Український математичний вісник
Том 18 **(2021)**, № 3, 338 – 358
https://doi.org/10.37069/1810-3200-2021-18-3-4

Schwartz-type boundary value problems for canonical domains in a biharmonic plane

Serhii V. Gryshchuk, Sergiy A. Plaksa

(Presented by V. Ryazanov)

Dedicated to V.Ya. Gutlyanskii on the occasion of his 80th birthday

Abstract. A commutative algebra $\mathbb{B}$ over the complex field with a basis $\{e_1, e_2\}$ satisfying the conditions $(e_1^2+e_2^2)^2 = 0$, $e_1^2+e_2^2 \neq 0$, is considered. This algebra is associated with the 2-D biharmonic equation. We consider Schwartz-type boundary value problems on finding a monogenic function of the type $\Phi(xe_1 + ye_2) = U_1(x,y)\,e_1 + U_2(x,y)\,ie_1 + U_3(x,y)\,e_2 + U_4(x,y)\,ie_2$, $(x,y) \in D$, when values of two components either U_1, U_3 or U_1, U_4 are given on the boundary of a domain D lying in the Cartesian plane xOy. For solving these boundary value problems for a half-plane and for a disk, we develop methods, which are based on expressions of solutions by means of Schwartz-type integrals, and obtain solutions in the explicit forms.

2010 MSC. 30G35, 31A30.

Key words and phrases. Biharmonic equation, biharmonic algebra, biharmonic plane, monogenic function, Schwartz-type boundary value problem, biharmonic Schwartz-type integral.

1. Monogenic functions in a biharmonic plane

We say that an associative commutative two-dimensional algebra $\mathbb{B}$ with unit 1 over the field of complex numbers $\mathbb{C}$ is *biharmonic* (this notion is proposed in [1]) if in $\mathbb{B}$ there exists a *biharmonic* basis, i.e., a basis $\{e_1, e_2\}$ satisfying the conditions

$$(e_1^2+e_2^2)^2 = 0, \qquad e_1^2+e_2^2 \neq 0\,. \tag{1.1}$$

V. F. Kovalev and I. P. Mel'nichenko [1] found a multiplication table for a biharmonic basis $\{e_1, e_2\}$:

$$e_1 = 1, \qquad e_2^2 = e_1 + 2ie_2, \tag{1.2}$$

Received 19.08.2021

where i is the imaginary complex unit.

E. Study [2] proved that there exist only two type (up to isomorphism) of two-dimensional algebra with 1 over the field $\mathbb{C}$. In the paper [3] I. P. Mel'nichenko proved that there exists the unique biharmonic algebra $\mathbb{B}$ and constructed all biharmonic bases in $\mathbb{B}$. It means that only one of algebras considered by E. Study [2] is biharmonic. Note that the algebra $\mathbb{B}$ is isomorphic to four-dimensional over the field of real numbers $\mathbb{R}$ algebras considered by A. Douglis [4] and L. Sobrero [5].

Consider a *biharmonic plane* $\mu := \{\zeta = x\,e_1 + y\,e_2 : x, y \in \mathbb{R}\}$ which is a linear span over the field $\mathbb{R}$ of the elements of biharmonic basis $\{e_1, e_2\}$ which satisfies (1.2).

With a domain D of the Cartesian plane xOy we associate the congruent domain $D_\zeta := \{\zeta = xe_1 + ye_2 : (x, y) \in D\}$ in the biharmonic plane μ and the congruent domain $D_z := \{z = x + iy : (x, y) \in D\}$ in the complex plane $\mathbb{C}$. Their boundaries are denoted by ∂D, ∂D_ζ and ∂D_z, respectively. Let $\overline{D_\zeta}$ (or $\overline{D_z}$, $\overline{D}$) be the closure of domain D_ζ (or D_z, D, respectively).

In what follows, $\zeta = x\,e_1 + y\,e_2$ and $z = x + iy$, where $(x, y) \in \overline{D}$.

Any function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ has an expansion

$$\Phi(\zeta) = U_1(x, y)\,e_1 + U_2(x, y)\,ie_1 + U_3(x, y)\,e_2 + U_4(x, y)\,ie_2\,, \qquad (1.3)$$

where $U_l\colon D \longrightarrow \mathbb{R}$, $l = \overline{1, 4}$, are real-valued component-functions. We use the following denotation:

$$\mathrm{U}_l\,[\Phi] := U_l,\ l = \overline{1, 4}.$$

We use also the euclidian norm $\|a\| := \sqrt{|z_1|^2 + |z_2|^2}$ in the algebra $\mathbb{B}$, where $a = z_1e_1 + z_2e_2$ and $z_1, z_2 \in \mathbb{C}$.

We say that a function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ is *monogenic* in a domain D_ζ if it has the classical derivative

$$\Phi'(\zeta) := \lim_{h\to 0,\, h\in\mu} \big(\Phi(\zeta + h) - \Phi(\zeta)\big)\,h^{-1}$$

at every point $\zeta \in D_\zeta$.

By *biharmonic functions* we call functions $W\colon D \longrightarrow \mathbb{R}$ which have continuous partial derivatives up to the fourth order inclusively and satisfy the biharmonic equation in the domain D:

$$\Delta^2\,W(x, y) \equiv \frac{\partial^4 W(x, y)}{\partial x^4} + 2\frac{\partial^4 W(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 W(x, y)}{\partial y^4} = 0\,. \qquad (1.4)$$

It is established in [6, 7] that every monogenic function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ has derivatives $\Phi^{(n)}(\zeta)$ of all orders n in the domain D_ζ and, therefore, it satisfies the two-dimensional biharmonic equation (1.4) in the domain D due to the relations (1.1) and the equality

$$\Delta^2\Phi(\zeta) = \Phi^{(4)}(\zeta)\,(e_1^2 + e_2^2)^2.$$

Therefore, we named in [8] such a function Φ *biharmonic monogenic function* in D_ζ.

Every component $U_l\colon D \longrightarrow \mathbb{R}$, $l = \overline{1,4}$, of the expansion (1.3) of a biharmonic monogenic function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ satisfies the equation (1.4) also, i.e., U_l is a biharmonic function in the domain D.

At the same time, every biharmonic in D function $U(x,y)$ is the first component $U_1 \equiv U$ in the expression (1.3) of a certain biharmonic monogenic function $\Phi : D_\zeta \longrightarrow \mathbb{B}$ and, moreover, all such functions Φ are found in [6, 7] in an explicit form.

We shall consider also a non-biharmonic basis $\{1, \rho\}$ with the nilpotent element

$$\rho = 2e_1 + 2ie_2 \tag{1.5}$$

for which $\rho^2 = 0$.

Every monogenic function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ is expressed via two corresponding analytic functions $F\colon D_z \longrightarrow \mathbb{C}$, $F_0\colon D_z \longrightarrow \mathbb{C}$ of the complex variable z in the form (cf., e.g., [6, 7]):

$$\Phi(\zeta) = F(z)e_1 - \left(\frac{iy}{2}\,F'(z) - F_0(z)\right)\rho \quad \forall\,\zeta \in D_\zeta\,. \tag{1.6}$$

The equality (1.6) establishes one-to-one correspondence between monogenic functions Φ in the domain D_ζ and pairs of complex-valued analytic functions F, F_0 in the domain D_z.

2. Schwarz-type boundary value problems for monogenic functions

Consider a boundary value problem on finding a function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ which is monogenic in a domain D_ζ when limiting values of two component-functions in (1.3) are given on the boundary ∂D_ζ, i.e., the following boundary conditions are satisfied:

$$U_k(x_\circ, y_\circ) = u_k(\zeta_\circ)\,, \quad U_m(x_\circ, y_\circ) = u_m(\zeta_\circ) \quad \forall\,\zeta_\circ = x_\circ e_1 + y_\circ e_2 \in \partial D_\zeta$$

for $1 \le k < m \le 4$, where

$$U_l(x_\circ, y_\circ) = \lim_{\zeta\to\zeta_\circ,\zeta\in D_\zeta} \mathrm{U}_l\left[\Phi\left(\zeta\right)\right], \qquad l \in \{k, m\},$$

and u_k, u_m are given continuous functions.

We demand additionally the existence of finite limits

$$\lim_{\|\zeta\|\to\infty,\,\zeta\in D_\zeta} \mathrm{U}_l\left[\Phi(\zeta)\right], \qquad l\in\{k,m\},$$

in the case where the domain D_ζ is unbounded as well as the assumption that every given function u_l, $l\in\{k,m\}$, has a finite limit

$$u_l(\infty):=\lim_{\|\zeta\|\to\infty,\,\zeta\in\partial D_\zeta} u_l(\zeta)$$

if ∂D_ζ is unbounded.

We call such a problem by the (k-m)-problem.

V. F. Kovalev [9] considered (k-m)-problems with additional assumptions that the sought-for function $\Phi:\overline{D_\zeta}\longrightarrow\mathbb{B}$ is continuous in $\overline{D_\zeta}$ and has the limit

$$\lim_{\|\zeta\|\to\infty,\,\zeta\in D_\zeta}\Phi(\zeta)=:\Phi(\infty)\in\mathbb{B}$$

in the case where the domain D_ζ is unbounded. He named such problems as *biharmonic Schwarz problems* owing to their analogy with the classic Schwarz problem on finding an analytic function of a complex variable when values of its real part are given on the boundary of domain. In [10,11], we called problems of such a type as (k-m)-*problems in the sense of Kovalev.*

V. F. Kovalev [9] established that all (k-m)-problems are reduced to the main three problems: with $k=1$ and $m\in\{2,3,4\}$, respectively.

It is shown in [9] (see also [12,13]) that the main biharmonic problem is reduced to the (1-3)-problem. A relation between the (1-4)-problem and boundary value problems of the plane elasticity theory is established in [14] (see also [10,15]).

(k-m)-problems in the sense of Kovalev are investigated in the papers [8,9,12–17].

In particular, using the biharmonic Cauchy type integral, in [13] we reduced the (1-3)-problem in the sense of Kovalev to a system of integral equations and established sufficient conditions under which this system has the Fredholm property. It was made for the case where the given boundary functions satisfy the Dini condition and the boundary of domain belongs to a class being wider than the class of Lyapunov curves that was usually required in the plane elasticity theory (cf., e.g., [18–22]). The similar is done for the (1-4)-problem in the sense of Kovalev in [15]. For cases where D_ζ is either a half-plane or a unit disk in the biharmonic plane, the solutions of the (1-3)-problem in the sense of Kovalev and the (1-4)-problem in the sense of Kovalev were found in explicit forms with using of integrals analogous to the classic Schwarz integral in [12] and [16], respectively.

In [11], improving the reduction method of the papers [13] and [15] for the (1-3)-problem and the (1-4)-problem, respectively, we have weakened conditions on the boundary of domain and the given boundary functions in comparison with corresponding results of the papers [13, 15]. Especially, it was shown that the Dini condition on the given boundary functions can be withdrawn, and these functions can only be assumed to be continuous. It was also shown in [10] that the formula of solutions of the (1-4)-problem for a half-plane is the same as one for the (1-4)-problem in the sense of Kovalev, but it is obtained under the mentioned weakening of the assumptions with respect to the given boundary functions.

Below, we shall show that the similar facts are realized for the (1-3)-problem in a half-plane and for both the (1-3)-problem and (1-4)-problem in a unit disk.

3. (1-3)-problem for a half-plane

Consider the (1-3)-problem in the case where the domain D_ζ is the half-plane $\Pi^+ := \{\zeta = xe_1 + ye_2 : y > 0\}$. It is natural to identify the boundary $\partial\Pi^+$ with the real axis $\mathbb{R}$ as well as to consider $\mathbb{R}$ as a subset of the complex plane $\mathbb{C}$.

We shall find solutions of the (1-3)-problem for the half-plane Π^+ in the class $\mathcal{M}_{\Pi^+}$ of functions represented in the form

$$\Phi(\zeta) = \frac{1}{\pi i}\int\limits_{-\infty}^{+\infty} \frac{u(t)(1+t\zeta)}{(t^2+1)}(t-\zeta)^{-1}\,dt =: S_{\Pi^+}[u](\zeta) \quad \forall\,\zeta\in\Pi^+\,, \quad (3.1)$$

where the function $u : \overline{\mathbb{R}} \longrightarrow \mathbb{B}$ is continuous on the extended real axis $\overline{\mathbb{R}} := \mathbb{R}\cup\{\infty\}$.

Here and in what follows, all integrals along the real axis are understood in the sense of their Cauchy principal values, i.e.,

$$\int\limits_{-\infty}^{+\infty} g(t,\cdot)\,dt := \lim_{N\to+\infty}\int\limits_{-N}^{N} g(t,\cdot)\,dt\,,$$

Note that in the case where $u : \overline{\mathbb{R}} \longrightarrow \mathbb{R}$ is a continuous real-valued function, the function $S_{\Pi^+}[u](\zeta)$ is the principal extension (see [23, p. 165]) into the half-plane Π^+ of the complex Schwarz integral

$$S[u](z) := \frac{1}{\pi i}\int\limits_{-\infty}^{+\infty} \frac{u(t)(1+tz)}{(t^2+1)(t-z)}\,dt\,,$$

which determines a holomorphic function in the half-plane $\{z = x + iy : y > 0\}$ of the complex plane with the given boundary values $u(t)$ of real part on the real axis. Furthermore, the equality

$$S_{\Pi^+}[u](\zeta) = S[u](z)e_1 - \frac{y}{2\pi}\,\rho \int\limits_{-\infty}^{\infty} \frac{u(t)}{(t-z)^2}dt \qquad \forall\,\zeta \in \Pi^+ \tag{3.2}$$

holds, and the following relations were proved within the proof of Theorem 1 in [12]:

$$y \int\limits_{-\infty}^{\infty} \frac{u(t)}{(t-z)^2}dt \leq 4\,\omega_{\mathbb{R}}(u, 2y)$$

$$+2\,y \int\limits_{2y}^{\infty} \frac{\omega_{\mathbb{R}}(u,\eta)}{\eta^2}\,d\eta \to 0, \quad z \to \xi\,, \quad \forall\,\xi \in \mathbb{R}\,, \tag{3.3}$$

where

$$\omega_{\mathbb{R}}(u, \varepsilon) = \sup_{t_1,t_2\in\mathbb{R}:|t_1-t_2|\leq\varepsilon} |u(t_1) - u(t_2)|$$

is the modulus of continuity of the function u.

In addition, using the change of variables $t = -1/t_1$ and $z = -1/z_1$, we obtain the following relation in the similar way as (3.3):

$$y \int\limits_{-\infty}^{\infty} \frac{u(t)}{(t-z)^2}dt \to 0, \quad z \to \infty\,. \tag{3.4}$$

Thus, it follows from the relations (3.2)–(3.4) that

$$U_1\Big[S_{\Pi^+}[u](\zeta)\Big] \to u(\xi)\,, \quad \zeta \to \xi, \quad \forall\,\xi \in \overline{\mathbb{R}}\,, \tag{3.5}$$

i.e., $S_{\Pi^+}[u]$ is the *biharmonic Schwarz integral* for the half-plane Π^+.

To describe all solutions of (1-3)-problem for Π^+ in the class $\mathcal{M}_{\Pi^+}$, first consider the homogeneous (1-3)-problem.

Theorem 3.1. *All solutions $\Phi \in \mathcal{M}_{\Pi^+}$ of the homogeneous* (1-3)*-problem for Π^+ with zero data $u_1 = u_3 \equiv 0$ are expressed in the form*

$$\Phi(\zeta) = a_1\, ie_1 + a_2\, ie_2\,, \tag{3.6}$$

where a_1, a_2 are any real constants.

Proof. For a monogenic function $\Phi \in \mathcal{M}_{\Pi^+}$ we use the expression (1.6) via two holomorphic functions F and F_0, where $D_\zeta = \Pi^+$.

Let us consider the linear functional $f : \mathbb{B} \to \mathbb{C}$ such that $f(e_1) = 1$ and $f(\rho) = 0$. It is well known [23, p. 135] that the functional f is also continuous and multiplicative due to the fact that its kernel is a maximal ideal of the algebra $\mathbb{B}$. Therefore, from the equalities (1.6), (3.1) it follows that

$$F(z) = f(\Phi(\zeta)) = f(S_{\Pi^+}[u](\zeta)) = S[v](z) \quad \forall\, z \in \mathbb{C} : \operatorname{Im} z > 0\,,$$

where $v(t) := f(u(t))$ for all $t \in \mathbb{R}$. Then

$$F'(z) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{v(t)}{(t-z)^2}\, dt \qquad \forall\, z \in \mathbb{C} : \operatorname{Im} z > 0\,,$$

and for the function $yF'(z)$ the relations of the form (3.3), (3.4) hold. As a result, we obtain the equality

$$\lim_{z \to \xi,\, y>0} yF'(z) = 0 \qquad \forall\, \xi \in \overline{\mathbb{R}}\,. \tag{3.7}$$

Further, using the equality (1.5), we rewrite the expansion (1.6) for all $\zeta \in \Pi^+$ in the basis $\{e_1, e_2\}$:

$$\Phi(\zeta) = \Big(F(z) - iyF'(z) + 2F_0(z)\Big)e_1 + i\Big(2F_0(z) - iyF'(z)\Big)e_2\,. \tag{3.8}$$

Thus, taking into account the relation (3.7), one can deduce that the homogeneous (1-3)-problem for Π^+ is reduced to finding holomorphic functions F, F_0 by solving two classical Schwarz problem for the half-plane $\{z = x + iy : y > 0\}$ with the following boundary conditions:

$$\operatorname{Re}\Big(F(\xi) + 2F_0(\xi)\Big) = 0, \qquad \operatorname{Re}\Big(2iF_0(\xi)\Big) = 0 \qquad \forall\, \xi \in \overline{\mathbb{R}}.$$

In such a way we obtain $F_0(z) \equiv a_2/2$ and $F(z) \equiv -a_2 + ia_1$, where a_1, a_2 are any real constants.

Finally, substituting the obtained functions F, F_0 into the equality (3.8), we obtain the equality (3.6). □

In the following theorem we establish the formula of solutions of the (1-3)-problem for the half-plane Π^+ in the class $\mathcal{M}_{\Pi^+}$.

Theorem 3.2. *Let the functions $u_1 : \overline{\mathbb{R}} \longrightarrow \mathbb{R}$ and $u_3 : \overline{\mathbb{R}} \longrightarrow \mathbb{R}$ be continuous. Then the* (1-3)*-problem for Π^+ is solvable in the class $\mathcal{M}_{\Pi^+}$, and the general solution is expressed in the form*

$$\Phi(\zeta) = S_{\Pi^+}[u_1](\zeta)\, e_1 + S_{\Pi^+}[u_3](\zeta)\, e_2 + a_1\, ie_1 + a_2\, ie_2, \tag{3.9}$$

where a_1, a_2 are any real constants.

Proof. It follows from the relation (3.5) that the function

$$S_{\Pi^+}[u_1](\zeta)\, e_1 + S_{\Pi^+}[u_3](\zeta)\, e_2$$

is a particular solution of the (1-3)-problem.

Now, it is obvious that the formula (3.9) represents the solution of the (1-3)-problem for Π^+ in the class $\mathcal{M}_{\Pi^+}$ as the sum of the mentioned particular solution and the general solution (3.6) of the homogeneous (1-3)-problem. □

In Theorem 3 [12] we obtained the general solution of (1-3)-problem in the sense of Kovalev in the form (3.9) but under complementary assumptions that for every given function $u_l\colon \overline{\mathbb{R}} \longrightarrow \mathbb{R}$, $l \in \{1,3\}$, its modulus of continuity and the local centered (with respect to the infinitely remote point) modulus of continuity satisfy Dini conditions.

4. A biharmonic analogue of Schwarz integral for a disk

In what follows, $D_\zeta := \{\zeta = xe_1 + ye_2 : \|\zeta\| \le 1\}$ is the unit disk in the biharmonic plane μ and $D_z := \{z = x + iy : |z| \le 1\}$ is the unit disk in the complex plane $\mathbb{C}$.

For a continuous function $u\colon \partial D_\zeta \longrightarrow \mathbb{R}$, by $\widehat{u}$ we denote the function defined on the unit circle ∂D_z of the complex plane $\mathbb{C}$ by the equality $\widehat{u}(z) = u(\zeta)$ for all $z \in \partial D_z$.

Consider the integral

$$S_{D_\zeta}[u](\zeta) := \frac{1}{2\pi i} \int\limits_{\partial D_\zeta} u(\tau)(\tau+\zeta)(\tau-\zeta)^{-1}\, \tau^{-1}\, d\tau \qquad \forall\, \zeta \in D_\zeta \tag{4.1}$$

that is an analogue of the complex Schwarz integral

$$S_0[\widehat{u}](\zeta) := \frac{1}{2\pi i} \int\limits_{\partial D_z} \widehat{u}(t)\, \frac{t+z}{t-z}\, \frac{dt}{t} \qquad \forall\, z \in D$$

which determines a holomorphic function in the disk D_z with the given boundary values $\widehat{u}(t)$ of real part on the circle ∂D_z.

Some limiting properties of the integral (4.1) when ζ tends to a point of the circle ∂D_ζ are studied in [12] under the assumption that the modulus of continuity of the function u satisfies the Dini condition.

Now, supposing only the continuity of the function u, we shall consider some limiting properties of the integral (4.1) and shall prove some auxiliary statements.

Lemma 4.1. *If a function* $u\colon \partial D_\zeta \longrightarrow \mathbb{R}$ *is continuous, then the integral* (4.1) *is expressed in the form*

$$S_{D_\zeta}[u](\zeta) = S_0[\widehat{u}](z)\, e_1 + (I_3(z) + I_4(z))\, \frac{i\,\rho}{2} \quad \forall\, \zeta \in D_\zeta\,. \tag{4.2}$$

where

$$I_3(z) := \frac{1}{2\pi i} \int\limits_{\partial D_z} \frac{\widehat{u}(t)(xt_2 - t_1 y)}{t^2(t-z)}\, dt\,, \tag{4.3}$$

$$I_4(z) := \frac{1}{2\pi} \int\limits_{\partial D_z} \frac{\widehat{u}(t)(t+z)(t_2(t_2-y) + t_1(t_1-x))}{t^2(t-z)^2}\, dt\,, \tag{4.4}$$

and $t_1 := \mathrm{Re}\, t\,,\ \ t_2 := \mathrm{Im}\, t\,.$

Proof. Let $\tau := t_1e_1 + t_2e_2 \in \partial D_\zeta$ and $t := t_1 + it_2 \in \partial D_z$, where t_1, t_2 are real. Taking into account the relations $\zeta = ze_1 - \frac{iy}{2}\rho$, $\tau = te_1 - \frac{it_2}{2}\rho$ and the equalities (cf., e.g., [6])

$$(\tau - \zeta)^{-1} = \frac{1}{t-z}\, e_1 + \frac{(t_2 - y)}{(t-z)^2}\, \frac{i\rho}{2}\,, \qquad \tau^{-1} = \frac{1}{t}\, e_1 + \frac{t_2}{t^2}\, \frac{i\rho}{2}\,,$$

we obtain a chain of equalities

$$(\tau + \zeta)(\tau - \zeta)^{-1}\, \tau^{-1}$$

$$= \left((t+z)e_1 - (t_2+y)\frac{i\rho}{2}\right)\left(\frac{1}{t-z}\, e_1 + \frac{(t_2-y)}{(t-z)^2}\, \frac{i\rho}{2}\right)\left(\frac{1}{t}\, e_1 + \frac{t_2}{t^2}\, \frac{i\rho}{2}\right)$$

$$= \left(\frac{t+z}{t-z}\, e_1 - \left(\frac{t_2+y}{t-z} - \frac{(t+z)(t_2-y)}{(t-z)^2}\right)\frac{i\rho}{2}\right)\left(\frac{1}{t}\, e_1 + \frac{t_2}{t^2}\, \frac{i\rho}{2}\right)$$

$$= \frac{t+z}{t(t-z)}\, e_1 + \left(\frac{t_2(t+z)}{t^2(t-z)} - \frac{t_2+y}{t(t-z)} + \frac{(t+z)(t_2-y)}{t(t-z)^2}\right)\frac{i\rho}{2}$$

$$= \frac{t+z}{t(t-z)}\, e_1 + \left(\frac{xt_2 - t_1y}{t^2(t-z)} + \frac{(t+z)(t_2-y)}{t(t-z)^2}\right)\frac{i\rho}{2}\,. \tag{4.5}$$

Substituting (4.5) and the expression $d\tau = e_1 dt - \frac{i}{2}\rho\, dt_2$ into (4.1), we obtain

$$S_{D_\zeta}[u](\zeta) = S_0[\widehat{u}](z)\, e_1 + \frac{i\,\rho}{2}\, I_3(z)$$

$$+\frac{i\,\rho}{2}\, \frac{1}{2\pi i} \int\limits_{\partial D_z} \widehat{u}(t)\left(\frac{(t+z)(t_2-y)}{t(t-z)^2}\, dt - \frac{t+z}{t(t-z)}\, dt_2\right)$$

$$= S_0[\widehat{u}](z)\, e_1 + \frac{i\,\rho}{2}\, I_3(z) + \frac{i\,\rho}{2}\, \frac{1}{2\pi i} \int\limits_{\partial D_z} \widehat{u}(t)\left(\frac{(t+z)\big((t_2-y)\, dt_1 - (t_1-x)\, dt_2\big)}{t(t-z)^2}\right).$$

Finally, to complete the proof, it remains to use the relations $dt_1 = \frac{it_2}{t}\, dt\,,\ \ dt_2 = -\frac{it_1}{t}\, dt\,.$ □

In the next statement we use a singular integral which is understood in the sense of its Cauchy principal value, i.e.

$$\int\limits_{\partial D_z} \frac{g(t,\cdot)}{(t-z_\circ)^2}\,dt := \lim_{\varepsilon\to 0+0} \int\limits_{\{t\in\partial D_z\,:\,|t-z_\circ|\geq\varepsilon\}} \frac{g(t,\cdot)}{(t-z_\circ)^2}\,dt \ \ \forall\, z_\circ\in\partial D_z\,.$$

Lemma 4.2. *If a function* $\widehat{u}\colon \partial D_z \longrightarrow \mathbb{R}$ *is continuous, then the integral* (4.4) *is continuously extended from* D_z *onto the boundary* ∂D_z*, and the following equality holds:*

$$\lim_{z\to z_\circ,\, z\in D_z} I_4(z) = I_4(z_\circ) \quad \forall\, z_\circ\in\partial D_z, \tag{4.6}$$

where

$$I_4(z_\circ) := \frac{1}{2\pi}\int\limits_{\partial D_z} \frac{\widehat{u}(t)(t+z)(t_2(t_2-y_\circ)+t_1(t_1-x_\circ))}{t^2(t-z_\circ)^2}\,dt\,, \tag{4.7}$$

and $t_1 := \mathrm{Re}\, t\,,\ \ t_2 := \mathrm{Im}\, t\,,\ \ x_\circ := \mathrm{Re}\, z_\circ\,,\ \ y_\circ := \mathrm{Im}\, z_\circ\,.$

Proof. Let $z_* = x_* + iy_*$, where $x_*, y_* \in \mathbb{R}$, be the point of ∂D_z which is the nearest to a point $z\in D_z$, $z\neq 0$. Let us represent the integral (4.4) in the form

$$I_4(z) = \frac{1}{2\pi}\int\limits_{\partial D_z} \frac{G(t,z)-G(t,z_*)}{(t-z)^2}\,dt + \frac{1}{2\pi}\int\limits_{\partial D_z} \frac{G(t,z_*)}{(t-z)^2}\,dt =:$$

$$=: I_4'(z) + I_4''(z)\,,$$

where $G(t,z) := \widehat{u}(t)(t+z)\big(t_2(t_2-y)+t_1(t_1-x)\big)/t^2$.

Consider the difference

$$G(t,z)-G(t,z_*)$$

$$= \left(\frac{\widehat{u}(t)(t+z)}{t^2} - \frac{\widehat{u}(t)(t+z_*)}{t^2}\right)\big(t_2(t_2-y_*)+t_1(t_1-x_*)\big)$$

$$-\left(\frac{\widehat{u}(t)(t+z)}{t^2} - \frac{\widehat{u}(z_*)(z_*+z_*)}{z_*^2}\right)\big(t_2(y-y_*)+t_1(x-x_*)\big)$$

$$-\frac{2\widehat{u}(z_*)}{z_*}\big(t_2(y-y_*)+t_1(x-x_*)\big) =: G_1(t,z)+G_2(t,z)+G_3(t,z)$$

and represent $I_4'(z)$ as the following sum:

$$I_4'(z) = \frac{1}{2\pi}\sum_{m=1}^{3}\int\limits_{\partial D_z} \frac{G_m(t,z)}{(t-z)^2}\,dt\,.$$

Denote $\Gamma_1 := \{t \in \partial D_z : |t - z_*| \le 2\left(1 - |z|\right)\}$, $\Gamma_2 := \partial D_z \setminus \Gamma_1$.

Assume by c to denote constants whose values are independent of t, z and z_*, but, generally speaking, may be different even within a single chain of inequalities.

Taking into account the inequalities $|t-z| \ge \left(1-|z|\right)$ for all $t \in \partial D_z$, $|t - z| \ge |t - z_*|/2$ for all $t \in \Gamma_2$, $|G_1(t,z)| \le c\left(1 - |z|\right)|t - z_*|$ for all $t \in \partial D_z$, we obtain the relations

$$\left|\int\limits_{\partial D_z} \frac{G_1(t,z)}{(t-z)^2}\,dt\right| \le \int\limits_{\Gamma_1} \frac{|G_1(t,z)|}{|t-z|^2}\,|dt| + \int\limits_{\Gamma_2} \frac{|G_1(t,z)|}{|t-z|^2}\,|dt|$$

$$\le c\left(1-|z|\right)\left(\int\limits_{\Gamma_1} \frac{|t-z_*|}{\left(1-|z|\right)^2}\,|dt| + \int\limits_{\Gamma_2} \frac{|t-z_*|}{|t-z_*|^2}\,|dt|\right)$$

$$\le c\left(1-|z|\right)\,\ln\frac{1}{\left(1-|z|\right)} \to 0, \quad |z| \to 1. \tag{4.8}$$

For the function G_2, the following estimate is fulfilled:

$$|G_2(t,z)| \le c\left(1-|z|\right)\,\left(\omega(u, |t-z_*|) + |t-z_*| + |t-z|\right),$$

where

$$\omega(u,\varepsilon) = \sup_{t_1,t_2 \in \partial D_z : |t_1 - t_2| \le \varepsilon} |u(t_1) - u(t_2)|$$

is the modulus of continuity of the function u. Therefore, in a similar way as (4.8), we obtain

$$\left|\int\limits_{\partial D_z} \frac{G_2(t,z)}{(t-z)^2}\,dt\right| \le c\left(\omega\Big(u, 2\big(1-|z|\big)\Big) + \left(1-|z|\right)\int\limits_{1-|z|}^{2} \frac{\omega(u,\eta)}{\eta^2}\,d\eta\right.$$

$$\left. +\left(1-|z|\right)\ln\frac{1}{\left(1-|z|\right)}\right) \to 0, \quad |z| \to 1.$$

We have also

$$\left|\int\limits_{\partial D_z} \frac{G_3(t,z)}{(t-z)^2}\,dt\right|$$

$$= \left|\frac{2\widehat{u}(z_*)}{z_*}\right|\left|\int\limits_{\partial D_z} \frac{(t_2-y_*)(y-y_*) + (t_1-x_*)(x-x_*))}{(t-z)^2}\,dt\right|,$$

and further we obtain the same relations as (4.8).

Thus,

$$I_4'(z) \to 0, \quad |z| \to 1\,. \tag{4.9}$$

Before to consider $I_4''(z)$, let us point out an estimate for the function G. Using the change of variables $t = \exp(i\theta)$, $z_* = \exp(i\theta_*)$, we obtain the relations

$$|G(t, z_*)| \le c\,\big|t_2(t_2 - y_*) + t_1(t_1 - x_*)\big|$$

$$= c\,\big|1 - \cos(\theta - \theta_*)\big| \le c\,|t - z_*|^2 \tag{4.10}$$

which imply the existence of singular integral (4.7).

Now, let us fix a positive ε such that $\varepsilon \ge 2\,|z - z_\circ|$. Denote $\Gamma_\varepsilon := \{t \in \partial D_z : |t - z_\circ| \le \varepsilon\}$ and represent $I_4''(z)$ in the form

$$I_4''(z) = \frac{1}{2\pi}\int\limits_{\Gamma_\varepsilon} \frac{G(t, z_*)}{(t - z)^2}\,dt + \frac{1}{2\pi}\int\limits_{\partial D_z \setminus \Gamma_\varepsilon} \frac{G(t, z_*)}{(t - z)^2}\,dt =: J_1(z) + J_2(z)\,.$$

Taking into account the estimate (4.10) and the inequality $|t - z_*| \le 2|t - z|$ for all $t \in \Gamma_\varepsilon$, we obtain the estimate

$$|J_1(z)| \le c_1\,\varepsilon\,, \tag{4.11}$$

where the constant c_1 does not depend on z, z_* and ε.

Furthermore, it is evident that

$$J_2(z) \to \frac{1}{2\pi}\int\limits_{\partial D_z \setminus \Gamma_\varepsilon} \frac{G(t, z_\circ)}{(t - z_\circ)^2}\,dt, \quad z \to z_\circ\,. \tag{4.12}$$

Now, taking into account the relations (4.9), (4.11), (4.12), we can state that for any $\varepsilon > 0$ there exists a positive number $\delta < \varepsilon/2$ such that for all $z \in D_z : |z - z_\circ| < \delta$ the following inequality is fulfilled:

$$\left| I_4(z) - \frac{1}{2\pi}\int\limits_{\partial D_z \setminus \Gamma_\varepsilon} \frac{G(t, z_\circ)}{(t - z_\circ)^2}\,dt \right| \le c_0\,\varepsilon\,, \tag{4.13}$$

where c_0 is a fixed constant independent of z, $z_\circ$ and ε.

Finally, the inequality (4.13) implies the equality (4.6) as $\varepsilon \to 0$. □

Now, we can prove the main lemma of this section.

Lemma 4.3. *If a function* $u\colon \partial D_\zeta \longrightarrow \mathbb{R}$ *is continuous, then the real-valued component-functions* $U_1[S_{D_\zeta}[u]]$, $U_3[S_{D_\zeta}[u]]$, $U_4[S_{D_\zeta}[u]]$ *of integral* (4.1) *are continuously extended to the boundary* ∂D, *and the following equalities hold:*

$$\lim_{\zeta\to\zeta_\circ,\,\zeta\in D_\zeta} U_1[S_{D_\zeta}[u](\zeta)] = u(\zeta_\circ) + Bx_\circ - Ay_\circ + D \quad \forall\,\zeta_\circ \in \partial D_\zeta\,, \tag{4.14}$$

$$\lim_{\zeta\to\zeta_\circ,\,\zeta\in D_\zeta} U_3[S_{D_\zeta}[u](\zeta)] = Ax_\circ + By_\circ + C \quad \forall\,\zeta_\circ \in \partial D_\zeta\,, \tag{4.15}$$

$$\lim_{\zeta\to\zeta_\circ,\,\zeta\in D_\zeta} U_4[S_{D_\zeta}[u](\zeta)] = Bx_\circ - Ay_\circ + D \quad \forall\,\zeta_\circ \in \partial D_\zeta\,, \tag{4.16}$$

where real $x_\circ$, $y_\circ$ *such that* $\zeta_\circ = x_\circ e_1 + y_\circ e_2$, *and*

$$A := \frac{1}{2\pi}\mathrm{Re}\int\limits_{\partial D_z} \frac{\widehat{u}(t)}{t^2}\,dt, \quad B := \frac{1}{2\pi}\mathrm{Im}\int\limits_{\partial D_z} \frac{\widehat{u}(t)}{t^2}\,dt,$$
$$C := \frac{1}{2\pi}\mathrm{Re}\int\limits_{\partial D_z} \frac{\widehat{u}(t)}{t^3}\,dt, \quad D := \frac{1}{2\pi}\mathrm{Im}\int\limits_{\partial D_z} \frac{\widehat{u}(t)}{t^3}\,dt\,.$$

Proof. Let us use the equality (4.2). It is evident that $U_3[S_0[\widehat{u}](z)e_1] \equiv U_4[S_0[\widehat{u}](z)e_1] \equiv 0$ and

$$U_1\Big[S_0[\widehat{u}](z)e_1\Big] \to u(\zeta_\circ)\,, \quad \zeta \to \zeta_\circ, \quad \forall\,\zeta_\circ \in \partial D_\zeta\,.$$

In order to consider the integral (4.3), let us denote $\Omega(t,z) := \widehat{u}(t)(xt_2 - t_1 y)/t^2$. The following estimates are fulfilled:

$$|\Omega(t,z)| \le c\left|t_2(x-t_1) + t_1(t_2-y)\right| \le c\,|t-z|\,,$$

$$|\Omega(t,z) - \Omega(t,z_0)| \le c\,|z-z_0|\,,$$

where the constant c does not depend on t, z and z_0. Therefore, by virtue of Lemma 2 [11], the integral $I_3(z)$ is continuously extended from D_z onto the boundary ∂D_z, and the following equality holds:

$$\lim_{z\to z_\circ,\,z\in D_z} I_3(z) = I_3(z_\circ) \quad \forall\,z_\circ \in \partial D_z,$$

where

$$I_3(z_\circ) := \frac{1}{2\pi i}\int\limits_{\partial D} \frac{\widehat{u}(t)(x_\circ t_2 - t_1 y_\circ)}{t^2(t-z_\circ)}\,dt \tag{4.17}$$

exists as an improper integral and $x_\circ = \mathrm{Re}\,z_\circ$, $y_\circ = \mathrm{Im}\,z_\circ$.

The integral $I_4(z)$ is also continuously extended from D_z onto the boundary ∂D_z and the equality (4.6) holds due to Lemma 4.2.

The computations of integrals (4.7) and (4.17) are done in Theorem 4 [12], viz.,

$$I_3(z_\circ) = I_4(z_\circ) = \frac{-x_\circ + iy_\circ}{4\pi} \int\limits_{\partial D_z} \frac{\widehat{u}(t)}{t^2}\, dt - \frac{1}{4\pi} \int\limits_{\partial D_z} \frac{\widehat{u}(t)}{t^3}\, dt\,.$$

Finally, to complete the proof of equalities (4.14) – (4.16), it remains to single out corresponding component-functions for the hypercomplex function $(I_3(z_\circ) + I_4(z_\circ))i\rho/2 \equiv (I_3(z_\circ) + I_4(z_\circ))(ie_1 - e_2)\,.$ □

5. (1-3)-problem for the unit disk

Before to consider the (1-3)-problem for the unit disk D_ζ, let us make some comments.

V. F. Kovalev [9] shown that the biharmonic problem (cf., e.g., [24, p. 194] and [18, p. 13]) on finding a biharmonic function $U : D \longrightarrow \mathbb{R}$ with given limiting values of its partial derivatives $\partial U/\partial x$ and $\partial U/\partial y$ on the boundary ∂D can be reduced to the (1-3)-problem (see also [12, 13]).

It is well-known that the biharmonic problem has a necessary condition of solvability (cf., e.g., [24, p. 195]) which can be rewritten in the terms of (1-3)-problem as the following (cf., e.g., [10]):

$$\int\limits_{\partial D_\zeta} u_1(\zeta)\, dx + u_3(\zeta)\, dy = 0. \tag{5.1}$$

Below, we shall prove that under the condition (5.1), the (1-3)-problem for D_ζ is solvable in the class $\mathcal{M}_{1,3}$ of functions represented in the form

$$\Phi(\zeta) = S_{D_\zeta}[g_1](\zeta)e_1 + S_{D_\zeta}[g_3](\zeta)e_2 \quad \forall\, \zeta \in D_\zeta\,, \tag{5.2}$$

where the functions $g_1 : \partial D_\zeta \longrightarrow \mathbb{R}$, $g_3 : \partial D_\zeta \longrightarrow \mathbb{R}$ are continuous.

The solvability of the (1-3)-problem for D_ζ in the class $\mathcal{M}_{1,3}$ is described in the following theorem.

Theorem 5.1. *Let the functions* $u_1 \colon \partial D_\zeta \longrightarrow \mathbb{R}$ *and* $u_3 \colon \partial D_\zeta \longrightarrow \mathbb{R}$ *be continuous. Then* (1-3)*-problem for* D_ζ *is solvable in the class* $\mathcal{M}_{1,3}$ *if and only if the condition* (5.1) *is satisfied. The general solution is expressed in the form*

$$\Phi(\zeta) = S_{D_\zeta}[u_1](\zeta)\, e_1 + S_{D_\zeta}[u_3](\zeta)\, e_2 + b\zeta + b_1 e_1 + b_2 e_2 + ia\zeta\,, \tag{5.3}$$

where

$$b := -\frac{1}{2\pi}\, \mathrm{Re} \int\limits_{\partial D_z} \frac{\widehat{u}_3(t)}{t^2}\, dt - \frac{1}{2\pi}\, \mathrm{Im} \int\limits_{\partial D_z} \frac{\widehat{u}_1(t)}{t^2}\, dt\,,$$

$$b_1 := -\frac{1}{2\pi}\,\mathrm{Re}\int\limits_{\partial D_z}\frac{\widehat{u}_3(t)}{t^3}\,dt - \frac{1}{2\pi}\,\mathrm{Im}\int\limits_{\partial D_z}\frac{\widehat{u}_1(t)}{t^3}\,dt\,,$$

$$b_2 := \frac{1}{2\pi}\,\mathrm{Im}\int\limits_{\partial D_z}\frac{\widehat{u}_3(t)}{t^3}\,dt - \frac{1}{2\pi}\,\mathrm{Re}\int\limits_{\partial D_z}\frac{\widehat{u}_1(t)}{t^3}\,dt\,,$$

and a is any real constant.

Proof. Let the function (5.2) be a solution of the (1-3)-problem for D_ζ, i.e., the following boundary conditions are satisfied:

$$\lim_{\zeta\to\zeta_\circ,\,\zeta\in D_\zeta} U_k\left[S_{\partial D_\zeta}[g_1](\zeta)\,e_1 + S_{\partial D_\zeta}[g_3](\zeta)\,e_2\right]$$

$$= u_k(\zeta_\circ) \quad \forall\zeta_\circ\in\partial D_\zeta\,,\ \ k=1,3\,. \tag{5.4}$$

Our strategy is to find the functions g_1 and g_3 with using the method of indefinite coefficients and two equations (5.4) with respect to sought-for functions g_1, g_3.

Taking into account the equalities (4.14), (4.15), (5.4) and the multiplication rules (1.2), we deduce that the functions g_1, g_3 can be expressed in the form

$$g_k(\zeta) = u_k(\zeta) + a_{k,1}\,x + a_{k,2}\,y + a_{k,0} \qquad \forall\zeta\in\partial D_\zeta, \tag{5.5}$$

where unknown coefficients $a_{k,m}$ are real numbers for $k = 1,3$ and $m = 0,1,2$.

Let us denote

$$A_k := \frac{1}{2\pi}\,\mathrm{Re}\int\limits_{\partial D_z}\frac{\widehat{u}_k(t)}{t^2}\,dt, \quad B_k := \frac{1}{2\pi}\,\mathrm{Im}\int\limits_{\partial D_z}\frac{\widehat{u}_k(t)}{t^2}\,dt,$$

$$C_k := \frac{1}{2\pi}\,\mathrm{Re}\int\limits_{\partial D_z}\frac{\widehat{u}_k(t)}{t^3}\,dt, \quad D_k := \frac{1}{2\pi}\,\mathrm{Im}\int\limits_{\partial D_z}\frac{\widehat{u}_k(t)}{t^3}\,dt, \qquad k=1,3\,,$$

and substitute the expressions (5.5) into the equations (5.4). Taking into account the relations (4.14) – (4.16), the equalities $\mathrm{U}_1\,[e_2v] = \mathrm{U}_3\,[v]$ and $\mathrm{U}_3\,[e_2v] = \mathrm{U}_1\,[v] - 2\mathrm{U}_4\,[v]$ for all $v\in\mathbb{B}$ and

$$\frac{1}{2\pi}\int\limits_{\partial D_z}\frac{x}{z^2}\,dz = \frac{i}{2}\,, \qquad \frac{1}{2\pi}\int\limits_{\partial D_z}\frac{y}{z^2}\,dz = \frac{1}{2}\,,$$

$$\int\limits_{\partial D_z}\frac{x}{z^3}\,dz = \int\limits_{\partial D_z}\frac{y}{z^3}\,dz = 0\,,$$

we obtain the equalities

$$\left(\frac{3}{2}\,a_{1,1}+\frac{1}{2}\,a_{3,2}+A_3+B_1\right)x_\circ + \left(\frac{1}{2}\,a_{1,2}+\frac{1}{2}\,a_{3,1}-A_1+B_3\right)y_\circ + a_{1,0}+C_3+D_1=0,$$

$$\left(\frac{1}{2}\,a_{1,2}+\frac{1}{2}\,a_{3,1}+A_1-B_3\right)x_\circ + \left(\frac{1}{2}\,a_{1,1}+\frac{3}{2}\,a_{3,2}+A_3+B_1\right)y_\circ + a_{3,0}+C_1-D_3=0\,,$$

where real $x_\circ$, $y_\circ$ such that $\zeta_\circ = x_\circ e_1 + y_\circ e_2$.

Consequently, we have a system of six equations with six real unknowns $a_{k,m}$ (with $k = 1, 3$ and $m = 0, 1, 2$):

$$\begin{aligned} a_{1,0} &= -C_3 - D_1\,, \\ a_{3,0} &= -C_1 + D_3\,, \\ a_{1,2} + a_{3,1} &= 2A_1 - 2B_3\,, \\ a_{1,2} + a_{3,1} &= -2A_1 + 2B_3\,, \\ 3a_{1,1} + a_{3,2} &= -2A_3 - 2B_1\,, \\ a_{1,1} + 3a_{3,2} &= -2A_3 - 2B_1\,. \end{aligned} \tag{5.6}$$

It is obvious that this system is solvable if and only if $A_1 - B_3 = B_3 - A_1$, i.e. $A_1 = B_3$ that is equivalent to the condition (5.1). If this condition is satisfied, then the general solution of the system (5.6) contains an arbitrary real number $a_{1,2}$ and is of the form:

$$\begin{aligned} a_{1,0} &= -C_3 - D_1\,, \\ a_{3,0} &= -C_1 + D_3\,, \\ a_{3,1} &= -a_{1,2}\,, \\ a_{1,1} &= -\frac{1}{2}\,A_3 - \frac{1}{2}\,B_1\,, \\ a_{3,2} &= -\frac{1}{2}\,A_3 - \frac{1}{2}\,B_1\,. \end{aligned} \tag{5.7}$$

Thus, the function (5.2) is the general solution of (1-3)-problem for D_ζ in the class $\mathcal{M}_{1,3}$ if the functions g_1, g_3 are of the form (5.5), where the coefficients $a_{k,m}$ with $k = 1, 3$ and $m = 0, 1, 2$ are determined by the equalities (5.7).

Finally, with using the equalities

$$S_{D_\zeta}[1](\zeta) = e_1,\ S_{D_\zeta}[x](\zeta) = \frac{1}{2}(3e_1 + ie_2)\zeta,\ S_{D_\zeta}[y](\zeta) = \frac{1}{2}(-3ie_1 + e_2)\zeta$$

for any $\zeta \in D_\zeta$, the formula (5.2) is reduced to the form (5.3). □

In Theorem 5 [12] we obtained the general solution of (1-3)-problem in the sense of Kovalev for ∂D_ζ in a somewhat wider class of functions but under a complementary assumption that for the given functions $u_k\colon \partial D_\zeta \longrightarrow \mathbb{R}$, $k \in \{1, 3\}$, their moduli of continuity satisfy the Dini condition. In comparison with the formula (5.3), the solution in [12] contains an additional summand $i(a_1 e_1 + a_2 e_2)$ (where a_1, a_2 are any real constants) that does not belong to the class $\mathcal{M}_{1,3}$, as it follows from the proof of Theorem 5.1.

6. (1-4)-problem for the unit disk

Consider the (1-4)-problem for the unit disk D_ζ.

In contradistinction to the (1-3)-problem, which is solvable if and only if the condition (5.1) is satisfied, it is shown in the next theorem that the (1-4)-problem is solvable unconditionally, and an explicit formula for solution of the (1-4)-problem for D_ζ is obtained.

Theorem 6.1. *Let the functions $u_1\colon \partial D_\zeta \longrightarrow \mathbb{R}$ and $u_4\colon \partial D_\zeta \longrightarrow \mathbb{R}$ be continuous. Then the general solution of* (1-4)*-problem is expressed in the form*

$$\Phi(\zeta) = S_{D_\zeta}[u_1](\zeta)\, e_1 + S_{D_\zeta}[u_4](\zeta)\, ie_2$$

$$+\Big((d_1 + id_2)\zeta + d\Big)(e_1 + ie_2) + a_1\, ie_1 + a_2\, e_2\,, \tag{6.1}$$

where

$$d_1 := -\frac{1}{2\pi}\,\mathrm{Im} \int\limits_{\partial D_z} \frac{\widehat{u}_1(t) - \widehat{u}_4(t)}{t^2}\, dt\,,\ \ d_2 := -\frac{1}{2\pi}\,\mathrm{Re} \int\limits_{\partial D_z} \frac{\widehat{u}_1(t) - \widehat{u}_4(t)}{t^2}\, dt\,,$$

$$d := -\frac{1}{2\pi}\,\mathrm{Im} \int\limits_{\partial D_z} \frac{\widehat{u}_1(t) - \widehat{u}_4(t)}{t^3}\, dt\,,$$

and a_1, a_2 are any real constants.

Proof. First, note that the general solution of the homogeneous (1-4)-problem with zero data $u_1 = u_4 \equiv 0$ is expressed by the formula

$$\Phi_0(\zeta) = a_1 ie_1 + a_2 e_2 \quad \forall\, \zeta \in D_\zeta\,, \tag{6.2}$$

where a_1, a_2 are any real constants (see Theorem 5.2 in [10]).

Further, let us prove that there exists a particular solution of the (1-4)-problem in the form

$$\Phi_p(\zeta) = S_{D_\zeta}[u_1](\zeta)\, e_1 + S_{D_\zeta}[u_4](\zeta)\, ie_2$$

$$+\Big(d_1 e_1 + d_2 ie_1 + d_3 e_2 + d_4 ie_2\Big)\zeta + c_1\, e_1 + c_2\, ie_2 \quad \forall\, \zeta \in D_\zeta\,, \tag{6.3}$$

where unknown real coefficients $d_1, d_2, d_3, d_4, c_1, c_2$ are need to be found to satisfy the following boundary conditions

$$\lim_{\zeta\to\zeta_\circ,\, \zeta\in D_\zeta} U_k\left[\Phi_p(\zeta)\right] = u_k(\zeta_\circ) \quad \forall\, \zeta_\circ \in \partial D_\zeta\,, \;\; k = 1, 4\,. \tag{6.4}$$

Denoting

$$A_k := \frac{1}{2\pi}\,\mathrm{Re} \int\limits_{\partial D_z} \frac{\widehat{u}_k(t)}{t^2}\, dt\,, \quad B_k := \frac{1}{2\pi}\,\mathrm{Im} \int\limits_{\partial D_z} \frac{\widehat{u}_k(t)}{t^2}\, dt\,,$$

$$D_k := \frac{1}{2\pi}\,\mathrm{Im} \int\limits_{\partial D_z} \frac{\widehat{u}_k(t)}{t^3}\, dt, \qquad k = 1, 4\,,$$

and taking into account the relations (4.14), (4.16) and the equalities

$$\lim_{\zeta\to\zeta_\circ,\, \zeta\in D_\zeta} U_1\left[\Phi_p(\zeta)\right] = u_1(\zeta_\circ) +$$
$$+(B_1 - B_4 + d_1)x_\circ - (A_1 - A_4 - d_3)y_\circ + D_1 - D_4 + c_1\,,$$

$$\lim_{\zeta\to\zeta_\circ,\, \zeta\in D_\zeta} U_4\left[\Phi_p(\zeta)\right] = u_4(\zeta_\circ) +$$
$$+(B_1 - B_4 + d_4)x_\circ - (A_1 - A_4 - d_2 - 2d_3)y_\circ + D_1 - D_4 + c_2$$

for $\zeta_\circ \in \partial D_z$, where real $x_\circ$, $y_\circ$ such that $\zeta_\circ = x_\circ e_1 + y_\circ e_2$.

Now, it is clear that the identities (6.4) hold if and only if $d_4 = d_1 = -(B_1 - B_4)$, $d_3 = -d_2 = A_1 - A_4$, $c_1 = c_2 = -(D_1 - D_4)$.

Finally, substituting the found values for the coefficients d_1, d_2, d_3, d_4, c_1, c_2 to the partial solution (6.3) of the inhomogeneous (1-4)-problem and adding the general solution (6.2) of the homogeneous (1-4)-problem, after evident identical transformations we obtain the formula (6.1) for the general solution of the (1-4)-problem for the unit disk. □

In Theorem 4 [16] we obtain the general solution of (1-4)-problem in the sense of Kovalev for ∂D_ζ in the form (6.1) but under a complementary assumption that for the given functions $u_k\colon \partial D_\zeta \longrightarrow \mathbb{R}$, $k \in \{1, 4\}$, their moduli of continuity satisfy the Dini condition.

References

[1] Kovalev, V.F., Mel'nichenko, I.P. (1981). Biharmonic functions on the biharmonic plane. *Reports Acad. Sci. USSR, ser. A.*(8), 25–27 (in Russian).

[2] Study, E. (1890). Über systeme complexer zahlen und ihre anwendungen in der theorie der transformationsgruppen. *Monatshefte für Mathematik, 1*(1), 283–354.

[3] Mel'nichenko, I.P. (1986). Biharmonic bases in algebras of the second rank. *Ukr. Mat. Zh, 38*(2), 224–226; transl. in (1986). *Ukr. Math. J., 38*(2), 252–254.

[4] Douglis, A. (1953). A function-theoretic approach to elliptic systems of equations in two variables. *Communications on Pure and Applied Mathematics, 6*(2), 259–289.

[5] Sobrero, L. (1934). Nuovo metodo per lo studio dei problemi di elasticità, con applicazione al problema della piastra forata. *Ricerche di Ingegneria, 13*(2), 255–264.

[6] Gryshchuk, S.V., Plaksa, S.A. (2009). Monogenic functions in a biharmonic algebra. *Ukr. Mat. Zh., 61*(12), 1587–1596; transl. in (2009). *Ukr. Math. J., 61*(12), 1865–1876.

[7] Gryshchuk, S.V., Plaksa, S.A. (2013). *Basic Properties of Monogenic Functions in a Biharmonic Plane.* In: Agranovskii, M.L., Ben-Artzi, M., Galloway, G., Karp, L., Maz'ya, V. (eds.) Complex Analysis and Dynamical Systems V. Contemporary Mathematics, 591. Amer. Math. Soc., Providence, RI, 127–134.

[8] Gryshchuk, S.V., Plaksa, S.A. (2019). *Biharmonic monogenic functions and biharmonic boundary value problems.* Lindahl K.-O. et al. (eds.). Analysis, Probability, Applications, and Computation. Trends in Mathematics. Proc. of 11th International ISAAC Congress, Växjö (Sweden) 2017, Birkhaüser, 153–160.

[9] Kovalev, V.F. (1986). *Biharmonic Schwarz problem.* Preprint No. 86.16, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (in Russian).

[10] Gryshchuk, S.V., Plaksa, S.A. (2019). *Schwartz-Type Boundary Value Problems for Monogenic Functions in a Biharmonic Algebra.* Rogosin S., Çelebi A. (eds.). Analysis as a Life, Dedicated to Heinrich Begehr on the Occasion of his 80th Birthday. Trends in Mathematics. Birkhaüser, 193–211.

[11] Gryshchuk, S.V., Plaksa, S.A. (2021). *A Hypercomplex Method for Solving Boundary Value Problems for Biharmonic Functions.* In: Hošková-Mayerová Š., Flaut C., Maturo F. (eds). Algorithms as a Basis of Modern Applied Mathematics. Studies in Fuzziness and Soft Computing, 404. Springer, Cham, 231–255, https://doi.org/10.1007/978-3-030-61334-1_12.

[12] Gryshchuk, S.V., Plaksa, S.A. (2013). Schwartz-type integrals in a biharmonic plane. *International Journal of Pure and Applied Mathematics, 83*(1), 193–211.

[13] Gryshchuk, S.V., Plaksa, S.A. (2016). Monogenic functions in the biharmonic boundary value problem. *Math. Methods Appl. Sci., 39*(11), 2939–2952.

[14] Gryshchuk, S.V. (2016). $\mathbb{B}$*-valued monogenic functions and their applications to boundary value problems in displacements of 2-D Elasticity.* In: Analytic Methods of Analysis and Differential Equations: AMADE 2015. Rogosin S.V., Dubatovskaya, M.V., Belarusian State University, Minsk, Belarus (Eds.), Cambridge Scientic Publishers Ltd, UK, 37–47.

[15] Gryshchuk, S.V., Plaksa, S.A. (2017). Reduction of a Schwartz-type boundary value problem for biharmonic monogenic functions to Fredholm integral equations. *Open Math., 15*(1), 374–381.

[16] Gryshchuk, S.V., Plaksa, S.A. (2017). A Schwartz-Type Boundary Value Problem in a Biharmonic Plane. *Lobachevskii Journal of Mathematics, 38*(3), 435–442.

[17] Gryshchuk, S.V. (2017). One-dimensionality of the kernel of the system of Fredholm integral equations for a homogeneous biharmonic problem. *Zb. Pr. Inst. Mat. NAN Ukr., 14*(1), 128–139.

[18] Mikhlin, S.G. (1935). *The plane problem of the theory of elasticity.* In: Trans. Inst. of seismology, Acad. Sci. USSR., no. 65, Acad. Sci. USSR Publ. House, Moscow–Leningrad (in Russian).

[19] Muskhelishvili, N.I. (1977). *Some basic problems of the mathematical theory of elasticity. Fundamental equations, plane theory of elasticity, torsion and bending.* English transl. from the 4th Russian edition by R.M. Radok, Noordhoff International Publishing, Leiden.

[20] Lurie, A.I. (2005). *Theory of Elasticity.* Springer-Verlag, Berlin etc.

[21] Kahniashvili, N.S. (1953). Research of the plain problems of the theory of elasticity by the method of the theory of potentials. *Trudy Tbil. Univer., 50.* Publ. by Tbil. Univer., Tbilisi (in Russian).

[22] Bogan, Yu.A. (2001). On Fredholm integral equations in two-dimensional anisotropic theory of elasticity. *Sib. Zh. Vychisl. Mat., 4*(1), 21–30 (in Russian).

[23] Hille, E., Phillips, R.S. (2000). *Functional Analysis and Semi-Groups.* Colloquium Publications, 31. American Mathematical Society, Providence.

[24] Smirnov, V.I. (1964). *A course of higher mathematics, 3, Part 2.* Pergamon Press, Oxford.

Contact information

Serhii Viktorovych Gryshchuk
Department of Complex Analysis and Potential Theory, Institute of Mathematics of the National Academy of Science of Ukraine, Kyiv, Ukraine
E-Mail: serhii.gryshchuk@gmail.com

Sergiy Anatolijovych Plaksa
Department of Complex Analysis and Potential Theory, Institute of Mathematics of the National Academy of Science of Ukraine, Kyiv, Ukraine
E-Mail: plaksa62@gmail.com