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On boundary-value problems for semi-linear equations in the plane

V. Gutlyanskiĭ, O. Nesmelova, V. Ryazanov, A. Yefimushkin

Abstract. The study of the Dirichlet problem with arbitrary measurable data for harmonic functions in the unit disk $\mathbb{D}$ is due to the dissertation of Luzin. Later on, the known monograph of Vekua has been devoted to boundary-value problems only with Hölder continuous data for the generalized analytic functions, i.e., continuous complex-valued functions $f(z)$ of the complex variable $z = x+iy$ with generalized first partial derivatives by Sobolev satisfying the equations of the form $\partial_{\bar{z}} f + af + b\overline{f} = c$, where it was assumed that the complex-valued functions a, b and c belong to the class L^p with some $p > 2$ in smooth enough domains D in $\mathbb{C}$.

Our last paper [12] contained theorems on the existence of nonclassical solutions of the Hilbert boundary-value problem with arbitrary measurable data (with respect to logarithmic capacity) for generalized analytic functions $f : D \to \mathbb{C}$ such that $\partial_{\bar{z}} f = g$ with the real-valued sources. On this basis, it was established the corresponding existence theorems for the Poincare problem on directional derivatives and, in particular, for the Neumann problem to the Poisson equations $\triangle U = G \in L^p, p > 2$, with arbitrary measurable boundary data over logarithmic capacity.

The present paper is a natural continuation of the article [12] and includes, in particular, theorems on the existence of solutions of the Hilbert boundary-value problem with arbitrary measurable data for the corresponding nonlinear equations of the Vekua type $\partial_{\bar{z}} f(z) = h(z)q(f(z))$. On this basis, it is also established existence theorems for the Poincare boundary-value problem and, in particular, for the Neumann problem to the nonlinear Poisson equations of the form $\triangle U(z) = H(z)Q(U(z))$ with arbitrary measurable boundary data over logarithmic capacity. The Dirichlet problem was investigated by us for the given equations, too.

Our approach is based on the interpretation of boundary values in the sense of angular (along nontangential paths) limits that are a traditional tool of the geometric function theory.

As consequences, we give applications to some concrete semi-linear equations of mathematical physics, arising from modelling various physical processes. These results can be also applied to semi-linear equations of mathematical physics in anisotropic and inhomogeneous media.

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1. Introduction

The research of boundary-value problems with arbitrary measurable data is due to the famous dissertation of Luzin, see its original text [17], and its reprint [18] with comments of his pupils Bari and Men'shov. Namely, he has established that, for each measurable a.e. finite 2π-periodic function $\varphi(\vartheta) : \mathbb{R} \to \mathbb{R}$, there is a harmonic function U in the unit disk $\mathbb{D}$ such that $U(z) \to \varphi(\vartheta)$ for a.e. ϑ as $z \to \zeta := e^{i\vartheta}$ along all nontangential paths to $\partial\mathbb{D}$. The latter was based on his other deep result on the antiderivatives stated that, for any measurable function $\psi : [0,1] \to \mathbb{R}$, there is a continuous function $\Psi : [0,1] \to \mathbb{R}$ with $\Psi' = \psi$ a.e., see e.g. his papers [16] and [19], Theorem VII(2.3) in the Saks monograph [22].

The well–known monograph of Vekua [24] has been devoted to the theory of the **generalized analytic functions**, i.e., continuous complex-valued functions $h(z)$ of the complex variable $z = x + iy$ with generalized first partial derivatives by Sobolev satisfying equations of the form

$$\partial_{\bar{z}} h \;+\; ah \;+\; b\overline{h} \;=\; c\,, \qquad \partial_{\bar{z}} \;:=\; \frac{1}{2}\left(\frac{\partial}{\partial x} \;+\; i\cdot\frac{\partial}{\partial y}\right), \tag{1.1}$$

where it was assumed that the complex-valued functions a, b and c belong to the class L^p with some $p > 2$ in the corresponding domains $D \subseteq \mathbb{C}$.

Our paper [12] contained theorems on the existence of nonclassical solutions of the Hilbert boundary-value problem with arbitrary measurable data (with respect to logarithmic capacity) for generalized analytic functions $f : D \to \mathbb{C}$ such that $\partial_{\bar{z}} f = g$ with the real-valued sources g. On this basis, it was established the corresponding existence theorems for the Poincare problem on directional derivatives and, in particular, for the Neumann problem to the Poisson equations $\triangle U = G \in L^p, p > 2$ with arbitrary measurable boundary data.

The present paper is a natural continuation of the article [12], where the reader can find further historical comments, and includes, in particular, theorems on the existence of solutions of the Hilbert boundary-value problem with arbitrary measurable data for the corresponding nonlinear equations of the Vekua type $\partial_{\bar{z}} f(z) = h(z) q(f(z))$. On this basis, it is also established existence theorems for the Poincare boundary-value problem and, in particular, for the Neumann problem to the nonlinear Poisson equations of the form $\triangle U(z) = H(z)Q(f(z))$ with arbitrary measurable boundary data.

Our approach is based on the interpretation of boundary values in the sense of angular (along nontangential paths) limits that are a traditional tool of the geometric function theory.

As consequences, we give applications to some concrete semi-linear equations of mathematical physics arising under modelling various physical processes.

These results can be also applied to semi-linear equations of mathematical physics in anisotropic and inhomogeneous media that will be published elsewhere.

Many definitions for the following sections can be found in our last paper [12].

2. On the Dirichlet problem with measurable data for harmonic functions

Let us start with the following analog of the Luzin theorem on the antiderivatives in terms of logarithmic capacity, see Theorem 3.1 in [7].

Lemma 1. *Let $\varphi : [a, b] \to \mathbb{R}$ be a measurable function with respect to logarithmic capacity. Then there is a continuous function $\Phi : [a, b] \to \mathbb{R}$ with $\Phi'(x) = \varphi(x)$ q.e. on (a, b). Furthermore, Φ can be chosen with $\Phi(a) = \Phi(b) = 0$ and $|\Phi(x)| \leq \varepsilon$ under arbitrary prescribed $\varepsilon > 0$ for all $x \in [a, b]$.*

Remark 1. In view of arbitrariness of $\varepsilon > 0$ in Lemma 1, for each φ, there is the infinite collection of such Φ. Furthermore, it is easy to see by Lemma 3.1 in [7] that the space of such functions Φ has the infinite dimension.

Corollary 1. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a measurable function with respect to logarithmic capacity. Then the space of continuous functions $\Phi : \partial\mathbb{D} \to [-1, 1]$ with $\Phi(1) = 0$, $|\Phi(\zeta)| \leq \varepsilon$ under arbitrary prescribed $\varepsilon > 0$ for all $\zeta \in \partial\mathbb{D}$, and $\Phi'(e^{it}) = \varphi(e^{it})$ q.e. on $\mathbb{R}$ has the infinite dimension.*

On this basis, we obtain the following result, see e.g. Theorem 4.1 in [7].

Proposition 1. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a measurable function with respect to logarithmic capacity. Then there is a space of harmonic functions U in the unit disk $\mathbb{D}$ of the infinite dimension with the angular limits*

$$\lim_{z\to\zeta} u(z) = \varphi(\zeta) \qquad \text{q.e. on } \partial\mathbb{D}\ . \tag{2.1}$$

Remark 2. By the proof of Theorem 4.1 in [7], $u(z) = \frac{\partial}{\partial \vartheta} U(z)$, where

$$U(re^{i\vartheta}) = \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{1-r^2}{1-2r\cos(\vartheta - t)+r^2} \, \Phi(e^{it}) \, dt \, , \tag{2.2}$$

i.e., for any function Φ from Corollary 1, u can be calculated in the explicit form

$$u(re^{i\vartheta}) = -\frac{r}{\pi} \int\limits_0^{2\pi} \frac{(1-r^2)\sin(\vartheta - t)}{(1-2r\cos(\vartheta - t)+r^2)^2} \, \Phi(e^{it}) \, dt \, . \tag{2.3}$$

Later on, it was shown by Theorems 1 and 3 in [21] that the functions $u(z)$ can be represented as the **Poisson–Stieltjes integrals**

$$\mathbb{U}_\Phi(z) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} P_r(\vartheta - t) \, d\,\Phi(e^{it}) \quad \forall \; z = re^{i\vartheta}, \; r \in (0,1) \; , \; \vartheta \in [-\pi, \pi] \; , \tag{2.4}$$

where $P_r(\Theta) = (1-r^2)/(1-2r\cos\Theta + r^2), r < 1, \Theta \in \mathbb{R}$, is the **Poisson kernel**.

The corresponding analytic functions $\mathcal{A}(z)$ in $\mathbb{D}$ with the real parts $u(z)$ can be represented as the **Schwartz–Stieltjes integrals**

$$\mathbb{S}_\Phi(z) = \frac{1}{2\pi} \int\limits_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} \, d\,\Phi(\zeta) \, , \qquad z \in \mathbb{D} \, , \tag{2.5}$$

because of the Poisson kernel is the real part of the (analytic in the variable z) **Schwartz kernel** $(\zeta + z)/(\zeta - z)$. Integrating (2.5) by parts, see Lemma 1 and Remark 1 in [21], we obtain also the more convenient form of the representation

$$\mathbb{S}_\Phi(z) = \frac{z}{\pi} \int\limits_{\partial\mathbb{D}} \frac{\Phi(\zeta)}{(\zeta - z)^2} \, d\zeta \, , \qquad z \in \mathbb{D} \, . \tag{2.6}$$

By Corollary 1 the spaces of solutions of the Dirichlet problem in the classes of harmonic and analytic functions generating by integral operators $\mathbb{U}_\Phi$ and $\mathbb{S}_\Phi$, correspondingly, under each fixed boundary date φ that is measurable with respect to logarithmic capacity have the infinite dimension.

3. On completely continuous Hilbert operators

First of all, recall that a **completely continuous** mapping from a metric space M_1 into a metric space M_2 is defined as a continuous mapping on M_1 which takes bounded subsets of M_1 into relatively compact ones of M_2, i.e. with compact closures in M_2. When a continuous mapping takes M_1 into a relatively compact subset of M_1, it is nowadays said to be **compact** on M_1.

The notion of completely continuous (compact) operators is due essentially, in the simplest partial cases, to Hilbert and Riesz F., see the corresponding comments of Section VI.12 in [5], and to Leray and Schauder in the general case, see the paper [15].

In paper [12], we considered **generalized analytic functions** f **with sources** $g \in L^p, p > 2$, that have generalized first derivatives by Sobolev and satisfy the equation

$$\frac{\partial f}{\partial \bar{z}} \ = \ g \ , \qquad \frac{\partial}{\partial \bar{z}} \ := \ \frac{1}{2}\left(\frac{\partial}{\partial x} \ + \ i \cdot \frac{\partial}{\partial y} \right) \ , \quad z = x + iy \ , \qquad (3.1)$$

and studied for them the Hilbert boundary-value problem under arbitrary boundary data that are measurable with respect to the logarithmic capacity.

In particular, Theorem 1 in [12] stated, for $\lambda : \partial\mathbb{D} \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, with countable bounded variation, $\varphi : \partial\mathbb{D} \to \mathbb{R}$ which is measurable with respect to logarithmic capacity and $g : \mathbb{D} \to \mathbb{R}$ in $L^p(\mathbb{D})$, $p > 2$, there exist generalized analytic functions $f : D \to \mathbb{C}$ with the source g that have the angular limits

$$\lim_{z\to\zeta} \ \mathrm{Re} \ \left\{ \overline{\lambda(\zeta)} \cdot f(z) \right\} \ = \ \varphi(\zeta) \qquad \text{q.e. on } \ \partial\mathbb{D} \ . \qquad (3.2)$$

Furthermore, the space of such functions f has the infinite dimension.

Thus, the Hilbert boundary-value problem always has many solutions in the given sense for each such coefficient λ, boundary date φ and source g. Of course, axiom of choice by Zermelo makes it possible to choose one of such correspondence named further as a Hilbert operator but the latter with such a random choice can be completely discontinuous. Later on, to apply the approach of Leray–Schauder for extending Theorem 1 in [12] to the generalized analytic functions, satisfying nonlinear equations of the Vekua type, we need just the complete continuity of such correspondence.

Now we show that, fixing only one antiderivative Φ for the function φ from Corollary 1, it is possible to obtain a completely continuous Hilbert operator. For this purpose, let us analyze the construction of solutions of equation (3.1) with the Hilbert boundary condition (3.2) from the proof of Theorem 1 in [12].

There we often applied the **logarithmic (Newtonian) potential $\mathcal{N}_G$ of sources** $G \in L^p(\mathbb{C})$, $p > 2$, with compact supports given by the formula:

$$\mathcal{N}_G(z) \ := \ \frac{1}{2\pi} \int\limits_{\mathbb{C}} \ln|z-w|\, G(w)\, dm(w)\ . \tag{3.3}$$

However, by the linearity of the operator N_G with respect to G, we extend here the definition (3.3) in a natural way to the complex-valued sources G, as usual, interpreting the imaginary parts of G as distributed currents. Recall also that by Lemma 3 in [9], see also Theorem 2 in [11], $\mathcal{N}_G \in W^{2,p}_{\rm loc}(\mathbb{C}) \cap C^{1,\alpha}_{\rm loc}(\mathbb{C})$ with $\alpha := (p-2)/p$ and $\triangle \mathcal{N}_G \ = \ G$ a.e.

Let us consider equation (3.1) in the unit disk $\mathbb{D}$. Extending g by zero outside of $\mathbb{D}$ and setting $P = \mathcal{N}_G$ with $G = 2g$, $U = P_x$ and $V = -P_y$, we have that

$$H \ := \ U + iV \ = \ 2\cdot \frac{\partial P}{\partial z}\ , \qquad \frac{\partial}{\partial z} \ := \ \frac{1}{2}\left(\frac{\partial}{\partial x} \ - \ i\cdot \frac{\partial}{\partial y}\right)\ , \quad z = x+iy\ , \tag{3.4}$$

is just a generalized analytic function with the source g because the Laplacian

$$\triangle \ := \ \frac{\partial^2}{\partial x^2} \ + \ \frac{\partial^2}{\partial y^2} \ = \ 4\cdot \frac{\partial^2}{\partial z \partial \overline{z}} \ = \ 4\cdot \frac{\partial^2}{\partial \overline{z} \partial z}\ . \tag{3.5}$$

Note also by the way the connection of H with the known Pompeiu integral operator, see e.g. the relation (2.21) in [10], that gives its representation in the explicit form

$$H(z) \ = \ T_g(z) \ := \ \frac{1}{\pi} \int\limits_{\mathbb{C}} g(w)\, \frac{dm(w)}{z-w}\ . \tag{3.6}$$

Remark 3. In view of relation (3.6), we have by Theorem 1.19 in [24] that

$$|H(z)| \ \le \ M_1\, \|g\|_p \qquad \forall\ z \in \mathbb{C}\ , \tag{3.7}$$

$$|H(z_1) - H(z_2)| \ \le \ M_2\, \|g\|_p\, |z_1 - z_2|^{\alpha} \qquad \forall\ z_1, z_2 \in \mathbb{C}\ , \tag{3.8}$$

where the constants M_1 and M_2 depend only on $p > 2$, and $\alpha = (p-2)/p$. Thus, the linear operator H is completely continuous on compact sets in $\mathbb{C}$ and, in particular, on $\overline{\mathbb{D}}$ by Arzela-Ascoli theorem, see e.g. Theorem IV.6.7 in [5].

Next, since $H \in C^{\alpha}_{\rm loc}(\mathbb{C})$, $\alpha := (p-2)/p$, the boundary function

$$\varphi_g(\zeta) \ := \ \lim_{z\to\zeta} \ {\rm Re}\ \left\{ \overline{\lambda(\zeta)} \cdot H(z) \right\} \ = \ {\rm Re}\ \left\{ \overline{\lambda(\zeta)} \cdot H(\zeta) \right\}\ , \qquad \forall\, \zeta \in \partial\mathbb{D}\ , \tag{3.9}$$

is measurable with respect to logarithmic capacity.

Consequently, the generalized analytic functions f with the source g satisfying the Hilbert condition (3.2) can be get as the sums $f = H + \mathcal{C}$ with analytic functions $\mathcal{C}$ satisfying, in the sense of angular limits, the Hilbert boundary condition

$$\lim_{z\to\zeta} \mathrm{Re}\,\{\overline{\lambda(\zeta)}\cdot\mathcal{C}(z)\} \;=\; \psi(\zeta) := \varphi(\zeta) - \varphi_g(\zeta) \qquad \text{q.e. on } \partial\mathbb{D}\,. \quad (3.10)$$

In turn, by the construction of Theorem 5.1 in [13], such analytic functions $\mathcal{C}$ can be obtained as the products of 2 analytic functions $\mathcal{A}$ and $\mathcal{B}$. The first

$$\mathcal{A}(z) \;=\; e^{ia(z)}\;, \quad a(z) \;:=\; \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \alpha_\lambda(\zeta)\,\frac{z+\zeta}{z-\zeta}\,\frac{d\zeta}{\zeta}\;, \qquad z\in\mathbb{D}\;, \quad (3.11)$$

with a function α_λ that is measurable with respect to the logarithmic capacity, bounded on $\partial\mathbb{D}$, of countable bounded variation and such that

$$\lambda(\zeta) \;=\; e^{i\alpha_\lambda(\zeta)} \qquad \text{q.e. on } \partial\mathbb{D}\;. \quad (3.12)$$

By Lemma 4.1 in [13] the angular limits of Im $a(z)$ as $z \to \zeta$ q.e. on $\partial\mathbb{D}$ form a function $\beta : \partial\mathbb{D} \to \mathbb{R}$ that is measurable with respect to the logarithmic capacity. By Remark 2 the second analytic function $\mathcal{B}$ can be obtained in the form

$$\mathcal{B}(z) \;=\; \mathbb{S}_\Psi(z) \;=\; \frac{z}{\pi}\int\limits_{\partial\mathbb{D}} \frac{\Psi(\zeta)}{(\zeta-z)^2}\,d\zeta\;, \quad z\in\mathbb{D}\;, \quad (3.13)$$

where Ψ is an antiderivative of the function ψe^β from Corollary 1.

Thus, analytic functions $\mathcal{C}$ can be represented in more convenient form

$$\mathcal{C}(z) \;=\; \mathcal{A}(z)\cdot[\,\mathbb{S}_\Phi(z) \;-\; \mathbb{S}_{\Phi_g}(z)\,]\;, \quad (3.14)$$

where Φ and Φ_g are antiderivatives of the functions of φe^β and $\varphi_* e^\beta$ from Corollary 1, correspondingly. Note that the analytic functions $\mathcal{A}$ and $\mathbb{S}_\Phi$ do not depend on the sources g at all. Let us choose the function Φ_g in a suitable way.

From this point on, we demand that all sources g have compact supports in the unit disk and belong to a disk $\mathbb{D}_\rho := \{z\in\mathbb{C} : |z| \le \rho\}$ with a radius $\rho \in (0,1)$. Then the function $H(z)$, $z \in \mathbb{C}$, is analytic in a neighborhood of the unit circle $\partial\mathbb{D}$ and, in particular, $H(\zeta)$ is continuously differentiable in the variable ϑ, $\zeta = e^{i\vartheta}$, $\vartheta \in \mathbb{R}$. Moreover, by relation (3.6) we have that

$$H_\vartheta(\zeta) \;=\; i\zeta H'(\zeta) \;=\; \frac{\zeta}{\pi i}\int\limits_{\mathbb{D}_\rho} g(w)\,\frac{dm(w)}{(\zeta-w)^2} \qquad \forall\,\zeta\in\partial\mathbb{D}\,. \quad (3.15)$$

Let us denote by Λ an antiderivative for the function $\overline{\lambda}e^{\beta}$ from Corollary 1.

Then the following function Φ_g is an antiderivative for the function $\varphi_g e^{\beta}$:

$$\Phi_g(\zeta) \ := \ \mathrm{Re} \left\{ \Lambda(\zeta)H(\zeta) - \int\limits_0^{\vartheta} \Lambda(\xi)H_\theta(\xi)\, d\,\theta + S(\vartheta) \right\} , \tag{3.16}$$

where $S : [0, 2\pi] \to \mathbb{C}$ is either zero or a singular function of the form

$$S(\vartheta) \ := \ C(\vartheta) \int\limits_0^{2\pi} \Lambda(\xi)H_\theta(\xi)\, d\,\theta\ , \quad \zeta = e^{i\vartheta},\ \xi = e^{i\theta},\ \vartheta,\ \theta \in [0, 2\pi]\ , \tag{3.17}$$

with a singular function $C : [0, 2\pi] \to [0, 1]$ of the Cantor ladder type, i.e., C is continuous, nondecreasing, $C(0) = 0$, $C(2\pi) = 1$ and $C' = 0$ q.e. on $[0, 2\pi]$. Recall that the existence of such functions C follows from Lemma 3.1 in [7].

Let us show that the Hilbert operator $\mathcal{H}_g$ generated by the sums $H+\mathcal{C}$ under the given choice of Φ and Φ_g in (3.14) is completely continuous on compact sets in $\mathbb{D}$. Recall that the analytic functions $\mathcal{A}$ and $\mathbb{S}_\Phi$ in the representation (3.14) of $\mathcal{C}$ do not depend on the sources g. Hence by Remark 3, it remains to show that the linear operator $\mathbb{S}_{\Phi_g}$ is completely continuous.

Indeed, by the construction of Φ_g in (3.16) and relations (3.6) and (3.15)

$$|\Phi_g(\zeta)| \ \leq \ \frac{1}{\pi} \cdot \frac{\|g\|_1}{1-\rho} \ + \ 2 \cdot \frac{\|g\|_1}{(1-\rho)^2} \ \leq \ c_\rho \cdot \|g\|_1 \ \leq \ C_\rho \cdot \|g\|_p \quad \forall\, \zeta \in \partial\mathbb{D} \tag{3.18}$$

with $c_\rho = 3/(1-\rho)^2$ and $C_\rho = 3\pi/(1-\rho)^2$, respectively. Hence, by (2.6)

$$|\mathbb{S}_{\Phi_g}(z)| \ \leq \ C_{\rho,r} \cdot \|g\|_p\ , \quad \forall\ z \in \mathbb{D}_r\ ,\ r \in (0,1)\ , \tag{3.19}$$

$$|\mathbb{S}_{\Phi_g}(z_1) - \mathbb{S}_{\Phi_g}(z_2)| \ \leq \ C^*_{\rho,r} \cdot \|g\|_p \cdot |z_1 - z_2|\ , \quad \forall\ z_1, z_2 \in \mathbb{D}_r\ ,\ r \in (0,1)\ , \tag{3.20}$$

where the constants $C_{\rho,r}$ and $C^*_{\rho,r}$ depend only on the radii ρ and $r \in (0, 1)$. Thus, the operator $\mathbb{S}_{\Phi_*}$ is completely continuous on compact sets in $\mathbb{D}$ again by the Arzela–Ascoli theorem. Combining it with Remark 3, we obtain the following conclusion.

Lemma 2. *Let* $\lambda : \partial\mathbb{D} \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, *be of countable bounded variation and let* $\varphi : \partial\mathbb{D} \to \mathbb{R}$ *be measurable with respect to the logarithmic capacity.*

Then there is a Hilbert operator $\mathcal{H}_g$ *over* $g : \mathbb{D} \to \mathbb{C}$ *in* $L^p(\mathbb{D})$, $p > 2$, *with compact supports in* $\mathbb{D}$, *generating generalized analytic functions* $f : \mathbb{D} \to \mathbb{C}$ *with the sources* g *and the angular limits*

$$\lim_{z\to\zeta} \operatorname{Re} \left\{ \overline{\lambda(\zeta)} \cdot f(z) \right\} = \varphi(\zeta) \qquad q.e.\ on\ \ \partial\mathbb{D}\ , \tag{3.21}$$

whose restriction to sources g *with* $\operatorname{supp} g \subseteq \mathbb{D}_\rho$ *is completely continuous over* $\mathbb{D}_r$ *for each* ρ *and* $r \in (0,1)$.

Remark 4. Note that the nonlinear operator $\mathcal{H}_g$ constructed above is not bounded except the trivial case $\Phi \equiv 0$ because then $\mathcal{H}_0 = \mathcal{A} \cdot \mathbb{S}_\Phi \neq 0$. However, the restriction of the operator $\mathcal{H}_g$ to $\mathbb{D}_r$ under each $r \in (0,1)$ is bounded at infinity in the sense that $\max\limits_{z\in\mathbb{D}_r} |\mathcal{H}_g(z)| \leq M \cdot \|g\|_p$ for some $M > 0$ and all g with large enough $\|g\|_p$. Note also that by Corollary 1 we are able always to choose Φ for any φ, including $\varphi \equiv 0$, which is not identically 0 in the unit disk $\mathbb{D}$.

4. On Hilbert problem for semi-linear equations

In this section we study the solvability of the Hilbert boundary-value problem for nonlinear equations of the Vekua type $\partial_{\bar{z}} f(z) = h(z)q(f(z))$ in the unit disk $\mathbb{D}$. The well-known Leray–Schauder approach allows us to reduce the problem to the study of the corresponding linear equation from our last paper [12] on the basis of Lemma 2 in the previous section on completely continuous Hilbert operator $\mathcal{H}_g$ and Remark 4 on its boundedness at infinity.

For the sake of completeness, we recall some definitions and basic facts of the celebrated paper [15].

First of all Leray and Schauder extend as follows the Brouwer degree to compact perturbations of the identity I in a Banach space B, i.e. a complete normed linear space. Namely, given an open bounded set $\Omega \subset B$, a compact mapping $F : B \to B$ and $z \notin \Phi(\partial\Omega)$, $\Phi := I - F$, the **(Leray–Schauder) topological degree** $\deg\,[\Phi, \Omega, z]$ of Φ in Ω over z is constructed from the Brouwer degree by approximating the mapping F over Ω by mappings F_ε with range in a finite-dimensional subspace B_ε (containing z) of B. It is showing that the Brouwer degrees $\deg\,[\Phi_\varepsilon, \Omega_\varepsilon, z]$ of $\Phi_\varepsilon := I_\varepsilon - F_\varepsilon$, $I_\varepsilon := I|_{B_\varepsilon}$, in $\Omega_\varepsilon := \Omega \cap B_\varepsilon$ over z stabilize for sufficiently small positive ε to a common value defining $\deg\,[\Phi, \Omega, z]$ of Φ in Ω over z.

This topological degree "algebraically counts" the number of fixed points of $F(\cdot) - z$ in Ω and conserves the basic properties of the Brouwer degree as additivity and homotopy invariance. Now, let a be an isolated fixed point of F. Then the **local (Leray–Schauder) index** of a is defined by $\operatorname{ind}[\Phi, a] := \deg[\Phi, B(a,r), 0]$ for small enough r > 0. If $a = 0$, then we say on the **index** of F. In particular, if $F \equiv 0$, correspondingly, $\Phi \equiv I$, then the index of F is equal to 1.

The fundamental Theorem 1 in [15] can be formulated in the following way: *Let B be a Banach space, and let $F(\cdot,\tau) : B \to B$ be a family of operators with $\tau \in [0,1]$. Suppose that the following hypotheses hold:*

(H1) *$F(\cdot,\tau)$ is completely continuous on B for each $\tau \in [0,1]$ and uniformly continuous with respect to the parameter $\tau \in [0,1]$ on each bounded set in B;*

(H2) *the operator $F := F(\cdot, 0)$ has finite collection of fixed points whose total index is not equal to zero;*

(H3) *the collection of all fixed points of the operators $F(\cdot,\tau)$, $\tau \in [0,1]$, is bounded in B.*

Then the collection of all fixed points of the family of operators $F(\cdot,\tau)$ contains a continuum along which τ takes all values in $[0,1]$.

In the proof of the next theorem the initial operator $F(\cdot) := F(\cdot,0) \equiv 0$. Hence F has the only one fixed point (at the origin) and its index is equal to 1 and, thus, hypothesis (H2) will be automatically satisfied.

Theorem 1. *Let $\lambda : \partial\mathbb{D} \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, be of countable bounded variation and let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable with respect to the logarithmic capacity.*

Suppose that $h : \mathbb{D} \to \mathbb{C}$ is a function in the class $L^p(\mathbb{D})$ for $p > 2$ with compact support in $\mathbb{D}$ and $q : \mathbb{C} \to \mathbb{C}$ is a continuous function with

$$\lim_{w\to\infty} \frac{q(w)}{w} \ = \ 0\ . \tag{4.1}$$

Then there is a function $f : \mathbb{D} \to \mathbb{C}$ in the class $C^{\alpha}_{\mathrm{loc}}(\mathbb{D})$ with $\alpha = (p-2)/p$ and generalized first derivatives by Sobolev such that

$$\partial_{\bar z} f(z) \ = \ h(z)\cdot q(f(z)) \qquad a.e.\ in\ \mathbb{D} \tag{4.2}$$

and, in addition, f is a generalized analytic function with a source $g \in L^p(\mathbb{D})$ and the angular limits

$$\lim_{z\to\zeta} \operatorname{Re}\left\{\overline{\lambda(\zeta)}\cdot f(z)\right\} \ = \ \varphi(\zeta) \qquad q.e.\ on\ \ \partial\mathbb{D}\ . \tag{4.3}$$

Moreover, $f = \mathcal{H}_g$, where $\mathcal{H}_g$ is the Hilbert operator described in the last section, and the support of g is in the support of h and the upper bound of $\|g\|_p$ depends only on $\|h\|_p$ and on the function q.

Proof. If $\|h\|_p = 0$ or $\|q\|_C = 0$, then any analytic function from Theorem 5.1 in [13] gives the desired solution of (4.2). Thus, we may assume that $\|h\|_p \neq 0$ and $\|q\|_C \neq 0$. Set $q_*(t) = \max\limits_{|w|\leq t} |q(w)|$, $t \in \mathbb{R}^+ := [0, \infty)$. Then the function $q_* : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing and, moreover, by (4.1)

$$\lim_{t\to\infty} \frac{q_*(t)}{t} \ = \ 0 \ . \tag{4.4}$$

By Lemma 2 and Remark 4 we obtain the family of operators $F(g;\tau) : L^p_h(\mathbb{D}) \to L^p_h(\mathbb{D})$, where $L^p_h(\mathbb{D})$ consists of functions $g \in L^p(\mathbb{D})$ with supports in the support of h,

$$F(g;\tau) \ := \ \tau h \cdot q(\mathcal{H}_g) \qquad \forall\, \tau \in [0,1] \tag{4.5}$$

which satisfies all groups of hypothesis H1–H3 of Theorem 1 in [15]. Indeed:

H1). First of all, by Lemma 2 the function $F(g;\tau) \in L^p_h(\mathbb{D})$ for all $\tau \in [0,1]$ and $g \in L^p_h(\mathbb{C})$ because the function $q(\mathcal{H}_g)$ is continuous and, furthermore, the operators $F(\cdot;\tau)$ are completely continuous for each $\tau \in [0,1]$ and even uniformly continuous with respect to the parameter $\tau \in [0,1]$.

H2). The index of the operator $F(g;0)$ is obviously equal to 1.

H3). Let us assume that solutions of the equations $g = F(g;\tau)$ is not bounded in $L^p_h(\mathbb{D})$, i.e., there is a sequence of functions $g_n \in L^p_h(\mathbb{D})$ with $\|g_n\|_p \to \infty$ as $n \to \infty$ such that $g_n = F(g_n;\tau_n)$ for some $\tau_n \in [0,1]$, $n = 1, 2, \ldots$.

However, then by Remark 4 we have that, for some constant $M > 0$,

$$\|g_n\|_p \ \leq \ \|h\|_p \; q_*\left(M \, \|g_n\|_p \right)$$

and, consequently,

$$\frac{q_*(M \,\|g_n\|_p)}{M \,\|g_n\|_p} \ \geq \ \frac{1}{M \,\|h\|_p} \ > \ 0 \tag{4.6}$$

for all large enough n. The latter is impossible by condition (4.4). The obtained contradiction disproves the above assumption.

Thus, by Theorem 1 in [15] there is a function $g \in L^p_h(D)$ with $F(g;1) = g$, and by Lemma 2 the function $f := \mathcal{H}_g$ gives the desired solution of (4.2). □

Remark 5. By the construction in the above proof, the source $g : \mathbb{D} \to \mathbb{C}$ is a fixed point of the (nonlinear) integral operator $\Omega_g := h{\cdot}q(\mathcal{H}_g) :$

$L^p_h(\mathbb{D}) \to L^p_h(\mathbb{D})$, where $L^p_h(\mathbb{D})$ consists of functions g in $L^p(\mathbb{D})$ with supports in the support of h.

In particular, choosing $\lambda \equiv 1$ in Theorem 1 we obtain the following consequence on the Dirichlet problem for the nonlinear equations of the Vekua type.

Corollary 2. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable with respect to the logarithmic capacity, $h : \mathbb{D} \to \mathbb{C}$ be a function in the class $L^p(\mathbb{D})$ for $p > 2$ with compact support in $\mathbb{D}$ and let $q : \mathbb{C} \to \mathbb{C}$ be a continuous function with condition* (4.1)*.*

Then there is a function $f : \mathbb{D} \to \mathbb{C}$ in the class $C^\alpha_{\mathrm{loc}}(\mathbb{D})$ with $\alpha = (p-2)/p$ and generalized first derivatives by Sobolev, satisfying equation (4.2) *a.e. that is a generalized analytic function with a source $g \in L^p(\mathbb{D})$ and the angular limits*

$$\lim_{z\to\zeta} \mathrm{Re}\, f(z) \;=\; \varphi(\zeta) \qquad \textit{q.e. on } \partial\mathbb{D}\ . \tag{4.7}$$

Moreover, $f = \mathcal{H}_g$, where $\mathcal{H}_g$ is the Hilbert operator described in the last section (with the simplest $\mathcal{A} \equiv 1$), and the support of g is in the support of h and the upper bound of $\|g\|_p$ depends only on $\|h\|_p$ and on the function q.

5. Extension of results to regular enough domains

Here we extend the above results to Jordan domains with the so–called quasihyperbolic boundary condition, see definitions in our last paper [12]. As known, such domains include, for instance, domains with quasiconformal boundaries and, in particular, domains with smooth and Lipschitz boundaries. However, quasiconformal curves can be even nowhere locally rectifiable.

Theorem 2. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, be in $\mathcal{CBV}(\partial D)$ and let $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose that $h : D \to \mathbb{C}$ is a function in the class $L^p(D)$ for $p > 2$ with compact support in D and $q : \mathbb{C} \to \mathbb{C}$ is a continuous function with

$$\lim_{w\to\infty} \frac{q(w)}{w} \;=\; 0\ . \tag{5.1}$$

Then there is a function $f : D \to \mathbb{C}$ in the class $C^\alpha_{\mathrm{loc}}(D)$ with $\alpha = (p-2)/p$ and generalized first derivatives by Sobolev such that

$$\partial_{\bar\xi} f(\xi) \;=\; h(\xi)\cdot q(f(\xi)) \qquad \textit{a.e. in } D \tag{5.2}$$

and, in addition, f is a generalized analytic function with a source $g \in L^p(D)$ and the angular limits

$$\lim_{\xi\to\omega} \operatorname{Re} \left\{ \overline{\lambda(\omega)} \cdot f(\xi) \right\} \; = \; \varphi(\omega) \qquad \text{q.e. on } \partial D \; . \tag{5.3}$$

Moreover, $f(\xi) = \mathcal{H}_{\tilde g}(c(\xi))$, where c is a conformal mapping of D onto $\mathbb{D}$, $\mathcal{H}_{\tilde g}$ is the Hilbert operator described in Section 3, $\tilde g = g \circ c^{-1}$, and the support of g is in the support of h and the upper bound of $\|g\|_p$ depends only on $\|h\|_p$, the function q and the domain D.

Proof. Let c be a conformal mapping of D onto $\mathbb{D}$ that exists by the Riemann mapping theorem, see e.g. Theorem II.2.1 in [8]. Now, by the Caratheodory theorem, see e.g. Theorem II.3.4 in [8], c is extended to a homeomorphism $\tilde c$ of $\overline{D}$ onto $\overline{\mathbb{D}}$. Furthermore, by Corollary of Theorem 1 in [4], $c_* := \tilde c|_{\partial D} : \partial D \to \partial\mathbb{D}$ and its inverse function are Hölder continuous. Then $\tilde\lambda := \lambda\circ c_*^{-1} \in \mathcal{CBV}(\partial\mathbb{D})$ and $\tilde\varphi := \varphi\circ c_*^{-1}$ is measurable with respect to the logarithmic capacity, see e.g. Remarks 1 and 2 in [12].

Now, set $\tilde h = h\circ C\cdot\overline{C'}$, where C is the inverse conformal mapping to c, $C := c^{-1} : \mathbb{D} \to D$. Then it is clear by the hypothesis of Theorem 2 that $\tilde h$ has compact support in $\mathbb{D}$ and belongs to the class $L^p(\mathbb{D})$. Consequently, by Theorem 1 there is a function $\tilde f : \mathbb{D} \to \mathbb{C}$ in the class $C^\alpha_{\text{loc}}(\mathbb{D})$ with $\alpha = (p-2)/p$ and generalized first derivatives by Sobolev such that

$$\partial_{\bar z}\tilde f(z) \; = \; \tilde h(z)\cdot q(\tilde f(z)) \qquad \text{a.e. in } \mathbb{D} \tag{5.4}$$

and $\tilde f$ is a generalized analytic function with a source $\tilde g \in L^p(\mathbb{D})$ and the angular limits

$$\lim_{z\to\zeta} \operatorname{Re} \left\{ \overline{\tilde\lambda(\zeta)} \cdot \tilde f(z) \right\} \; = \; \tilde\varphi(\zeta) \qquad \text{q.e. on } \partial\mathbb{D} \; , \tag{5.5}$$

moreover, $\tilde f = \mathcal{H}_{\tilde g}$, where $\mathcal{H}_{\tilde g}$ is the Hilbert operator described in Section 3, and the support of $\tilde g$ is in the support of $\tilde h$ and the upper bound of $\|\tilde g\|_p$ depends only on $\|\tilde h\|_p$ and on the function q.

Next, setting $f = \tilde f \circ c$, by simple calculations, see e.g. Section 1.C in [1], we obtain that $\frac{\partial f}{\partial \bar\xi} = \frac{\partial \tilde f}{\partial \bar z} \circ c \cdot \overline{c'}$ and, consequently, the function $f : D \to \mathbb{C}$ is in the class $C^\alpha_{\text{loc}}(D)$ with $\alpha = (p-2)/p$ and generalized first derivatives by Sobolev that satisfies equation (5.2), f is a generalized analytic function with a source $g \in L^p(D)$ and, moreover, $f(\xi) = \mathcal{H}_{\tilde g}(c(\xi))$, where $\mathcal{H}_{\tilde g}$ is the Hilbert operator described in Section 3, $\tilde g = g \circ c^{-1}$, and the support of g is in the support of h and the upper bound of $\|g\|_p$ depends only on $\|h\|_p$, the function q and the domain D.

It remains to show that f has the angular limits as $\xi \to \omega \in \partial D$ and satisfies the boundary condition (5.3) q.e. on ∂D. Indeed, by the Lindelöf theorem, see e.g. Theorem II.C.2 in [14], if ∂D has a tangent at a point ω, then arg $[c_*(\omega) - c(\xi)]$ − arg $[\omega - \xi] \to$ const as $\xi \to \omega$. In other words, the images under the conformal mapping c of sectors in D with a vertex at $\omega \in \partial D$ is asymptotically the same as sectors in $\mathbb{D}$ with a vertex at $\zeta = c_*(\omega) \in \partial\mathbb{D}$. Consequently, nontangential paths in D are transformed under c into nontangential paths in $\mathbb{D}$ and inversely q.e. on ∂D and $\partial\mathbb{D}$, respectively, because ∂D has a tangent q.e. and c_* and c_*^{-1} keep sets of logarithmic capacity zero. □

Remark 6. By the construction in the above proof, the source $g = \tilde{g} \circ c$, where c is a conformal mapping of D onto $\mathbb{D}$ and $\tilde{g} : \mathbb{D} \to \mathbb{C}$ is a fixed point of the (nonlinear) integral operator $\tilde{\Omega}_{g_*} := \tilde{h} \cdot q(\mathcal{H}_{g_*}) : L^p_{\tilde{h}}(\mathbb{D}) \to L^p_{\tilde{h}}(\mathbb{D})$, where $L^p_{\tilde{h}}(\mathbb{D})$ consists of functions g_* in $L^p(\mathbb{D})$ with supports in the support of $\tilde{h} := h \circ C \cdot \overline{C'}$, C is the inverse conformal mapping to c, $C := c^{-1} : \mathbb{D} \to D$.

In particular, choosing $\lambda \equiv 1$ in Theorem 2 we obtain the following consequence on the Dirichlet problem for the nonlinear equations of the Vekua type.

Corollary 3. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to the logarithmic capacity, $h : D \to \mathbb{C}$ be in the class $L^p(D)$, $p > 2$, with compact support in D, and let $q : \mathbb{C} \to \mathbb{C}$ be a continuous function with condition* (5.1)*.*

Then there is a function $f : D \to \mathbb{C}$ in the class $C^{\alpha}_{\text{loc}}(D)$ with $\alpha = (p-2)/p$ and generalized first derivatives by Sobolev, satisfying equation (5.2) *that is a generalized analytic function with a source $g \in L^p(D)$ and the angular limits*

$$\lim_{\xi \to \omega} \text{Re}\, f(\xi) \;=\; \varphi(\omega) \qquad \textit{q.e. on } \partial D\,. \tag{5.6}$$

Moreover, $f(\xi) = \mathcal{H}_{\tilde{g}}(c(\xi))$, where c is a conformal mapping of D onto $\mathbb{D}$, $\mathcal{H}_{\tilde{g}}$ is the Hilbert operator described in Section 3 (with the simplest $\mathcal{A} \equiv 1$), $\tilde{g} = g \circ c^{-1}$, and the support of g is in the support of h and the upper bound of $\|g\|_p$ depends only on $\|h\|_p$, the function q and the domain D.

6. On completely continuous Poincare operators

In Section 7 of [12], we considered the Poincare boundary-value problem on the directional derivatives and, in particular, the Neumann problem with arbitrary measurable boundary data over logarithmic capacity for the Poisson equations

$$\triangle U(z) \;=\; G(z) \tag{6.1}$$

with real valued functions G of classes $L^p(D)$ with $p > 2$ in Jordan's domains $D \subset \mathbb{C}$. Recall that a continuous solution U of (6.1) in the class $W^{2,p}_{\rm loc}$ was called by us in [12] a **generalized harmonic function with the source G** and that by the Sobolev embedding theorem such a solution belongs to the class C^1, see Theorem I.10.2 in [23].

As usual, here $\frac{\partial u}{\partial \nu}\,(\xi)$ denotes the derivative of u at the point $\xi \in D$ in the direction $\nu \in \mathbb{C}$, $|\nu| = 1$, i.e.,

$$\frac{\partial u}{\partial \nu}(\xi) \;:=\; \lim_{t\to 0}\; \frac{u(\xi + t\cdot\nu) - u(\xi)}{t}\ . \tag{6.2}$$

The Neumann boundary value problem is a special case of the Poincare problem on the directional derivatives with the unit interior normal $n = n(\omega)$ to ∂D at the point ω as $\nu(\omega)$, see Corollary 4 further.

By Theorem 5 in [12], for each $\nu : \partial\mathbb{D} \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, in $\mathcal{CBV}(\partial\mathbb{D})$, $\varphi : \partial\mathbb{D} \to \mathbb{R}$ that is measurable with respect to logarithmic capacity and $G : \mathbb{D} \to \mathbb{R}$ in $L^p(\mathbb{D})$, $p > 2$, there is a generalized harmonic function $U : \mathbb{D} \to \mathbb{R}$ with the source G that have the angular limits

$$\lim_{z\to\zeta}\; \frac{\partial U}{\partial \nu}\;(z) \;=\; \varphi(\zeta) \qquad \text{q.e. on} \;\; \partial\mathbb{D}\ . \tag{6.3}$$

Furthermore, the space of such functions U has the infinite dimension.

As it follows from its proof and Section 3 at the present paper with $\lambda = \bar{\nu}$, one of such functions U can be presented as a sum of the logarithmic potential $P = N_G$ with the source G, see (3.3), and the harmonic function

$$\gamma(z) \;:=\; \mathrm{Re}\int\limits_0^z \{\ \mathcal{H}_{G/2}(\xi)\;-\;T_{G/2}(\xi)\ \}\ d\,\xi\ , \tag{6.4}$$

where we assume that $G \in L^p(\mathbb{D})$, $p > 2$, and has compact support in $\mathbb{D}$.

Denoting by $\mathcal{P}_G$ the given correspondence between such sources G and the generalized harmonic functions with the sources G and the Poincare boundary condition (6.3), we see that $\mathcal{P}_G$ is a completely continuous operator over each disk $|z| < r < 1$ because the operators $\mathcal{H}_{G/2}$ and $T_{G/2}$ are so and, in addition, the indefinite integral as well as the operator of taking Re are bounded and linear. Thus, by Lemma 2 and Remark 4 we come to the following statements.

Lemma 3. *Let $\nu : \partial\mathbb{D} \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, be of countable bounded variation and let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable with respect to the logarithmic capacity.*

Then there is a Poincare operator $\mathcal{P}_G$ over the sources $G : \mathbb{D} \to \mathbb{R}$ in $L^p(\mathbb{D})$, $p > 2$, with compact supports in $\mathbb{D}$, generating generalized harmonic functions $U : \mathbb{D} \to \mathbb{R}$ with the sources G and the angular limits (6.3), *whose restriction to sources G with $\operatorname{supp} G \subseteq \mathbb{D}_\rho$ is completely continuous over $\mathbb{D}_r$ for each ρ and $r \in (0,1)$.*

Remark 7. Moreover, we may assume that the restriction of the operator $\mathcal{P}_G$ to $\mathbb{D}_r$ under each $r \in (0,1)$ is bounded at infinity in the sense that $\max\limits_{z\in\mathbb{D}_r} |\mathcal{P}_G(z)| \leq M \cdot \|G\|_p$ for some $M > 0$ and all G with large enough $\|G\|_p$.

7. On Poincare problem for semi-linear equations

In this section we study the solvability of the Poincare boundary-value problem for semi-linear Poisson equations of the form $\triangle U(z) = H(z) \cdot Q(U(z))$ in the unit disk $\mathbb{D}$. Again the Leray–Schauder approach allows us to reduce the problem to the study of the linear Poisson equation from our last paper [12] on the basis of Lemma 3 on completely continuous Poincare operator $\mathcal{P}_G$ and Remark 7 on its boundedness at infinity from the previous section.

In the proof of the next theorem the initial operator $F(\cdot) := F(\cdot, 0) \equiv 0$. Hence F has the only one fixed point (at the origin) and its index is equal to 1 and, thus, hypothesis (H2) in Section 4 will be automatically satisfied.

Theorem 3. *Let $\nu : \partial\mathbb{D} \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, be of countable bounded variation and let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable with respect to the logarithmic capacity.*

Suppose that $H : \mathbb{D} \to \mathbb{R}$ is a function in the class $L^p(\mathbb{D})$ for $p > 2$ with compact support in $\mathbb{D}$ and $Q : \mathbb{R} \to \mathbb{R}$ is a continuous function with

$$\lim_{t\to\infty} \frac{Q(t)}{t} = 0 . \tag{7.1}$$

Then there is a function $U : \mathbb{D} \to \mathbb{R}$ in the class $W^{2,p}_{\text{loc}}(\mathbb{D}) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{D})$ with $\alpha = (p-2)/p$ such that

$$\triangle U(z) = H(z) \cdot Q(U(z)) \qquad a.e.\ in\ \mathbb{D} \tag{7.2}$$

and, in addition, U is a generalized harmonic function with a source $G \in L^p(\mathbb{D})$ and the angular limits

$$\lim_{z\to\zeta} \frac{\partial U}{\partial \nu}(z) = \varphi(\zeta) \qquad q.e.\ on\ \partial\mathbb{D}\ . \tag{7.3}$$

Moreover, $U = \mathcal{P}_G$, where $\mathcal{P}_G$ is the Poincare operator described in the last section, and the support of G is in the support of H and the upper bound of $\|G\|_p$ depends only on $\|H\|_p$ and on the function Q.

Proof. If $\|H\|_p = 0$ or $\|Q\|_C = 0$, then any harmonic function from Theorem 7.2 in [13] gives the desired solution of (7.2). Thus, we may assume that $\|H\|_p \neq 0$ and $\|Q\|_C \neq 0$. Set $Q_*(t) = \max\limits_{|\tau|\le t} |Q(\tau)|$, $t \in \mathbb{R}^+ := [0,\infty)$. Then the function $Q_* : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing and, moreover, by (7.1)

$$\lim_{t\to\infty} \frac{Q_*(t)}{t} = 0\ . \tag{7.4}$$

By Lemma 3 and Remark 7 we obtain the family of operators $F(G;\tau) : L^p_H(\mathbb{D}) \to L^p_H(\mathbb{D})$, where $L^p_H(\mathbb{D})$ consists of functions $G \in L^p(\mathbb{D})$ with supports in the support of H,

$$F(G;\tau) := \tau H\cdot Q(\mathcal{P}_G) \qquad \forall\, \tau\in[0,1] \tag{7.5}$$

which satisfies all groups of hypothesis H1–H3 of Theorem 1 in [15]. Indeed:

H1). First of all, by Lemma 3 the function $F(G;\tau) \in L^p_H(\mathbb{D})$ for all $\tau \in [0,1]$ and $G \in L^p_H(\mathbb{C})$ because the function $Q(\mathcal{P}_G)$ is continuous and, furthermore, the operators $F(\cdot\,;\tau)$ are completely continuous for each $\tau\in[0,1]$ and even uniformly continuous with respect to the parameter $\tau \in [0,1]$.

H2). The index of the operator $F(\cdot\,;0)$ is obviously equal to 1.

H3). Let us assume that solutions of the equations $G = F(G;\tau)$ is not bounded in $L^p_H(\mathbb{D})$, i.e., there is a sequence of functions $G_n \in L^p_H(\mathbb{D})$ with $\|G_n\|_p \to \infty$ as $n\to\infty$ such that $G_n = F(G_n;\tau_n)$ for some $\tau_n \in [0,1]$, $n = 1,2,\ldots$. However, then by Remark 7 we have that, for some constant $M > 0$,

$$\|G_n\|_p \le \|H\|_p\, Q_*(\,M\,\|G_n\|_p)$$

and, consequently,

$$\frac{Q_*(\,M\,\|G_n\|_p)}{M\,\|G_n\|_p} \ge \frac{1}{M\,\|H\|_p} > 0 \tag{7.6}$$

for all large enough n. The latter is impossible by condition (4.4). The obtained contradiction disproves the above assumption.

Thus, by Theorem 1 in [15] there is a function $G \in L^p_H(D)$ with $F(G;1) = G$, and by Lemma 3 the function $U := \mathcal{P}_G$ gives the desired solution of (4.2). $\square$

Remark 8. By the construction in the above proof, the source $G : \mathbb{D} \to \mathbb{R}$ is a fixed point of the nonlinear operator $\Omega_G := H \cdot Q(\mathcal{P}_G) : L^p_H(\mathbb{D}) \to L^p_H(\mathbb{D})$, where $L^p_H(\mathbb{D})$ consists of functions G in $L^p(\mathbb{D})$ with supports in the support of H.

We are able to say more in Theorem 3 for the case of $\mathrm{Re}\, n(\zeta)\overline{\nu(\zeta)} > 0$, where $n(\zeta)$ is the inner normal to $\partial\mathbb{D}$ at the point ζ. Indeed, the latter magnitude is a scalar product of $n = n(\zeta)$ and $\nu = \nu(\zeta)$ interpreted as vectors in $\mathbb{R}^2$ and it has the geometric sense of projection of the vector ν into n. In view of (7.3), since the limit $\varphi(\zeta)$ is finite, there is a finite limit $U(\zeta)$ of $U(z)$ as $z \to \zeta$ in $\mathbb{D}$ along the straight line passing through the point ζ and being parallel to the vector ν because along this line

$$U(z) \ = \ U(z_0) \ - \int\limits_0^1 \frac{\partial U}{\partial \nu}\ (z_0 + \tau(z - z_0))\ d\tau\ . \tag{7.7}$$

Thus, at each point with condition (7.3), there is the directional derivative

$$\frac{\partial U}{\partial \nu}\ (\zeta) \ := \ \lim_{t \to 0} \ \frac{U(\zeta + t \cdot \nu) - U(\zeta)}{t} \ = \ \varphi(\zeta)\ . \tag{7.8}$$

In particular, in the case of the Neumann problem, $\mathrm{Re}\, n(\zeta)\overline{\nu(\zeta)} \equiv 1 > 0$, where $n = n(\zeta)$ denotes the unit inner normal to $\partial\mathbb{D}$ at the point ζ, and we have by Theorem 3 and Remark 8 the following significant result.

Corollary 4. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable with respect to logarithmic capacity, $H : \mathbb{D} \to \mathbb{R}$ be in $L^p(\mathbb{D})$, $p > 2$, with compact support in $\mathbb{D}$ and let $Q : \mathbb{R} \to \mathbb{R}$ be a continuous function with condition* (7.1).

Then one can find generalized harmonic functions $U : \mathbb{D} \to \mathbb{R}$ with a source $G \in L^p(\mathbb{D})$ satisfying equation (7.2) *such that q.e. on $\partial\mathbb{D}$ there exist:*

1) the finite limit along the normal $n(\zeta)$

$$U(\zeta) \ := \ \lim_{z \to \zeta} \ U(z)\ ,$$

2) the normal derivative

$$\frac{\partial U}{\partial n}\,(\zeta) \ := \ \lim_{t \to 0} \ \frac{U(\zeta + t \cdot n(\zeta)) - U(\zeta)}{t} \ = \ \varphi(\zeta)\ ,$$

3) the angular limit

$$\lim_{z\to\zeta} \frac{\partial U}{\partial n}(z) \ = \ \frac{\partial U}{\partial n}(\zeta)\ .$$

8. On Poincare problem in more general domains

Now we extend the above results to Jordan domains with the quasihyperbolic boundary condition. Recall once more that such domains include, for instance, domains with quasiconformal boundaries and, in particular, domains with smooth and Lipschitz boundaries, but in general quasiconformal curves can be even nowhere locally rectifiable.

Theorem 4. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\nu : \partial D \to \mathbb{C}$, $|\nu| \equiv 1$, be in $\mathcal{CBV}(\partial D)$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose that $H : D \to \mathbb{R}$ is a function in the class $L^p(D)$ for $p > 2$ with compact support in D and $Q : \mathbb{R} \to \mathbb{R}$ is a continuous function with

$$\lim_{t\to\infty} \frac{Q(t)}{t} \ = \ 0\ . \tag{8.1}$$

Then there is a function $U : D \to \mathbb{R}$ in the class $W^{2,p}_{\rm loc}(D) \cap C^{1,\alpha}_{\rm loc}(D)$ with $\alpha = (p-2)/p$ such that

$$\triangle U(\xi) \ = \ H(\xi)\cdot Q(U(\xi)) \qquad \text{a.e. in } D \tag{8.2}$$

and, in addition, U is a generalized harmonic function with a source $G \in L^p(D)$ and with the angular limits

$$\lim_{\xi\to\omega} \frac{\partial U}{\partial \nu}\ (\xi) \ = \ \varphi(\omega) \qquad \text{q.e. on } \ \partial D\ . \tag{8.3}$$

Moreover, $U(\xi) = \mathcal{P}_{\tilde G}(c(\xi))$, where c is a conformal mapping of D onto $\mathbb{D}$, $\mathcal{P}_{\tilde G}$ is the Poincare operator described in the previous section, $\tilde G = G \circ c^{-1}$, and the support of G is in the support of H and the upper bound of $\|G\|_p$ depends only on $\|H\|_p$, the function Q and the domain D.

Proof. Arguing similarly to the first item in the proof of Theorem 2, we see that $\tilde\nu := \nu \circ c_*^{-1} \in \mathcal{CBV}(\partial\mathbb{D})$ and $\tilde\varphi := \varphi \circ c_*^{-1}$ is measurable with respect to the logarithmic capacity, where $c_* := \tilde c|_{\partial D} : \partial D \to \partial\mathbb{D}$ is the restriction to the boundary of the homeomorphic extension of c to $\overline{D}$ onto $\overline{\mathbb{D}}$.

Now, set $\tilde{H} = |C'|^2 \cdot H \circ C$, where C is the inverse conformal mapping $C := c^{-1} : \mathbb{D} \to D$. Then it is clear by the hypothesis of Theorem 4 that $\tilde{H}$ has compact support in $\mathbb{D}$ and belongs to the class $L^p(\mathbb{D})$. Consequently, by Theorem 3 there is a function $\tilde{U} : \mathbb{D} \to \mathbb{R}$ in the class $W^{1,p}_{\mathrm{loc}}(\mathbb{D}) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{D})$ with $\alpha = (p-2)/p$ such that

$$\triangle \tilde{U}(z) \;=\; \tilde{H}(z) \cdot Q(\tilde{U}(z)) \qquad \text{a.e. in } \mathbb{D} \tag{8.4}$$

and $\tilde{U}$ is a generalized analytic function with a source $\tilde{G} \in L^p(\mathbb{D})$ and the angular limits

$$\lim_{z \to \zeta} \frac{\partial \tilde{U}}{\partial \tilde{\nu}}\,(z) \;=\; \tilde{\varphi}(\zeta) \qquad \text{q.e. on } \partial\mathbb{D}\,, \tag{8.5}$$

moreover, $\tilde{U} = \mathcal{P}_{\tilde{G}}$, where $\mathcal{P}_{\tilde{G}}$ is the Poincare operator described in Section 6, and the support of $\tilde{G}$ is in the support of $\tilde{H}$ and the upper bound of $\|\tilde{G}\|_p$ depends only on $\|\tilde{H}\|_p$ and on the function Q.

Next, setting $U = \tilde{U} \circ c$, by simple calculations, see e.g. Section 1.C in [1], we obtain that $\triangle U = |c'|^2 \cdot \triangle \tilde{U} \circ c$ and, consequently, the function $U : D \to \mathbb{C}$ is in the class $W^{1,p}_{\mathrm{loc}}(D) \cap C^{1,\alpha}_{\mathrm{loc}}(D)$ with $\alpha = (p-2)/p$ that satisfies equation (8.2), U is a generalized harmonic function with a source $G \in L^p(D)$ and, moreover, $U(\xi) = \mathcal{P}_{\tilde{G}}(c(\xi))$, where $\mathcal{P}_{\tilde{G}}$ is the Poincare operator from Section 6, $\tilde{G} = G \circ c^{-1}$, and the support of G is in the support of H and the upper bound of $\|G\|_p$ depends only on $\|H\|_p$, the function Q and the domain D.

Finally, arguing similarly to the last item in the proof of Theorem 2, we show that (8.5) implies (8.3). ☐

Remark 9. By the construction in the above proof, the source $G = \tilde{G} \circ c$, where c is a conformal mapping of D onto $\mathbb{D}$ and $\tilde{G} : \mathbb{D} \to \mathbb{R}$ is a fixed point of the nonlinear operator $\tilde{\Omega}_{G_*} := \tilde{H} \cdot Q(\mathcal{P}_{G_*}) : L^p_{\tilde{H}}(\mathbb{D}) \to L^p_{\tilde{H}}(\mathbb{D})$, where $L^p_{\tilde{H}}(\mathbb{D})$ consists of functions G_* in $L^p(\mathbb{D})$ with supports in the support of $\tilde{H} := |C'|^2 \cdot H \circ C$, where C is the inverse conformal mapping $C := c^{-1} : \mathbb{D} \to D$.

We are able to say more in Theorem 4 for the case of $\mathrm{Re}\, n(\zeta)\overline{\nu(\zeta)} > 0$, where $n(\zeta)$ is the inner normal to ∂D at the point ζ. Indeed, the latter magnitude is a scalar product of $n = n(\zeta)$ and $\nu = \nu(\zeta)$ interpreted as vectors in $\mathbb{R}^2$ and it has the geometric sense of projection of the vector ν into n. In view of (8.3), since the limit $\varphi(\zeta)$ is finite, there is a finite limit $U(\zeta)$ of $U(z)$ as $z \to \zeta$ in D along the straight line passing through the point ζ and being parallel to the vector ν because along this line

$$U(z) \;=\; U(z_0) \;-\; \int\limits_0^1 \frac{\partial U}{\partial \nu}\,(z_0+\tau(z-z_0))\;d\tau\;. \tag{8.6}$$

Thus, at each point with condition (8.3), there is the directional derivative

$$\frac{\partial U}{\partial \nu}\,(\zeta)\;:=\;\lim_{t\to 0}\;\frac{U(\zeta+t\cdot\nu)-U(\zeta)}{t}\;=\;\varphi(\zeta)\;. \tag{8.7}$$

In particular, in the case of the Neumann problem, Re $n(\zeta)\overline{\nu(\zeta)}\equiv 1>0$, where $n=n(\zeta)$ denotes the unit interior normal to ∂D at the point ζ, and we have by Theorem 4 and Remark 9 the following significant result.

Corollary 5. *Let D be a Jordan domain in $\mathbb{C}$ with the quasihyperbolic boundary condition, the unit inner normal $n(\zeta)$, $\zeta\in\partial D$, belong to the class $\mathcal{CBV}(\partial D)$ and $\varphi:\partial D\to\mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose that $H:D\to\mathbb{R}$ is in $L^p(D)$, $p>2$, with compact support in D. Then one can find a generalized harmonic function $U:D\to\mathbb{R}$ with a source $G\in L^p(D)$ satisfying equation (8.2) *such that q.e. on ∂D there exist:*

1) the finite limit along the normal $n(\zeta)$

$$U(\zeta)\;:=\;\lim_{z\to\zeta}\;U(z)\;,$$

2) the normal derivative

$$\frac{\partial U}{\partial n}\,(\zeta)\;:=\;\lim_{t\to 0}\;\frac{U(\zeta+t\cdot n(\zeta))-U(\zeta)}{t}\;=\;\varphi(\zeta)\;,$$

3) the angular limit

$$\lim_{z\to\zeta}\;\frac{\partial U}{\partial n}\,(z)\;=\;\frac{\partial U}{\partial n}\,(\zeta)\;.$$

9. Poincare and Neumann problems in physical applications

Theorem 4 and Corollary 5 on the Poincare and Neumann boundary-value problems, correspondingly, with arbitrary measurable boundary data over the logarithmic capacity can be applied to mathematical problems appearing under modeling various types of physical and chemical absorption with diffusion, plasma states, stationary burning etc.

The first circle of such applications is relevant to reaction-diffusion problems. Problems of this type are discussed in [6], p. 4, and, in detail, in [2]. A nonlinear system is obtained for the density U and the temperature T of the reactant. Upon eliminating T the system can be reduced to equations of the type (8.2),

$$\triangle U \;=\; \sigma \cdot Q(U) \tag{9.1}$$

with $\sigma > 0$ and, for isothermal reactions, $Q(U) = U^{\beta}$ where $\beta > 0$ that is called the order of the reaction. It turns out that the density of the reactant U may be zero in a subdomain called a **dead core.** A particularization of results in Chapter 1 of [6] shows that a dead core may exist just if and only if $\beta \in (0,1)$ and σ is large enough, see also the corresponding examples in [9]. In this connection, the following statements may be of independent interest.

Corollary 6. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\nu : \partial D \to \mathbb{C}$, $|\nu| \equiv 1$, be in $\mathcal{CBV}(\partial D)$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose that $H : D \to \mathbb{R}$ is a function in the class $L^p(D)$ for $p > 2$ with compact support in D.

Then there is a solution $U : D \to \mathbb{R}$ in the class $W^{2,p}_{\rm loc}(D) \cap C^{1,\alpha}_{\rm loc}(D)$ with $\alpha = (p-2)/p$ of the semi-linear Poisson equation

$$\triangle U(\xi) \;=\; H(\xi) \cdot U^{\beta}(\xi)\;, \quad 0 < \beta < 1\;, \qquad a.e.\ in\ D \tag{9.2}$$

satisfying the Poincare boundary condition on directional derivatives

$$\lim_{\xi \to \omega} \frac{\partial U}{\partial \nu}\,(\xi) \;=\; \varphi(\omega) \qquad q.e.\ on\ \ \partial D \tag{9.3}$$

in the sense of the angular limits.

Moreover, U is a generalized harmonic function with a source $G \in L^p(D)$ whose support is in the support of H and the upper bound of $\|G\|_p$ depends only on $\|H\|_p$, the function Q and the domain D.

Corollary 7. *In particular, in the case of the Neumann problem, i.e., if $\nu(\zeta)$ is the unit interior normal $n(\zeta)$ to ∂D at the point ζ, one can find a solution $U : D \to \mathbb{R}$ in the class $W^{2,p}_{\rm loc}(D) \cap C^{1,\alpha}_{\rm loc}(D)$ with $\alpha = (p-2)/p$ of the semi-linear Poisson equation* (9.2) *such that q.e. on ∂D there exist:*

1) the finite limit along the normal $n(\zeta)$

$$U(\zeta) \ := \ \lim_{z\to\zeta} \ U(z) \ ,$$

2) the normal derivative

$$\frac{\partial U}{\partial n}\,(\zeta) \ := \ \lim_{t\to 0} \ \frac{U(\zeta+t\cdot n(\zeta))-U(\zeta)}{t} \ = \ \varphi(\zeta) \ ,$$

3) the angular limit

$$\lim_{z\to\zeta} \ \frac{\partial U}{\partial n}\,(z) \ = \ \frac{\partial U}{\partial n}\,(\zeta) \ .$$

Note also that certain mathematical models of a thermal evolution of a heated plasma lead to nonlinear equations of the type (9.1). Indeed, it is known that some of them have the form $\triangle\psi(u)=f(u)$ with $\psi'(0)=\infty$ and $\psi'(u)>0$ if $u\neq 0$ as, for instance, $\psi(u)=|u|^{q-1}u$ under $0<q<1$, see e.g. [6]. With the replacement of the function $U=\psi(u)=|u|^q\cdot\mathrm{sign}\,u$, we have that $u=|U|^Q\cdot\mathrm{sign}\,U$, $Q=1/q$, and, with the choice $f(u)=|u|^{q^2}\cdot\mathrm{sign}\,u$, we come to the equation $\triangle U=|U|^q\cdot\mathrm{sign}\,U=\psi(U)$.

Corollary 8. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\nu:\partial D\to\mathbb{C}$, $|\nu|\equiv 1$, be in $\mathcal{CBV}(\partial D)$ and $\varphi:\partial D\to\mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose also that $H:D\to\mathbb{R}$ is a function in the class $L^p(D)$ for $p>2$ with compact support in D.

Then there is a solution $U:D\to\mathbb{R}$ in the class $W^{2,p}_{\rm loc}(D)\cap C^{1,\alpha}_{\rm loc}(D)$ with $\alpha=(p-2)/p$ of the semi-linear Poisson equation

$$\triangle U(\xi) \ = \ H(\xi)\cdot|U(\xi)|^{\beta-1}U(\xi) \ , \quad 0\ <\ \beta\ <\ 1 \ , \qquad a.e.\ in\ D \tag{9.4}$$

satisfying the Poincare boundary condition on directional derivatives (9.3).

Moreover, U is a generalized harmonic function with a source $G\in L^p(D)$ whose support is in the support of H and the upper bound of $\|G\|_p$ depends only on $\|H\|_p$, the function Q and the domain D.

Corollary 9. *In particular, in the case of the Neumann problem, i.e., if $\nu(\zeta)$ is the unit interior normal $n(\zeta)$ to ∂D at the point ζ, one can find a solution $U:D\to\mathbb{R}$ in the class $W^{2,p}_{\rm loc}(D)\cap C^{1,\alpha}_{\rm loc}(D)$ with $\alpha=(p-2)/p$ of the semi-linear Poisson equation* (9.4) *that q.e. on ∂D satisfies the conclusions 1)-3) of Corollary 7.*

Moreover, U is a generalized harmonic function with a source $G \in L^p(D)$ whose support is in the support of H and the upper bound of $\|G\|_p$ depends only on $\|H\|_p$, the function Q and the domain D.

Finally, we recall that in the combustion theory, see e.g. [3, 20] and the references therein, the following model equation

$$\frac{\partial u(z,t)}{\partial t} \;=\; \frac{1}{\delta}\cdot\triangle u \;+\; e^u\ , \quad t\geq 0,\ z\in D, \tag{9.5}$$

takes a special place. Here $u \geq 0$ is the temperature of the medium and δ is a certain positive parameter. We restrict ourselves here by the stationary case, although our approach makes it possible to study the parabolic equation (9.5), see [9]. Namely, the corresponding equation of the type (8.2) is appeared here after the replacement of the function u by $-u$ with the function $Q(u) = e^{-u}$ that is bounded at all.

Corollary 10. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\nu : \partial D \to \mathbb{C}$, $|\nu| \equiv 1$, be in $\mathcal{CBV}(\partial D)$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose also that $H : D \to \mathbb{R}$ is a function in the class $L^p(D)$ for $p > 2$ with compact support in D.

Then there is a solution $U : D \to \mathbb{R}$ in the class $W^{2,p}_{\rm loc}(D) \cap C^{1,\alpha}_{\rm loc}(D)$ with $\alpha = (p-2)/p$ of the semi-linear Poisson equation

$$\triangle U(\xi) \;=\; H(\xi)\cdot e^{U(\xi)}\ , \qquad a.e.\ in\ D \tag{9.6}$$

satisfying the Poincare boundary condition on directional derivatives (9.3).

Moreover, U is a generalized harmonic function with a source $G \in L^p(D)$ whose support is in the support of H and the upper bound of $\|G\|_p$ depends only on $\|H\|_p$, the function Q and the domain D.

Corollary 11. *In particular, in the case of the Neumann problem, i.e., if $\nu(\zeta)$ is the unit interior normal $n(\zeta)$ to ∂D at the point ζ, one can find a solution $U : D \to \mathbb{R}$ in the class $W^{2,p}_{\rm loc}(D) \cap C^{1,\alpha}_{\rm loc}(D)$ with $\alpha = (p-2)/p$ of the semi-linear Poisson equation* (9.6) *that q.e. on ∂D satisfies the conclusions 1)-3) of Corollary 7. Moreover, U is a generalized harmonic function with a source $G \in L^p(D)$ whose support is in the support of H and the upper bound of $\|G\|_p$ depends only on $\|H\|_p$, the function Q and the domain D.*

10. Dirichlet problem for Poisson's type equations

The situation with the Dirichlet problem is much simpler that makes possible to lower the degree of integrability of sources G.

As known, N_G for $G \in L^p$ with $p > 1$ and with support in $\mathbb{D}$ is continuous in $\mathbb{C}$, belongs to the class $W^{2,p}(\mathbb{D})$ and $\triangle N_G = G$ a.e. Moreover, $N_G \in W^{1,q}_{\rm loc}(\mathbb{C})$ for $q > 2$, consequently, N_G is locally Hölder continuous. If $G \in L^p(\mathbb{C})$, $p > 2$, then $N_G \in C^{1,\alpha}_{\rm loc}(\mathbb{C})$ for $\alpha := (p-2)/p$, and for all $\alpha \in (0,1)$ under $p = \infty$, see e.g. Lemma 3 in [10] or Theorem 2 in [11].

Furthermore, the collection $\{N_G\}$ is equicontinuous if the collection $\{G\}$ is bounded by the norm in $L^p(\mathbb{C})$. Moreover, on each compact set S in $\mathbb{C}$

$$\|N_G\|_C \ \leq \ M \cdot \|G\|_p \ , \tag{10.1}$$

where M is a constant depending only on S and, in particular, the restriction of N_G to $\overline{\mathbb{D}}$ is a completely continuous bounded linear operator, see e.g. Lemma 2 in [10] or Theorem 1 in [11].

By Proposition 1 there is a space of harmonic functions u in the unit disk $\mathbb{D}$ of the infinite dimension with the angular limits q.e. on $\partial\mathbb{D}$

$$\lim_{z\to\zeta} u(z) \ = \ \psi(\zeta) \ := \ \varphi(\zeta) \ - \ \varphi_G(\zeta) \ , \qquad \varphi_G(\zeta) := N_G(\zeta) \ . \tag{10.2}$$

Note that $U := u + N_G|_{\mathbb{D}}$ with such u are continuous solutions of the Poisson equation $\triangle U = G$ a.e. in the class $W^{2,p}_{\rm loc}(\mathbb{D})$ with the angular limits

$$\lim_{z\to\zeta} U(z) \ = \ \varphi(\zeta) \qquad \text{q.e. on } \partial\mathbb{D} \ . \tag{10.3}$$

By Remark 2 such a harmonic function $u : \mathbb{D} \to \mathbb{R}$ can be obtained in the form of the real part of the analytic function

$$\mathbb{S}_\Psi(z) \ := \ \frac{z}{\pi} \int\limits_{\partial\mathbb{D}} \frac{\Psi(\zeta)}{(\zeta - z)^2}\, d\zeta \ , \qquad z \in \mathbb{D} \ , \tag{10.4}$$

where Ψ is an antiderivative of the function ψ from Corollary 1.

Consequently, such a harmonic function u can be represented in the form

$$u(z) \ = \ u_0(z) - u_G(z) \ , \quad u_0(z) \ := \ \mathrm{Re}\, \mathbb{S}_\Phi(z) \ , \quad u_G(z) \ := \ \mathrm{Re}\, \mathbb{S}_{\Phi_G}(z) \ , \tag{10.5}$$

where Φ and Φ_G are antiderivatives of φ and φ_G in Corollary 1, correspondingly. Note that the harmonic function u_0 does not depend on the sources G at all.

Let us choose the function Φ_G in a suitable way to guarantee that the correspondence $G \mapsto u + N_G|_{\mathbb{D}}$ is a Dirichlet operator $\mathcal{D}_G$ that is completely continuous on compact sets in $\mathbb{D}$ generating solutions of the Poisson equation $\triangle U = G$ a.e. in the class $C \cap W^{2,p}_{\rm loc}(\mathbb{D})$ with the Dirichlet boundary condition (10.3).

Namely, the following function Φ_G is an antiderivative for the function φ_G:

$$\Phi_G(\zeta) \ := \int\limits_0^{\vartheta} N_G(e^{i\theta})\, d\theta \ - \ S(\vartheta)\ , \quad \zeta = e^{i\vartheta}, \quad \theta\ , \vartheta \in [0, 2\pi]\ , \tag{10.6}$$

where $S : [0, 2\pi] \to \mathbb{C}$ is either zero or a singular function of the form

$$S(\vartheta) \ := \ C(\vartheta) \int\limits_0^{2\pi} N_G(e^{i\theta})\, d\theta\ , \quad \zeta = e^{i\vartheta}, \quad \theta\ , \vartheta \in [0, 2\pi]\ , \tag{10.7}$$

with a singular function $C : [0, 2\pi] \to [0, 1]$ of the Cantor ladder type, i.e., C is continuous, nondecreasing, $C(0) = 0$, $C(2\pi) = 1$ and $C' = 0$ q.e. Recall that the existence of such functions C follows from Lemma 3.1 in [7].

Setting $u_G = \mathrm{Re}\ \mathbb{S}_{\Phi_G}$, it is easy to see by (10.1) that

$$|\Phi_G(\zeta)| \ \leq \ 4\pi M \cdot \|G\|_p \qquad \forall\, \zeta \in \partial\mathbb{D} \tag{10.8}$$

and by (2.6) that, for constants C_r and C_r^* depending only on $r \in (0, 1)$,

$$|u_G(z)| \ \leq \ |\mathbb{S}_{\Phi_G}(z)| \ \leq \ C_r \cdot \|G\|_p\ , \quad \forall\ z \in \mathbb{D}_r\ , \tag{10.9}$$

$$|u_G(z_1) - u_G(z_2)| \leq |\mathbb{S}_{\Phi_G}(z_1) - \mathbb{S}_{\Phi_G}(z_2)| \leq C_r^* \|G\|_p |z_1 - z_2|, z_1, z_2 \in \mathbb{D}_r. \tag{10.10}$$

Consequently, the operator $u_G := \mathrm{Re}\ \mathbb{S}_{\Phi_G}$ is completely continuous on compact sets in $\mathbb{D}$ by the Arzela–Ascoli theorem, see e.g. Theorem IV.6.7 in [5]. Thus, we obtain the next conclusion.

Lemma 4. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable over logarithmic capacity. Then there is a Dirichlet operator $\mathcal{D}_G$ over $G : \mathbb{D} \to \mathbb{C}$ in $L^p(\mathbb{D})$, $p > 1$, generating continuous solutions $U : \mathbb{D} \to \mathbb{R}$ of the Poisson equation $\triangle U = G$ in the class $W^{2,p}_{\mathrm{loc}}(\mathbb{D})$ with the Dirichlet boundary condition* (10.3) *in the sense of angular limits q.e. on $\partial\mathbb{D}$, that is completely continuous over $\mathbb{D}_r$ for each $r \in (0, 1)$.*

Remark 10. Note that the nonlinear operator $\mathcal{D}_G$ constructed above is not bounded except the trivial case $\Phi \equiv 0$ because then $\mathcal{D}_0 = \mathbb{S}_\Phi \neq 0$. However, the restriction of the operator $\mathcal{D}_G$ to $\mathbb{D}_r$ under each $r \in (0, 1)$ is bounded at infinity in the sense that $\max\limits_{z\in\mathbb{D}_r} |\mathcal{D}_G(z)| \leq M \cdot \|G\|_p$ for some $M > 0$ and all G with large enough $\|G\|_p$. Note also that by Corollary 1 we are able always to choose Φ for any φ, including $\varphi \equiv 0$, which is not identically 0 in the unit disk $\mathbb{D}$.

Moreover, by the above construction $U := \mathcal{D}_G$ belongs to the class $W^{1,q}_{\rm loc}(\mathbb{D})$ for some $q > 2$, consequently, U is locally Hölder continuous. If $G \in L^p(\mathbb{D})$, $p > 2$, then $U \in C^{1,\alpha}_{\rm loc}(\mathbb{D})$ for $\alpha := (p-2)/p$, and for all $\alpha \in (0,1)$ under $p = \infty$.

Lemma 4 and Remark 10 make it possible to obtain the following statement on the Dirichlet problem with arbitrary boundary data that are measurable over logarithmic capacity for semi-linear Poisson equations.

Theorem 5. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose that $H : D \to \mathbb{R}$ is a function in the class $L^p(D)$ for $p > 1$ with compact support in D and $Q : \mathbb{R} \to \mathbb{R}$ is a continuous function with

$$\lim_{t\to\infty} \frac{Q(t)}{t} \ = \ 0 \ . \tag{10.11}$$

Then there is a continuous solution U of the class $W^{2,p}_{\rm loc}(D)$ for the equation

$$\triangle U(z) \ = \ H(z) \cdot Q(U(z)) \tag{10.12}$$

a.e. in D with the angular limits q.e. on ∂D

$$\lim_{z\to\zeta} U(z) \ = \ \varphi(\zeta) \ . \tag{10.13}$$

Moreover, U belongs to the class $W^{1,q}_{\rm loc}(\mathbb{D})$ for some $q > 2$ and, consequently, U is locally Hölder continuous. Furthermore, if $G \in L^p(\mathbb{D})$ with $p > 2$, then $U \in C^{1,\alpha}_{\rm loc}(\mathbb{D})$ for $\alpha := (p-2)/p$, and for all $\alpha \in (0,1)$ under $p = \infty$.

The proof of Theorem 5 will be a simple exercise for the reader because of it is perfectly similar to one for Theorems 3 and 4 and its general scheme is based again on the Leray–Schauder approach.

Theorem 5 can be applied to the study of the physical phenomena discussed by us in the last section. In this connection, the particular cases of the function $Q(t)$ of the forms e^t, t^β and $|t|^{\beta-1}t$ with $\beta \in (0,1)$ will be useful.

Finally, due to the factorization theorem in [9], we are able by the quasiconformal replacements of variables to extend the above results to semi–linear equations of the Poisson type describing the corresponding physical phenomena in anisotropic and inhomogeneous media, too, that shall be published elsewhere.

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Contact information

Vladimir Gutlyanskiĭ
Institute of Applied Mathematics
and Mechanics of the NAS of Ukraine,
Slavyansk, Ukraine
E-Mail: vgutlyanskii@gmail.com

Olga Nesmelova
Institute of Applied Mathematics
and Mechanics of the NAS of Ukraine,
Slavyansk, Ukraine
E-Mail: star-o@ukr.net

Vladimir Ryazanov
Institute of Applied Mathematics
and Mechanics of the NAS of Ukraine,
Slavyansk, Ukraine,
Bogdan Khmelnytsky National
University of Cherkasy,
Cherkasy, Ukraine
E-Mail: Ryazanov@nas.gov.ua,
vl.ryazanov1@gmail.com

Artyem Yefimushkin
Institute of Applied Mathematics
and Mechanics of the NAS of Ukraine,
Slavyansk, Ukraine
E-Mail: a.yefimushkin@gmail.com