n \to \mathbb{C}$ is ω-continuous function if

$$\|f\|_\omega = \sup_{x\neq y}\frac{|f(x)-f(y)|}{\omega(|x-y|)} < +\infty. \tag{6}$$

The class of such functions is denoted by Λ_ω, clearly it is a vector space. Since, as noted, T is not bounded on L^∞, we are going to work within the following subclasses of Λ_ω:

$$\Lambda_\omega^R = \{f \in \Lambda_\omega : \mathrm{supp} f \subset B(0,R)\}, \qquad R > 0. \tag{7}$$

We impose the following two conditions on majorant ω:

$$\int_0^\delta \frac{\omega(t)}{t}\,dt \leq A_1\omega(\delta), \qquad 0 < \delta \leq 1 \tag{8}$$

$$\int_\delta^1 \frac{\omega(t)}{t^2}\,dt \leq A_2\frac{\omega(\delta)}{\delta}, \qquad 0 < \delta \leq 1. \tag{9}$$

Let us note that condition (8) is satisfied by $\omega(t) = t^\alpha$ for all $0 < \alpha \leq 1$, however condition (9) is satisfied by $\omega(t) = t^\alpha$ only if $0 < \alpha < 1$. It is easy to see that if we replace requirement $0 < \delta \leq 1$ in (8) and (9) by a more general one $0 < \delta \leq R$ and integrate from δ to R in (9), then the same inequalities hold, but with different constants $A_1 = A_1(R,\omega)$ and $A_2 = A_2(R,\omega)$.

It is easy to show that the limit in (4) exists for all $x \in \mathbb{R}^n$ if $f \in \Lambda_\omega^R$. Indeed, the cancellation property (3) imples that the existence of the limit

$$\lim_{\varepsilon\to 0}\int_{B(0,|x|+R)} K_\varepsilon(y)[f(x-y)-c]dy, \tag{10}$$

is independent of the choice of a constant c and if it exists is equal to $Tf(x)$. In particular, by choosing $c = f(x)$ and using spherical coordinates we get

$$\begin{aligned}\int_{B(0,|x|+R)} |K(y)||f(x-y)-f(x)|dy &\leq \|\Omega\|_\infty \|f\|_\omega \int_{B(0,|x|+R)} \frac{\omega(|y|)}{|y|^n}\,dy \\ &= \|\Omega\|_\infty \|f\|_\omega |\mathbb{S}^{n-1}| \int_0^{|x|+R} \frac{\omega(t)}{t}\,dt \\ &< +\infty \end{aligned} \tag{11}$$

since the last integral is convergent by (8). Therefore we have

$$Tf(x) = \int K(y)[f(x-y)-f(x)]\,dy, \qquad x \in \mathbb{R}^n \tag{12}$$

where integration is extended over any ball $B(0,\rho)$ containing the support of $f(x-y)$. The main result of this section is Theorem 1 below. The proof relies on methods developed in foundational papers by Calderon and Zygmund, in particular in [3], where Hölder continuity was treated in the case of periodic functions.

A different approach is taken in [9], the authors treat a more general case where the kernel K is of the form $K(x,(x-y)/|x-y|)$ and derive global estimates. However, they assume condition (9), drop condition (8) and assume the following additional condition:

$$\omega_\alpha(t) = t^{-\alpha}\omega(t) \quad \text{is increasing for some} \quad 0 < \alpha < 1. \tag{13}$$

In [10] the authors made some corrections to their previous paper [9] cited above.

Theorem 1. *Assume a majorant ω satisfies conditions* (8) *and* (9). *Then for every $R > 0$ there is a constant $C = C(R, n, \Omega, \omega)$ such that*

$$\|Tf\|_\omega \leq C\|f\|_\omega, \qquad f \in \Lambda_\omega^R. \tag{14}$$

We note the following well known estimate for the kernel K:

$$|K(x+h) - K(x)| \leq C(\Omega)\frac{|h|}{|x|^{n+1}}, \qquad x \neq 0, \quad |h| \leq |x|/3. \tag{15}$$

Proof. Let us choose $f \in \Lambda_\omega^R$ and $x, x+h \in \mathbb{R}^n$, where $|h| \leq 1$. From (12) we obtain

$$Tf(x) = \left(\int_{|y|\leq 3|h|} + \int_{3|h|\leq y \leq |x|+R}\right) K(y)[f(x-y)-f(x)]\,dy = I_1 + I_2. \tag{16}$$

Analogously to the argument leading to (11) we estimate I_1:

$$|I_1| \le \|\Omega\|_\infty \|f\|_\omega |\mathbb{S}^{n-1}| \int_0^{3|h|} \frac{\omega(t)}{t}\, dt \le C(n,\Omega,\omega)\|f\|_\omega \omega(|h|), \tag{17}$$

where we used condition (8). Replacing x with $x+h$ in (16) we obtain

$$\begin{aligned} Tf(x+h) &= \left(\int_{|y|\le 3|h|} + \int_{3|h|\le y \le |x+h|+R} \right) K(y) \\ &\quad \times [f(x+h-y) - f(x+h)]\, dy \\ &= J_1 + J_2 \end{aligned} \tag{18}$$

where $|J_1| \le C(n,\Omega,\omega)\|f\|_\omega \omega(|h|)$ and

$$\begin{aligned} J_2 &= \int_{3|h|\le|y|\le|x+h|+R} K(y)[f(x+h-y) - f(x+h)]\, dy \\ &= \int_{3|h|\le|y|\le|x+h|+R} K(y)[f(x+h-y) - f(x)]\, dy \\ &= \int_{3|h|\le|z+h|\le|x+h|+R} K(z+h)[f(x-z) - f(x)]\, dz \\ &= \int_{3|h|\le|z|\le|x|+R} K(z+h)[f(x-z) - f(x)]\, dz + E = \tilde{J}_2 + E. \end{aligned} \tag{19}$$

Note that cancellation property enabled replacement of $f(x+h)$ with $f(x)$. Now we estimate the error term E, which results from a change of domain of integration from one spherical ring $A(-h; 3|h|, |x+h|+R)$ to another one $A(0; 3|h|, |x|+R)$. Regarding the change of inner limits, the size of $K(z+h)$ is estimated by $C(n)\|\Omega\|_\infty |h|^{-n}$, the measure of the symmetric difference of $B(0, 3|h|)$ and $B(-h, 3|h|)$ is estimated by $C(n)|h|^n$ and the size of $f(x-z) - f(x)$ is estimated by $C(\omega)\|f\|_\omega \omega(|h|)$. Hence the error due to the change of inner limits is estimated by $C(n,\Omega,\omega)\|\Omega\|_\infty \omega(|h|)$. Regarding the change of outer limits, the measure of the symmetric difference of domains of integration is estimated by $C(n)|h|(|x|+R)^{n-1}$, the size of $K(z+h)$ by $\|\Omega\|_\infty (|x|+R)^{-n}$ and the size of $f(x-z) - f(x)$ by $2\|f\|_\infty$, hence the contribution to the error term is bounded by $C\|\Omega\|_\infty \|f\|_\infty |h|$, where C is a constant depending only on n. Since $\|f\|_\infty \le C(\omega, R)\|f\|_\omega$ and $\delta \le C\omega(\delta)$ for $0 \le \delta \le 1$ we obtain

$$|E| \le C(R, n, \Omega, \omega)\|f\|_\omega \omega(|h|). \tag{20}$$

Combining (16), (17), (18), (19) and (20) we obtain

$$\begin{aligned}|Tf(x+h) - Tf(x)| &= |J_1 + \tilde{J}_2 + E - I_1 - I_2| \\ &\le |\tilde{J}_2 - I_2| + |E| + |I_1| + |J_1| \\ &= E_1 + E_2 \qquad \text{where}\end{aligned}$$

$$E_2 = |E| + |I_1| + |J_1| \le C(R, n, \Omega, \omega)\|f\|_\omega \omega(|h|) \qquad \text{and}$$

$$E_1 = \left| \int_{3|h| \le |z| \le |x|+R} [K(z+h) - K(z)][f(x-z) - f(x)]\, dz \right|.$$

Since f is supported in $B(0,R)$ we can assume $\omega(t)$ is constant for $t \ge 2R$, in particular $\omega(t) \le \omega(2R)$. We use (15) and (9) to estimate E_1:

$$\begin{aligned}E_1 &\le C(\Omega) \int_{3|h| \le |z| \le |x|+R} \frac{|h|}{|z|^{n+1}} |f(x-z) - f(z)|\, dz \\ &\le C(\Omega)|h||\mathbb{S}^{n-1}|\|f\|_\omega \int_{3|h|}^{|x|+R} \frac{\omega(t)}{t^2}\, dt \\ &= |h| C(n, \Omega)\|f\|_\omega \left(\int_{3|h|}^1 + \int_1^\infty \right) \frac{\omega(t)}{t^2}\, dt \\ &\le C_n \|f\|_\omega (A_2 \omega(|h|) + \omega(2R)|h|) = C(R, n, \Omega, \omega)\|f\|_\omega \omega(|h|).\end{aligned}$$

Note that this was the only estimate in the proof that relied on (9). This gives desired estimate for $|h| \le 1$, the estimate for $|h| > 1$ follows easily from the vanishing of Tf at infinity. In fact, since the support of f is compact we have the following asymptotics: $Tf(x) = O(|x|^{-n})$ as $|x| \to +\infty$; we leave details to the reader. □

3. An application to Beltrami equation

In this section ω denotes a majorant satisfying conditions (8) and (9). If $f : G \to \mathbb{C}$ is continuous on a domain $G \subset \mathbb{R}^n$ and $K \subset G$ is compact we set

$$\|f\|_{\omega,K} = \sup_{x \ne y, x,y \in K} \frac{|f(x) - f(y)|}{\omega(|x-y|)}. \tag{21}$$

The space $\Lambda_{\omega,loc}(G)$ consists of all $f \in C(G)$ such that $\|f\|_{\omega,K} < +\infty$ for every compact $K \subset G$. Next, for integer $k \ge 0$, we define a space $\Lambda^k_{\omega,loc}(G)$ as a vector space of all functions $f \in C^k(G)$ such that $\partial^\alpha f \in \Lambda_{\omega,loc}(G)$ for every multiindex α, $|\alpha| \le k$. It is easily verified that $\phi f \in \Lambda^k_{\omega,loc}(G)$ if $\phi \in C^{k+1}(G)$ and $f \in \Lambda^k_{\omega,loc}(G)$.

Proposition 1. *Let $\mu \in \Lambda_{\omega,loc}(G)$, where G is a domain in $\mathbb{C}$. If f is a solution of Beltrami equation $f_{\overline{z}} = \mu f_z$ in G such that f and $f_{\overline{z}}$ (weak derivatives) belong to $\Lambda_{\omega,loc}(G)$, then f is in $\Lambda^1_{\omega,loc}(G)$.*

Proof. Let us note that it suffices to prove that f_z is in $\Lambda_{\omega,loc}(G)$. For a given compact $K \subset G$ we choose a compactly supported $\varphi \in C^\infty(G)$ which is equal to 1 in a neighborhood of K. Since $(\varphi f)_{\overline{z}} = \varphi_{\overline{z}} f + \varphi f_{\overline{z}}$ we see that $(\varphi f)_{\overline{z}} \in \Lambda^R_\omega$ for some $R > 0$. Now we use crucial property of the Ahlfors–Beurling operator S, namely it transforms weak $\overline{z}$ - derivative to weak z - derivative: $S(g_{\overline{z}}) = g_z$ for a compactly supported g. This gives $(\varphi f)_z = S[(\varphi f)_{\overline{z}}]$. Since $(\varphi f)_{\overline{z}} \in \Lambda^R_\omega$, Theorem 1 gives $(\varphi f)_z \in \Lambda_\omega$. Since φ is equal to 1 in a neighborhood of K, this shows that $\|f_z\|_{\omega,K} < \infty$ which, together with assumptions on f and $f_{\overline{z}}$, gives desired result. □

Regarding the study of general Beltrami equations with singularities we refer reader to a monograph [5] where further references can be found. The authors of [5] use a geometric approach they deveolped, which is based on the notions of modulus and capacity, to derive the main existence theorems, including sophisticated and more general existence theorems that have been recently established. It seems a natural question for further research whether our initial results concerning Beltrami equations can be further developed using the methods from [5]. In particular it is a plausible hypothesis that assumptions on regularity of f and $f_{\overline{z}}$ in Proposition 1 can be dropped. This is certainly true in the case of α - Hölder continuity.

Indeed, in that case global results, i.e. up to the boundary, are available. For example, the following result is contained in [6].

Theorem 2. ([6], Theorem 1.2) *Let f be a quasiconformal mapping between bounded planar domains D and G with $C^{1,\alpha}$ boundaries, where $0 < \alpha < 1$. Then the following three conditions are equivalent.*

(A) *$f \in C^{1,\alpha}(\overline{D})$ and $f^{-1} \in C^{1,\alpha}(\overline{G})$.*

(B) *The complex dilatation μ_f is α Hölder continuous on D.*

(C) *The complex dilatation $\mu_{f^{-1}}$ is α Hölder continuous on G.*

The author states, without proof, that condition (A) implies the following estimate: $0 < c \leq |f_{\overline{z}}| < |f_z| \leq C < +\infty$. The estimate from below for $f_{\overline{z}}$ does not hold, as the case of a conformal map f demonstrates, however it is not used later on in the proof of the theorem. The two sided estimate for f_z follows easily from

$$(1-k^2)|f_z|^2 \leq |f_z|^2 - |f_{\overline{z}}|^2 = J_f(z) \leq |f_z|^2, \qquad \text{where} \quad k = \|\mu_f\|_\infty < 1$$

and the two-sided estimate of the Jacobian $0 < c_1 \leq J_f \leq C_1$ which in itself follows from the assumption that f and f^{-1} are continuously differentiable up to the boundary. Assuming (A), the estimate $0 < c \leq |f_z| \leq C < +\infty$ easily gives, as stated in [6], that $\mu_f = f_{\overline{z}}/f_z$ is α Hölder continuous on D, by symmetry the same is true for $\mu_{f^{-1}}$ on G. Note that this gives also α Hölder continuity of $f_z/|f_z|$ and therefore α Hölder continuity of the second complex dilatation $\nu_f = (f_z/|f_z|)^2 \mu_f$ of f.

The argument present in [6] in fact gives equivalence of (A) and

(BC) The complex dilatation μ_f *and* the second complex dilatation ν_f are α Hölder continuous on D.

The author proves that (B) implies $f \in C^{1,\alpha}(\overline{D})$, and therefore (C) implies $f^{-1} \in C^{1,\alpha}(\overline{G})$. However, the proof that (B) implies (C) relies on the wrong formula $\mu_{f^{-1}} = -\mu_f \circ f^{-1}$, the correct one is $\mu_{f^{-1}} = -\nu_f \circ f^{-1}$, see [1].

Related to this, we present an example which shows that the conditions $f : \mathbb{D} \to G$ is quasiconformal, $f \in C^{1,\alpha}(\overline{\mathbb{D}})$ imply neither α - Hölder continuity of μ_f nor even C^1 smoothness of the boundary of G.

Example 1. Let $f(z) = (1-z)^{\alpha+1} + k(1-\overline{z})^{\alpha+1}$ where $0 < \alpha < 1$, $0 < k < 1$ and $z \in \mathbb{D}$. Then $p = f_z = (\alpha+1)(1-z)^{\alpha}$ and $q = f_{\overline{z}} = k(\alpha+1)(1-\overline{z})^{\alpha} = k\overline{p}$. Clearly, $f \in C^{1,\alpha}(\overline{\mathbb{D}})$. Let us show that f is injective. In fact, it suffices to prove that the mapping $g(w) = w^{\alpha+1} + k\overline{w^{\alpha+1}}$ is injective on the right half plane $\Pi = \{w \in \mathbb{C} : \Re w > 0\}$. The last fact is obvious, since $w \mapsto w^{\alpha+1}$ is injective on Π and $\zeta \mapsto \zeta + k\overline{\zeta}$ is also injective on $\mathbb{C}$.

Since $|\mu_f| = |q/p| = k < 1$ we conclude that f is a quasiconformal map on $\mathbb{D}$ which is in $C^{1,\alpha}(\overline{\mathbb{D}})$.

However, $\arg \mu_f(z) = 2\alpha \arg(1-z)$, hence μ_f has no continuous extension at point $1 \in b\mathbb{D}$. In particular, it is not α Hölder continuous on $\mathbb{D}$. It is easily verified that $G = f(\mathbb{D})$ does not have smooth boundary at point 0 in bG.

4. Local and global injectivity

As mentioned in Introduction, here we present a sufficient condition for a continuous map f from a domain $D \subset \mathbb{C}$ to $\mathbb{C}$ to be a homeomorphism. For application to Beltrami equation on $\mathbb{C}$ the case $D = \mathbb{C}$ suffices. We point out that we have two assumptions in the following proposition, one is local in nature (local homeomorphism) and the other one is global (on cluster sets). In the usual expositions of Beltrami equation, for example in [1], the second assumption comes from normalization at infinity. There are other approaches to injectivity problems, in the context of harmonic mappings, which are analytical in nature, see for example [8].

Proposition 2. *Let $D \subset \mathbb{C}$ be a simply connected domain and assume $f : D \to \mathbb{C}$ is a local homeomorphism. Set $W = f(D)$ and let F_∞ be the component of $F = \overline{\mathbb{C}} \setminus W$ which contains ∞. Assume that for any sequence z_n in D with no accumulation point in D the sequence $f(z_n)$ has all its accumulation points in the set bF_∞. Then $f : D \to W$ is a homeomorphism. In particular, W is simply connected.*

Proof. We start by proving that W is simply connected. Indeed, otherwise there is a bounded component K of F. Let us choose w in bK and select a sequence w_n in W such that $w_n \to w$. There is a sequence z_n in D such that $f(z_n) = w_n$. By assumption z_n has an accumulation point z in D, hence there is a subsequence $z_{n_k} = \zeta_k$ which converges to a point $z \in D$. In particular, $w = \lim_k f(\zeta_k) = f(z) \in W$, which is a contradiction.

The assumption on cluster sets and simple connectedness of W imply that f is a proper map (the inverse images of compact sets are compact sets). This, and the assumption that f is a local homeomorphism, implies that $f : D \to W$ is a covering map, see [4]. Since W and D are simply connected domains this covering map is a homeomorphism. □

5. Weakly closed and weakly exact $1-$ forms

Proposition 3. *Let $\omega = \sum_j a_j(x)dx_j$ be a weakly closed differential form with continuous coefficients on a simply connected domain $\Omega \subset \mathbb{R}^n$, i.e.*

$$\partial a_j/\partial x_k = \partial a_k/\partial x_j \quad \textit{in the sense of distributions,} \quad 1 \le j,k \le n.$$

Then there is a function $f \in C^1(\Omega)$ such that $df = w$.

Moreover, for a fixed $x_0 \in \Omega$, f is uniquelly determined by the condition $f(x_0) = 0$ and given by a formula

$$f(x) = \int_\gamma \omega, \tag{22}$$

where γ is any piecewise smooth path from x_0 to x.

Proof. It suffices to prove the proposition in the special case where Ω is a ball $B = B(y,R)$, the general case follows easily by standard topological arguments.

Let, for $\varepsilon > 0$, $\omega_\varepsilon = \omega * \chi_\varepsilon$. By this we mean $\omega_\varepsilon = \sum_j a_{j,\varepsilon}dx_j$ where $a_{j,\varepsilon} = a_j * \chi_\varepsilon$ are C^∞ functions in $B_{-\varepsilon} = B(y, R-\varepsilon)$ for $1 \le j \le n$. Then, for $1 \le j,k \le n$ we have:

$$\frac{\partial a_{j,\varepsilon}}{\partial x_k} = \frac{\partial}{\partial x_k}(a_j * \chi_\varepsilon) = \left(\frac{\partial a_j}{\partial x_k}\right) * \chi_\varepsilon = \left(\frac{\partial a_k}{\partial x_j}\right) * \chi_\varepsilon = \frac{\partial a_{k,\varepsilon}}{\partial x_j}.$$

Therefore, ω_ε is a C^∞ closed form in $B_{-\varepsilon}$. Hence, setting

$$f_\varepsilon(x) = \int_{\gamma_x} \omega_\varepsilon, \tag{23}$$

where γ_x is any smooth path connecting a fixed point x_0 to x within $B_{-\varepsilon}$, we obtain $f_\varepsilon \in C^\infty(B_{-\varepsilon})$ such that $df_\varepsilon = \omega_\varepsilon$ and $f_\varepsilon(x_0) = 0$.

It is an immediate consequence of continuity of coefficients a_j that the limit $\lim_{\varepsilon\to 0} \omega_\varepsilon = \omega$ is locally uniform in B. Hence the convergence of df_ε to ω as ε tends to 0 is locally uniform. Since $f_\varepsilon(x_0) = 0$, this gives locally uniform convergence of f_ε to a C^1 function f on B, moreover $df = \omega$. A simple passage to the limit in (23) shows that f is given by (22). □

Corollary 1. *Assume p and q are continuous complex valued functions on a simply connected domain $D \subset \mathbb{C}$ satisfying*

$$\frac{\partial}{\partial \overline{z}} p(z) = \frac{\partial}{\partial z} q(z) \tag{24}$$

in the sense of distribution theory. Then there is a continuously differentiable function $f : D \to \mathbb{C}$ such that $df = pdz + qd\overline{z}$.

Proof. The assumption (24) can be written in the following form:

$$p_x + ip_y = q_x - iq_y \qquad \text{weakly.}$$

This means that the form $pdz + qd\overline{z} = p(dx+idy) + q(dx-idy)$ is weakly closed. Hence the statement follows from Proposition 3. □

Proposition 4. *Let D, p, q and f be as in Corollary 1. Moreover, assume that $|p| > 0$ and $|q| \le k|p|$ for some constant $0 \le k < 1$. Then f is a C^1 local homeomorphism, $f(D) = W$ is open and $f : D \to W$ is quasiregular. In particular, if f is proper then it is quasiconformal.*

Proof. Since $J_f = |p|^2 - |q|^2 > 0$, the Inverse Function Theorem implies that f is a C^1 local homeomorphism and thus W is open. The condition $|q| \le k|p|$ gives quasiregularity of f. The last statement follows from Proposition 2. □

Remark 1. Concerning our discussion on Theorem 2 from [6], D. Kalaj published, based on communication with the second author, a Corrigendum to [6], see [7].

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