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Maximal coefficient functionals on univalent functions

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Abstract. Recently the author presented a new approach to solving the coefficient problems for holomorphic functions based on the features of Bers' fiber spaces for punctured Riemann surfaces. The holomorphy of functionals causes strong rigid constrains.

This paper extends the previous results to a broad class of plurisubharmonic coefficient functionals and provides new extremal features of the Koebe function.

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1. Introductory remarks. Main result

This paper continues the recent authors research on solving the classical coefficient problems for holomorphic functions by applying a new approach based on lifting the functionals onto the Bers fiber space over the universal Teichmüller space.

Estimating holomorphic functionals depending on the Taylor coefficients of univalent holomorphic functions is an old problem arising in various geometric and physical applications of complex analysis. The indicated approach has been offered in [14, 15].

The goal of this paper is to extend the obtained results to subharmonic functionals, to avoid the strong rigid constrains caused by holomorphy.

We consider on the canonical class S of univalent functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

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on the unit disk $\mathbb{D} = \{|z| < 1\}$ the general plurisubharmonic polynomial coefficient functionals of the form

$$J(f) = J(\alpha_{n_1,\dots,n_m})$$

$$= \max_{-\pi \le \vartheta_1,\dots,\vartheta_m \le \pi} \Big| \sum_{|n|=2}^{N} \alpha_{n_1,\dots,n_m} a_{n_1} \dots a_{n_m} \; e^{i(n_1\vartheta_1 + \dots + n_m\vartheta_m)} \Big|, \qquad (1)$$

where $|n| = n_1 + \dots + n_m$ and the factors $\alpha_{n_1,\dots,n_m}$ do not depend on coefficients a_j. For such functionals, we have the following general distortion theorem.

Theorem. *Any nonconstant plurisubharmonic coefficient functional* (1) *is maximized on the class S only by the rotations*

$$w_{\tau,\theta}(z) = e^{-i\theta} w(e^{i\tau} z) \qquad (2)$$

of the Koebe function

$$\kappa_0(z) = \frac{z}{(1-z)^2} = z + \sum_2^\infty n z^n. \qquad (3)$$

This theorem shows that the rigid constrains to holomorphic functionals in [14, 15] (the rotational invariance, restrictions to location of the zero set), are omitted in the case of plurisubharmonic functionals.

In particular, the theorem gives, taking $J(f) = |a_n|$, the estimate $|a_n| \le n$ on S stated by the Bieberbach conjecture (see [3, 8, 14]).

We also mention another important consequence from this theorem.

Corollary. *The Koebe function (and its rotations) maximizes every trigonometric polynomial*

$$\sum_{|n|=2}^{N} a_{n_1} \dots a_{n_m} \; e^{i(n_1\vartheta_1 + \dots + n_m\vartheta_m)}$$

of a degree $N \ge 2$ generated by the coefficients of functions $f \in S$.

2. Functions with quasiconformal extension

In a similar way, one can consider plurisubharmonic functionals on rotationally invariant subclasses $\mathcal{X}$ of S satisfying some essential additional conditions and extend to this classes the results of [15].

A completely different situation arises for univalent functions admitting quasiconformal extension across the unit circle $\mathbf{S}^1 = \{|z| = 1\}$ to the whole Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with a prescribed bound for their dilatations. Such functions are intrinsically connected with the Teichmüller space theory bridging it with geometric function theory and play a crucial role in both these fields. But this class does not obey the conditions indicated above.

Consider, for example, the subclass $S(k)$ of S, which consists of $f \in S$ admitting k-quasiconformal extensions $\widehat{f}$ onto the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with the additional normalization $f(\infty) = \infty$. These extensions satisfy in the complementary disk

$$\mathbb{D}^* = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} = \{z \in \widehat{\mathbb{C}} : \ |z| > 1\}$$

the Beltrami equation $\partial_{\bar{z}} w = \mu(z) \partial_z w$ with $\|\mu\|_\infty \le k < 1$. The quantity $k(w) = \|\mu_w\|_\infty$ is called **quasiconformal dilatation** of the map w.

There is a lot of holomorphic and subharmonic polynomial functionals $J(f) : S(k) \to \mathbb{C}$ satisfying

$$\max_{S(k)} |J(f)| = k.$$

Such functionals arise, for example, from the **Grunsky coefficients** $\alpha_{n,n}(f)$ of functions $f \in S(k)$ with equal Grunsky and Teichmüller norms (see, e.g., [11, 17]).

Recall that due to the classical Grunsky theorem, a holomorphic function $f(z) = z + a_2 z^2 + \dots$ in a neighborhood of the origin $z = 0$ can be extended to a univalent function on $\mathbb{D}$ if and only if its Grunsky coefficients α_{mn} satisfy the inequality

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| \le 1, \tag{4}$$

where α_{mn} are defined by

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=1}^{\infty} \alpha_{mn} z^m \zeta^n, \quad (z, \zeta) \in (\mathbb{D})^2,$$

the sequence $\mathbf{x} = (x_n)$ runs over the unit sphere $S(l^2)$ of the Hilbert space l^2 with norm $\|\mathbf{x}\|^2 = \sum\limits_1^\infty |x_n|^2$, and the principal branch of the logarithmic function is chosen (cf. [7]). The quantity

$$\varkappa(f) = \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| : \ \mathbf{x} = (x_n) \in S(l^2) \Big\} \le 1$$

is called the **Grunsky norm** of f.

For the functions with k-quasiconformal extensions ($k < 1$), we have instead of (4) a stronger bound

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| \le k \quad \text{for any} \ \ \mathbf{x} = (x_n) \in S(l^2), \qquad (5)$$

established first by Kühnau in [18]. Then $\varkappa(f) \le k(f)$, where $k(f)$ denotes the **Teichmüller norm** of f which is equal to the infimum of dilatations $k(w^\mu) = \|\mu\|_\infty$ of quasiconformal extensions of f to $\widehat{\mathbb{C}}$.

Each coefficient α_{mn} is represented as a polynomial of a finite number of the initial coefficients $a_2, \dots, a_s$ of f; hence it depends holomorphically on Beltrami coefficients of quasiconformal extensions of f as well as on the Schwarzian derivatives S_f.

It is technically more convenient to deal with the inverted functions

$$F(z) = 1/f(1/z) = z + b_0 + b_1 z^{-1} + \cdots : \ \mathbb{D}^* \to \widehat{\mathbb{C}}, \qquad (6)$$

which are univalent on $\mathbb{D}^*$ and have the same Grunsky coefficients as f.

The Grunsky univalence criterion was extended by Milin [20] to arbitrary finitely connected domains $D \ni \infty$, with appropriate generalization of the Grunsky coefficients and the Grunsky norm. The general quasiconformal features of this theory, in particular, the case $\varkappa(f) = k(f)$ are described in [12].

For most functions $f \in S(k)$, we have the strong inequality $\varkappa(f) < k(f)$ (moreover, the functions satisfying this inequality form a dense subset of S; see, e.g., [12,17]), while the functions with the equal Teichmüller and Grunsky norms play a crucial role in many applications.

Taking $f \in S(k)$ with $\varkappa(f) = k(f) = k$, one obtains for some $n \ge 1$ after appropriate choice of parameters $(x_n) \in l^2$ that the corresponding Grunsky coefficient $\alpha_{nn}(f)$ of this function satisfies $|\alpha_{nn}(f)| = k/n$ (and $|\alpha_{nn}(f)| \le k/n$ for other f).

An explicit example is given by the functions $f_{n-1}(z) = \{f_1(z^{n-1})\}^{1/(n-1)}$, where

$$f_1(z) = z/(1 - ktz)^2, \quad |z| < 1, \ |t| = 1.$$

The diagonal Grunsky coefficient $\alpha^{(2)}_{n-1,n-1}$ of $\sqrt{f_{n-1}(z^2)}$ satisfies (see, e.g. [11], Section 5)

$$|\alpha^{(2)}_{n-1,n-1}| = k/(n-1).$$

3. Proof of Theorem

The idea of the proof actually is the same as of the main results in [14, 15]; thus we only outline briefly the main steps.

1^0. Consider the collection $\widehat{S}(1)$ of univalent functions on the disk $\mathbb{D}$ which is the completion in the topology of locally uniform convergence on $\mathbb{D}$ of the set of univalent functions

$$w(z) = a_1 z + a_2 z^2 + \dots \quad \text{with} \ \ |a_1| = 1,$$

having quasiconformal extensions across the unit circle $\mathbb{S}^1$ to $\widehat{\mathbb{C}}$, which satisfy $w(1) = 1$. Equivalently, this collection is a disjunct union

$$\widehat{S}(1) = \bigcup_{-\pi \le \theta < \pi} S_\theta(1),$$

where $S_\theta(1)$ consists of univalent functions $w(z) = e^{i\theta} z + a_2 z^2 + \dots$ with quasiconformal extensions to $\widehat{\mathbb{C}}$ satisfying $w(1) = 1$ (also completed in the indicated weak topology).

Every $f \in S$ has its representative $\widehat{f}$ in $\widehat{S}(1)$ (not necessarily unique) obtained by pre and post compositions of f with rotations $z \mapsto e^{i\alpha} z$ about the origin, related via (2), i.e.,

$$f_{\tau,\theta}(z) = e^{-i\theta} f(e^{i\tau} z) \quad \text{with} \ \ \tau = \arg z_0,$$

where z_0 is a point for which $f(z_0) = e^{i\theta}$ is a common point of the unit circle and the boundary of domain $f(\mathbb{D})$. (The existence of such a point is a consequence of Schwarz's lemma.)

The **Schwarzian derivatives**

$$S_f(z) = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2, \quad z \in \mathbb{D},$$

belong to the complex Banach space $\mathbf{B}$ of hyperbolically bounded holomorphic functions φ (more precisely, of holomorphic quadratic differentials $\varphi(z) dz^2$ on $\mathbb{D}$ with the norm

$$\|\varphi\|_{\mathbf{B}} = \sup_D \lambda_{\mathbb{D}}^{-2}(z) |\varphi(z)|,$$

where $\lambda_{\mathbb{D}}(z) = 1/(1 - |z|^2)$. Accordingly, $\lambda_{\mathbb{D}}(z)|dz|$ is the hyperbolic metric on $\mathbb{D}$ of Gaussian curvature -4. This space $\mathbf{B}$ is dual to the space $A(\mathbb{D})$ of integrable holomorphic functions on $\mathbb{D}$ with L_1 norm.

The derivatives S_f of quasiconformally extendable functions f from any class $S_\theta(1)$ fill in the space $\mathbf{B}$ a path-wise bounded domain modelling the **universal Teichmüller space T**; then the Taylor coefficients of functions f and of their Schwarzians are holomorphic functions on this space.

The following lemma from [15] ensures the existence of univalent functions in the disk with quasiconformal extension satisfying the prescribed normalization of classes $S_\theta(1)$ and some other conditions. It concerns the solutions of the Beltrami equation $\partial_{\overline{z}} w = \mu(z) \partial_z w$ on $\mathbb{C}$ with coefficients μ supported in the disk $\mathbb{D}^*$, i.e., from the ball

$$\mathrm{Belt}(\mathbb{D}^*)_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu | \mathbb{D} = 0, \ \|\mu\| < 1\}.$$

Lemma 1. *For any $\mu \in \mathrm{Belt}(\mathbb{D}^*)_1$ and any $\theta \in [0, 2\pi)$, there exists a unique homeomorphic solution $w = w^\mu(z)$ of the equation $\partial_{\overline{z}} w = \mu(z) \partial_z w$ on $\widehat{\mathbb{C}}$ such that*

$$w(0) = 0, \quad w'(0) = e^{i\theta}, \quad w(1) = 1. \tag{7}$$

This solution is holomorphic on the unit disk $\mathbb{D}$, and hence, $w(z_0) = \infty$ at some point z_0 with $|z_0| \geq 1$ (so $w(z)$ does not have a pole in $\mathbb{D}$).

Note that for $\mu(z) = 0$ (almost everywhere on $\mathbb{D}^*$) the corresponding solution $w^\mu(z)$ satisfying (7) is the elliptic fractional linear transformation of $\widehat{\mathbb{C}}$ with the fixed points 0 and 1 given by

$$w = e^{-i\theta} z / [(e^{-i\theta} - 1) z + 1]; \tag{8}$$

it equals the identity map if $\theta = 0$.

The prescribed normalizing conditions $w(0) = 0, \ w'(0) = e^{i\theta}, \ w(1) = 1$ are compatible with existence and uniqueness of the corresponding conformal and quasiconformal maps and the Teichmüller space theory, ensure holomorphy of their Taylor coefficients, etc.

Note that we actually deal with the classical model of Teichmüller spaces via domains in the Banach spaces of Schwarzian dervatives S_w in $\mathbb{D}$ (or in the disk $\mathbb{D}^*$) of univalent holomorphic functions normalized either by fixing three boundary points on the unit circle $\mathbb{S}^1$ or via $w(0) = 0, \ w'(0) = 1, \ w(z_0) = z_0$, where $z_0 \in S^1$.

$\mathbf{2}^0$. The given polynomial functional $J(w)$, originally defined on S is naturally determined for functions f from any class $S_\theta(1)$ and on the ball $\mathrm{Belt}(\mathbb{D}^*)_1$ by

$$\widehat{J}(\mu) = J(w^\mu).$$

We now lift this functional onto the universal Teichmüller space $\mathbf{T}$ and the Teichmüller space $\mathbf{T}_1 = \mathrm{Teich}(\mathbb{D}_*)$ of the punctured disk $\mathbb{D}_* = \{0 < |z| < 1\}$, which covers $\mathbf{T}$.

Let us first consider this functional for the fixed values $\vartheta_j \in [-\pi, \pi]$, i.e., the holomorphic functionals

$$J_\vartheta(\mu) = \max_{-\pi \le \vartheta_1, \dots, \vartheta_m \le \pi} \Big| \sum_{|n|=2}^{N} \alpha_{n_1, \dots, n_m} \; a_{n_1} \dots a_{n_m} \varepsilon^{i(n_1 \vartheta_1 + \dots + n_m \vartheta_m)} \Big|, \; \vartheta_j \text{ fixed.} \tag{9}$$

We model the universal Teichmüller space $\mathbf{T}$ by the Schwarzians $S_w = \varphi$ of functions $w(z)$ from $S_\theta(1)$; then its base point $\varphi = \mathbf{0}$ corresponds to the function (8).

As was mentioned in the previous section, it is more convenient technically to deal with the inverted functions (6). In view of Lemma 1, we can model the space $\mathbf{T}$ using the inverted functions $W(z) = 1/w(1/z)$ for $w \in \widehat{S}(1)$.

These functions form the corresponding classes $\Sigma_\theta(1)$ of nonvanishing univalent functions on the disk $\mathbb{D}^*$ with expansions

$$W(z) = e^{-i\theta} z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots, \quad W(1) = 1,$$

and $\widehat{\Sigma}(1) = \bigcup_\theta \Sigma_\theta(1)$.

Simple computations yield that the coefficients a_n of $f \in S_\theta(1)$ and the corresponding coefficients b_j of $W(z) = 1/f(1/z) \in \Sigma_\theta(1)$ are related by

$$b_0 + e^{2i\theta} a_2 = 0, \quad b_n + \sum_{j=1}^{n} \epsilon_{n,j} b_{n-j} a_{j+1} + \epsilon_{n+2,0} a_{n+2} = 0, \quad n = 1, 2, \dots,$$

where $\epsilon_{n,j}$ are the entire powers of $e^{i\theta}$. This successively implies the representations of a_n by b_j via

$$a_n = (-1)^{n-1} \epsilon_{n-1,0} b_0^{n-1} - (-1)^{n-1} (n-2) \epsilon_{1,n-3} b_1 b_0^{n-3}$$
$$+ \text{lower terms with respect to } b_0. \tag{10}$$

This transforms either of functionals (1) and (9) into the coefficient functionals $\widetilde{J}(W)$ and $\widetilde{J}_\vartheta(W)$ on $\Sigma_\theta(1)$ depending on the corresponding coefficients b_j.

Note that the coefficients α_n of Schwarzians $S_w(z) = \sum_0^\infty \alpha_n z^n$ are represented as polynomials of $n+2$ initial coefficients of $w \in S_\theta(1)$ and, in view of (9), as polynomials of $n+1$ initial coefficients of the corresponding $W \in \Sigma_\theta(1)$, provided that θ is given and fixed and the number $e^{i\theta}$ is considered to be a constant (vice versa, the coefficients a_n and b_j are uniquely determined by α_n by solving the Schwarzian differential equation $S_w = \varphi$ or from the equation $S_W(z) = z^{-4} \varphi(1/z)$ and (10).

Recall that the canonical complex Banach structure on $\mathbf{T}$ is defined by factorization of the ball

$$\mathrm{Belt}(\mathbb{D})_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu|\mathbb{D}^* = 0, \ \|\mu\| < 1\}$$

of Beltrami coefficients (complex dilatations) vanishing on the complementary disk $\mathbb{D}^* = \{z \in \widehat{\mathbb{C}} : \ |z| > 1\}$.

The coefficients $\mu_1, \mu_2 \in \mathrm{Belt}(\mathbb{D})_1$ are called **equivalent** if the corresponding quasiconformal maps w^{μ_1}, w^{μ_2} (solutions to the Beltrami equation $\partial_{\overline{z}} w = \mu \partial_z w$ with $\mu = \mu_1, \mu_2$) coincide on the unit circle $\mathbb{S}^1 = \partial \mathbb{D}^*$ (hence, on $\overline{\mathbb{D}^*}$). Such μ and the corresponding maps w^μ are called $\mathbf{T}$-*equivalent*. The equivalence classes $[w^\mu]_{\mathbf{T}}$ are in one-to-one correspondence with the Schwarzian derivatives $S_{w^\mu}(z), \ z \in \mathbb{D}^*$, which belong to the space $\mathbf{B} = \mathbf{B}(\mathbb{D}^*)$ of hyperbolically bounded holomorphic functions on the disk $\mathbb{D}^*$ with norm $\|\varphi\|_{\mathbf{B}} = \sup_{D^*}(|z|^2 - 1)^2 |\varphi(z)|$. Note that $\varphi(z) = O(|z|^{-4})$ as $z \to \infty$.

The factorizing projection

$$\phi_{\mathbf{T}}(\mu) = S_{w^\mu} : \ \mathrm{Belt}(\mathbb{D})_1 \to \mathbf{T}$$

is a holomorphic map from $L_\infty(\mathbb{D})$ to $\mathbf{B}$. This map is a split submersion, which means that $\phi_{\mathbf{T}}$ has local holomorphic sections.

The points of Teichmüller space $\mathbf{T}_1$ of the punctured disk $\mathbb{D}_* = \{0 < |z| < 1\}$ are the classes $[\mu]_{\mathbf{T}_1}$ of $\mathbf{T}_1$**-equivalent** Beltrami coefficients $\mu \in \mathrm{Belt}(\mathbb{D})_1$ so that the corresponding quasiconformal automorphisms w^μ of the unit disk coincide on both boundary components (unit circle $\mathbb{S}^1 = \{|z| = 1\}$ and the puncture $z = 0$) and are homotopic on $\mathbb{D} \setminus \{0\}$.

By the quotient map

$$\phi_{\mathbf{T}_1} : \ \mathrm{Belt}(\mathbb{D})_1 \to \mathbf{T}_1, \quad \mu \to [\mu]_{\mathbf{T}_1},$$

each functional $\widetilde{J}_\vartheta(W^\mu)$ is pushed down to a bounded holomorphic functional $\mathcal{J}_\vartheta$ on the space $\mathbf{T}_1$ with the same range domain. Accordingly, $\widetilde{J}$ is pushed down to a plurisubharmonic functional $\mathcal{J}$ on $\mathbf{T}_1$.

Due to the Bers isomorphism theorem, *the space* $\mathbf{T}_1$ *is biholomorphically isomorphic to the Bers fiber space*

$$\mathcal{F}(\mathbf{T}) = \{(\phi_{\mathbf{T}}(\mu), z) \in \mathbf{T} \times \mathbb{C} : \ \mu \in \mathrm{Belt}(\mathbb{D})_1, \ z \in w^\mu(\mathbb{D})\}$$

over the universal space $\mathbf{T}$ *with holomorphic projection* $\pi(\psi, z) = \psi$ (see [2]).

This fiber space is a bounded hyperbolic domain in $\mathbf{B} \times \mathbb{C}$ and represents the collection of domains $D_\mu = w^\mu(\mathbb{D})$ as a holomorphic family over the space $\mathbf{T}$.

The indicated isomorphism between $\mathbf{T}_1$ and $\mathcal{F}(\mathbf{T})$ is induced by the inclusion map $j: \mathbb{D}_* \hookrightarrow \mathbb{D}$ forgetting the puncture at the origin via

$$\mu \mapsto (S_{w^{\mu_1}}, w^{\mu_1}(0)) \quad \text{with} \quad \mu_1 = j_*\mu := (\mu \circ j_0)\overline{j_0'}/j_0', \tag{11}$$

where j_0 is the lift of j to $\mathbb{D}$.

By Koebe's one-quarter theorem, for any univalent function $W(z) = z+b_0+b_1z^{-1}+\dots$ in $\mathbb{D}^*$, the boundary of domain $W(D^*)$ is located in the disk $\{|w-b_0| \le 2\}$. If $W(z) \ne 0$ in $\mathbb{D}^*$, its inversion $w(z) = z+a_2z^2+\dots$ is univalent in $\mathbb{D}$, and $b_0 = -a_2$ satisfies $|b_0| \le 2$. Using the maps W with quasiconformal extensions, one gets by the Bers theorem that the indicated domains D_μ are filled by the admissible values of $W^\mu(0)$; all these domains are located in the disk $\{|W| \le 4\}$.

3^0. Using the Bers theorem, we regard the points of the space $\mathbf{T}_1$ as the pairs $X_{W^\mu} = (S_{W^\mu}, W^\mu(0))$, taking $W^\mu \in S_\theta(1)$ with Beltrami coefficients $\mu \in \mathrm{Belt}(\mathbb{D})_1$ obeying $\mathbf{T}_1$-equivalence (hence, also $\mathbf{T}$-equivalence). This implies the plurisubharmonic functionals $|\mathcal{J}_\vartheta(X_{W^\mu})| = |\mathcal{J}_\vartheta(S_{W^\mu},\ t)|$ and $|\mathcal{J}(X_{W^\mu})| = \sup_\vartheta |\mathcal{J}_\vartheta(S_{W^\mu},\ t)|$ on the fiber space $\mathcal{F}(\mathbf{T})$, where $t = W^\mu(0)$ runs, respectively, over some domains $D_{\theta,\vartheta}$ and $D_\theta = \bigcup_\vartheta D_{\theta,\vartheta}$, both located in the disk $\{|t| \le 4\}$.

Now define on the union D_θ the corresponding functions

$$u_{\theta,\vartheta}(t) = \sup_{S_{W^\mu}} |\mathcal{J}_\vartheta(S_{W^\mu}, t)|,$$

where the supremum is taken over all $S_{W^\mu} \in \mathbf{T}$ admissible for a given $t = W^\mu(0) \in D_{\theta,\vartheta}$, and

$$u_\theta(t) = \sup_\vartheta\ u_{\theta,\vartheta}(t). \tag{12}$$

The last function plays a crucial role in our construction. The following basic lemma provides that this function inherits from $\mathcal{J}$ the subharmonicity.

Lemma 2. *The function $u_\theta(t)$ is subharmonic in the domain D_θ.*

The proof of this lemma follows the lines of the corresponding lemma in [14], with taking the maximum over ϑ, which provides the needed rotational invariance of u_θ. This proof involves a weak approximation of the underlying space $\mathbf{T}$ (and simultaneously of the space $\mathbf{T}_1$) by finite dimensional Teichmüller spaces of the punctured spheres in the topology of locally uniform convergence on $\mathbb{C}$ and using the increasing unions of the quotient spaces

$$\mathcal{T}_s = \bigcup_{j=1}^{s} \widehat{\Sigma}^0_{\theta_j}/\sim\ = \bigcup_{j=1}^{s}\{(S_{W_{\theta_j}}, W^\mu_\theta(0))\}\ \simeq \mathbf{T}_1 \cup \dots \cup \mathbf{T}_1, \tag{13}$$

where θ_j run over a dense subset $\Theta \subset [-\pi, \pi]$, the equivalence relation $\sim$ means $\mathbf{T}_1$-equivalence on a dense subset $\widehat{\Sigma}^0(1)$ in the union $\widehat{\Sigma}(1)$ formed by univalent functions $W_{\theta_j}(z) = e^{-i\theta_j} z + b_0 + b_1 z^{-2} + \dots$ on $\mathbb{D}^*$ with quasiconformal extension to $\widehat{\mathbb{C}}$ satisfying $W_{\theta_j}(1) = 1$, and

$$\mathbf{W}_\theta^\mu(0) = (W_{\theta_1}^{\mu_1}(0), \dots, W_{\theta_s}^{\mu_s}(0)).$$

The Beltrami coefficients $\mu_j \in \mathrm{Belt}(\mathbb{D})_1$ are chosen here independently. The corresponding collection $\beta = (\beta_1, \dots, \beta_s)$ of the Bers isomorphisms

$$\beta_j: \ \{(S_{W_{\theta_j}}, W_{\theta_j}^{\mu_j}(0))\} \to \mathcal{F}(\mathbf{T})$$

determines a holomorphic surjection of the space $\mathcal{T}_s$ onto $\mathcal{F}(\mathbf{T})$. The function (12) is determined by

$$u(t) = \sup_\Theta u_{\theta_s}(t),$$

where u_{θ_s} is obtained by maximization of type (12) over $\mathcal{T}_s$. For details see [14].

3^0. We pass from (12) to the function

$$u(t) = \sup_\theta u_\theta(t) \tag{14}$$

followed by its upper semicontinuous normalization, which implies a subharmonic function (still denoted by $u(t)$) in the domain $D = \cup_\theta D_\theta$ located in the disk $\{|t| \leq 4\}$. This domain is filled by the admissible values of $t = W^\mu(0)$. Our goal now is to show that this domain coincides with the disk $\{|t| \leq 4\}$.

This is established in the same way as in [14] by applying the following local existence theorem from [9], which we present as

Lemma 3. *Let D be a finitely connected domain on the Riemann sphere $\widehat{\mathbb{C}}$. Assume that there are a set E_0 of positive two-dimensional Lebesgue measure and a finite number of points $z_1, z_2, ..., z_m$ distinguished in D. Let $\alpha_1, \alpha_2, ..., \alpha_m$ be non-negative integers assigned to $z_1, z_2, ..., z_m$, respectively, so that $\alpha_j = 0$ if $z_j \in E_0$.*

Then, for a sufficiently small $\varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$, and for any given collection of numbers $w_{sj}, s = 0, 1, ..., \alpha_j,\ j = 1, 2, ..., m$, which satisfy the conditions $w_{0j} \in D$,

$$|w_{0j} - z_j| \leq \varepsilon, \quad |w_{1j} - 1| \leq \varepsilon, \quad |w_{sj}| \leq \varepsilon \ (s = 0, 1, \dots a_j,\ j = 1, ..., m),$$

there exists a quasiconformal automorphism h_ε *of domain* D *which is conformal on* $D \setminus E_0$ *and satisfies*

$$h_\varepsilon^{(s)}(z_j) = w_{sj} \quad \text{for all } s = 0, 1, ..., \alpha_j, \ j = 1, ..., m.$$

Moreover, the Beltrami coefficient $\mu_{h_\varepsilon}(z) = \partial_{\bar{z}} h_\varepsilon / \partial_z h_\varepsilon$ *of* h *on* E_0 *satisfies* $\|\mu_{h_\varepsilon}\|_\infty \le M\varepsilon$ *and depends holomorphically on* $\{w_{sj}\}$ *(with indicated values of* s *and* j*). The constants* ε_0 *and* M *depend only on the sets* D, E_0 *and the vectors* $(z_1, ..., z_m)$ *and* $(\alpha_1, ..., \alpha_m)$.

We apply this lemma to quasiconformally extendable functions from $\widehat{S}(1)$, which are dense in this subclass (in topology of locally uniform convergence on $\mathbb{D}$).

Now, let w_0 be an extremal of the given functional J on $\widehat{S}(1)$. Then $|J_\vartheta(w_0)| > 0$ for appropriate ϑ. Pass to the function

$$w_{0r}(z) = \frac{1}{r} w_0(rz) = re^{i\theta} z + r^2 a_2^0 z^2 + \dots$$

with r close to 1 and to its image $\widetilde{w}_{0r}$ in $\widehat{S}(1)$, using the corresponding rotations of type (2). This image is univalent and holomorphic on the closed disk $\overline{\mathbb{D}}$ and satisfy also the third normalization condition $w(1) = 1$.

Varying appropriately the coefficients $a_2, a_{m_1}, \dots, a_{m_s}$ of w_{0r} by Lemma 3 with quasiconformal variation h conformal on $w_{0r}(\mathbb{D})$, one derives that the maximal subharmonic function (14) must be positive on the whole disk $\{|t| \le 4\}$, and this function is circularly symmetric. Therefore, it attains its maximal value, which coincides with $\max |J(f)|$ on S, on the boundary circle $\{|t| = 4\}$. Since the points of this circle correspond only to rotations of the function (3), the assertion the theorem follows.

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