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On Approximation Properties of a Stancu Generalization of Szasz–Mirakyan–Bernstein Operators

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Abstract. In this paper, we have introduced a Stancu generalization of the Szasz–Mirakyan–Bernstein Operators defined on the space of continuous functions defined on a compact interval. We have given a general formula for the moments of that operators. We have used Korovkin's Theorem for uniform approximation under some restrictions. We have obtained some results for the approximation rates in terms of modulus of continuity. Finally, we gave some Voronovskaya-type theorems.

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1. Introduction

Approximation theory has been used in the theory of approximation of continuous functions in terms of sequences of positive and linear operators and even today it is an active area of study. There are a lot of operators and their generalizations used for approximate to continuous functions that their Korovkin type approximation properties and rates of convergence are examined. The most famous and useful of these operators are Bernstein operators. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. For each positive integer n, the n-th Bernstein operator of f, $B_n(f)$ is defined as

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,1]$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \le k \le n. \tag{1.1}$$

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The Bernstein polynomials was introduced by S.N. Bernstein [2] in 1912. With the help of this polynomials Bernstein both proved the Weierstrass Theorem and showed that it can be approximated with positive linear polynomial operators to arbitrary continuous function in a compact interval. In 1968, D.D. Stancu [10] introduced a linear positive operators which are known as Bernstein Stancu polynomials:

$$B_n^{\alpha,\beta}(f;x)=\sum_{k=0}^{n}p_{n,k}(x)f\left(\frac{k+\alpha}{n+\beta}\right)$$

where $f\in C[0,1]$, $x\in[0,1]$ and α,β are fixed real numbers such that $0\le\alpha\le\beta$. In the case of $\alpha=\beta=0$, they turn into the classical Bernstein polynomials. Some generalizations of Bernstein Stancu operatos can be found in [4,6,8,9]. In 1950, O. Szasz [11] introduced a generalization of Bernstein polynomials to the infinite interval, which are known as Szasz–Mirakjan operators in literature and defined as

$$S_n(f;x)=\sum_{m=0}^{\infty}q_{n,m}(x)f\left(\frac{m}{n}\right),\quad x\in[0,\infty)$$

where

$$q_{n,m}(x)=e^{-nx}\frac{(nx)^m}{m!},\quad m\in\mathbb{N}_0. \tag{1.2}$$

Some generalizations of Szasz–Mirakjan operators can be found in [1,3, 5,7].

Recently, Tunç and Şimşek [12] introduced a hybrid version of Bernstein and Szasz–Mirakyan operators called as Szasz–Mirakjan–Bernstein operators (SMB). They are defined as

$$L_n(f;x)=\sum_{m=1}^{\infty}q_{n,m-1}(x)\sum_{k=0}^{m\alpha_n}p_{m\alpha_n,k}(x)f\left(\frac{k}{m\alpha_n}\right),\quad x\in[0,1],$$

where $f\in C[0,1]$, $n\in\mathbb{N}$ and (α_n) is a nondecreasing sequence of positive integers, and $p_{n,k}$ and $q_{n,k}$ are defined in (1.1) and (1.2), respectively.

In this study, we introduce a Stancu type generalization of the Szasz–Mirakjan–Bernstein operators defined as follows:

$$L_n^{\alpha,\beta}(f;x)=\sum_{m=1}^{\infty}q_{n,m-1}(x)\sum_{k=0}^{m\alpha_n}p_{m\alpha_n,k}(x)f\left(\frac{k+\alpha}{m\alpha_n+\beta}\right),\quad x\in[0,1] \tag{1.3}$$

where $f\in C[0,1],n\in\mathbb{N}$ and (α_n) is a nondecreasing sequence of integers and $0\le\alpha\le\beta$, and investigate some approximation properties of these operators. We call the operators $L_n^{\alpha,\beta}(f;x)$ are Szasz–Mirakjan–Bernstein–Stancu (SMBS) operators. For $\alpha = \beta = 0$, SMBS operators become the SMB operators. In the case of $\alpha = 0$, we use the notation L_n^{β} instead of $L_n^{0,\beta}$.

In the next section, we will prove a general formula for the moments and obtain some central moments of SMBS operators. In the last section, we give some uniform approximation theorems using Korovkin's Theorem under some restrictions. Also, we obtain some results for approximation rates in terms of modulus of continuity and Voronovkaya type theorems.

2. Auxiliary Results

In order to provide simplicity in the formulas to be obtained in the sequel, we define the monomials as $e_k(x) =: x^k$, $k \in \mathbb{N}_0 =: \mathbb{N} \cup \{0\}$ and

$$H_n^{(k,i)}(x) =: \sum_{m=1}^{\infty} q_{n,m-1}(x) \frac{(m\alpha_n)^k}{(m\alpha_n + \beta)^i}, \qquad n, i \in \mathbb{N},\ k \in \mathbb{N}_0, \tag{2.1}$$

where (α_n) is a nondecreasing sequence of integers, $q_{n,k}$ are defined in (1.2) and $0 \le \alpha \le \beta$. The value of $H_n^{(k,i)}$ functions at the point $x = 0$ is as follows:

$$H_n^{(k,i)}(0) = \frac{\alpha_n^k}{(\alpha_n + \beta)^i}. \tag{2.2}$$

Lemma 2.1. *We have*

$$H_n^{(k,i)}(x) \le \frac{1}{2} \left(\frac{2}{n\alpha_n x} \right)^{i-k} \tag{2.3}$$

for all $x > 0$ and $k < i$.

Proof. Let $n, i \in \mathbb{N}$ and $k \in \mathbb{N}_0$. If $x > 0$ and $k < i$, then we have

$$\begin{aligned} H_n^{(k,i)}(x) &= \sum_{m=1}^{\infty} q_{n,m-1}(x) \frac{(m\alpha_n)^k}{(m\alpha_n + \beta)^i} \\ &\le \frac{e^{-nx}}{(n\alpha_n x)^{i-k}} \sum_{m=1}^{\infty} \frac{(nx)^{m+i-k-1}}{(m+i-k-1)!} \cdot \frac{\prod_{j=1}^{i-k-1} (m+j)}{m^{i-k-1}} \end{aligned}$$

$$= \frac{e^{-nx}}{(n\alpha_n x)^{i-k}} \sum_{m=1}^{\infty} \frac{(nx)^{m+i-k-1}}{(m+i-k-1)!} \cdot \prod_{j=1}^{i-k-1} \left(1 + \frac{j}{m^{i-k-1}}\right)$$
$$\leq \frac{2^{i-k-1}}{e^{nx}(n\alpha_n x)^{i-k}} \sum_{m=i-k}^{\infty} \frac{(nx)^m}{m!}$$
$$= \frac{2^{i-k-1}}{e^{nx}(n\alpha_n x)^{i-k}} \left(e^{nx} - \sum_{m=0}^{i-k-1} \frac{(nx)^m}{m!}\right)$$
$$= \frac{1}{2}\left(\frac{2}{n\alpha_n x}\right)^{i-k} \left(1 - e^{-nx} \sum_{m=0}^{i-k-1} \frac{(nx)^m}{m!}\right) \leq \frac{1}{2}\left(\frac{2}{n\alpha_n x}\right)^{i-k}.$$

□

Remark 2.1. If the sequence (α_n) is bounded, then the functions $H_n^{(k,i)}$ do not converge to zero at the point $x = 0$. However, if the sequence (α_n) diverges to infinity, the sequence $(H_n^{(k,i)})$ converges to zero for each $x \geq 0$.

Lemma 2.2. *We have*

$$1 - \frac{k\upsilon_n}{nx}\left(1 - e^{-nx}\right) \leq n\alpha_n x H_n^{(k-1,k)}(x) \leq 1 \tag{2.4}$$

for all $x > 0$ and $n, k \in \mathbb{N}$, where $\upsilon_n = [\![\beta/\alpha_n]\!] + 1$, ($[\![\cdot]\!]$ stands for the greatest integer function).

Proof. By using the inequality (2.3), we have

$$n\alpha_n x H_n^{(k-1,k)} \leq n\alpha_n x \frac{1}{2}\frac{2}{n\alpha_n x} = 1$$

that gives the right hand of (2.4). The inequality on the left is seen as a result of the following processes:

$$n\alpha_n x H_n^{(k-1,k)}(x) = n\alpha_n x e^{-nx} \sum_{m=1}^{\infty} \frac{(nx)^{m-1}}{(m-1)!} \frac{(m\alpha_n)^{k-1}}{(m\alpha_n + \beta)^k}$$
$$= e^{-nx} \sum_{m=1}^{\infty} \frac{(nx)^m}{(m-1)!} \frac{m^{k-1}}{(m + \beta/\alpha_n)^k}.$$

Taking $\upsilon_n = [\![\beta/\alpha_n]\!] + 1$ and applying Bernouille's inequality we obtain

$$e^{-nx} \sum_{m=1}^{\infty} \frac{(nx)^m}{(m-1)!} \frac{m^{k-1}}{(m + \frac{\beta}{\alpha_n})^k} \geq e^{-nx} \sum_{m=0}^{\infty} \frac{(nx)^m}{m!} \left(\frac{m}{m+\upsilon_n}\right)^k$$
$$= e^{-nx} \sum_{m=0}^{\infty} \frac{(nx)^m}{m!} \left(1 - \frac{\upsilon_n}{m+\upsilon_n}\right)^k$$

$$\geq e^{-nx}\sum_{m=0}^{\infty}\frac{(nx)^m}{m!}\left(1-\frac{k\upsilon_n}{m+\upsilon_n}\right)$$
$$=1-k\upsilon_n e^{-nx}\sum_{m=0}^{\infty}\frac{(nx)^m}{m!}\frac{1}{m+\upsilon_n}$$
$$\geq 1-\frac{k\upsilon_n}{nxe^{nx}}\sum_{m=0}^{\infty}\frac{(nx)^{m+1}}{(m+1)!}$$
$$=1-\frac{k\upsilon_n}{nx}\left(1-\frac{1}{e^{nx}}\right).$$

□

Remark 2.2. If $\lim_{n\to\infty}\alpha_n=\infty$ then υ_n is 1 for sufficiently large integers n.

The general formula for the moments of SMBS operators is given below.

Theorem 2.1. *For $k\in\mathbb{N}_0$, the equality*

$$L_n^{\alpha,\beta}(e_k;x)=\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{p=0}^{j}\binom{i}{j}(\alpha)_{i-j}\,S(k,i)\,s(j,p)\,x^jH_n^{(p,k)}(x) \tag{2.5}$$

holds, where $s(j,p)$ are first kind Stirling numbers, $S(k,i)$ are second kind Stirling numbers and $(\cdot)_m$ is Pochhemmer symbol.

Proof. For $k\in\mathbb{N}_0$, since we have

$$L_n^{\alpha,\beta}(e_k;x)=\sum_{m=1}^{\infty}q_{n,m-1}(x)\,B_{m\alpha_n}^{\alpha,\beta}(e_k;x)$$

for SMBS operators, where $B_m^{\alpha,\beta}$ are Bernstein–Stancu operators, using general moment formula for them we get

$$L_n^{\alpha,\beta}(e_k;x)=\sum_{m=1}^{\infty}\frac{q_{n,m-1}(x)}{(m\alpha_n+\beta)^k}\sum_{i=0}^{k}\sum_{j=0}^{i}S(k,i)\binom{i}{j}(\alpha)_{i-j}(m\alpha_n)_jx^j.$$

By simple calculations, we obtain

$$L_n^{\alpha,\beta}(e_k;x) \tag{2.6}$$
$$=\sum_{m=1}^{\infty}q_{n,m-1}(x)\sum_{i=0}^{k}\sum_{j=0}^{i}S(k,i)\binom{i}{j}(\alpha)_{i-j}\sum_{p=0}^{j}s(j,p)(m\alpha_n)^px^j$$
$$=\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{p=0}^{j}\binom{i}{j}(\alpha)_{i-j}S(k,i)\,s(j,p)\,x^jH_n^{(p,k)}(x).$$

□

Let us write clearly the first four moment formulae necessary for the proof of the results to be given in the next section. The following formulae are easily obtained from the Theorem 2.1 after the necessary arrangements are made.

Corollary 2.1. *For $x \in [0,1]$ and $n \in \mathbb{N}$, we have*

$$\begin{aligned}
L_n^{\alpha,\beta}(e_0;x) &= 1\\
L_n^{\alpha,\beta}(e_1;x) &= x + (\alpha - \beta x) H_n^{(0,1)}(x)\\
L_n^{\alpha,\beta}(e_2;x) &= x^2 + \left(x\left(1+2\alpha\right) - x^2\left(1+2\beta\right)\right) H_n^{(1,2)}(x)\\
&\quad + \left(\alpha^2 - x^2\beta^2\right) H_n^{(0,2)}(x)\\
L_n^{\alpha,\beta}(e_3;x) &= x^3 + \left(3x^2\left(1+\alpha\right) - 3x^3\left(1+\beta\right)\right) H_n^{(2,3)}(x)\\
&\quad + \left(\left(1+3\alpha+3\alpha^2\right)x - x^3\left(3\beta^2-2\right) - 3x^2\left(1+\alpha\right)\right) H_n^{(1,3)}(x)\\
&\quad + \left(\alpha^3 - x^3\beta^3\right) H_n^{(0,3)}(x)\\
L_n^{\alpha,\beta}(e_4;x) &= x^4\left(x^3\left(6+4\alpha\right) - x^4\left(6+4\beta\right)\right) H_n^{(3,4)}(x)\\
&\quad + \left(x^2\left(7+12\alpha+6\alpha^2\right) - x^3\left(18+12\alpha\right) - x^4\left(6\beta^2-11\right)\right) H_n^{(2,4)}(x)\\
&\quad + \left(x\left(1+4\alpha+6\alpha^2+4\alpha^3\right) - x^2\left(7+12\alpha+6\alpha^2\right) + x^3\left(12+8\alpha\right)\right.\\
&\quad \left. - x^4\left(6+4\beta^3\right)\right) H_n^{(1,4)}(x) + \left(\alpha^4 - x^4\beta^4\right) H_n^{(0,4)}(x).
\end{aligned}$$

For $\alpha = 0$, the following equations are true for the moments of the SMBS operators.

Corollary 2.2. *For $x \in [0,1]$ and $n \in \mathbb{N}$, we have*

$$\begin{aligned}
L_n^{\beta}(e_0;x) &= 1\\
L_n^{\beta}(e_1;x) &= x - \beta x H_n^{(0,1)}(x)\\
L_n^{\beta}(e_2;x) &= x^2 + \left(x - x^2\left(1+2\beta\right)\right) H_n^{(1,2)}(x) - x^2\beta^2 H_n^{(0,2)}(x)\\
L_n^{\beta}(e_3;x) &= x^3 + \left(3x^2 - 3x^3\left(1+\beta\right)\right) H_n^{(2,3)}(x)\\
&\quad + \left(x - 3x^2 - x^3\left(3\beta^2-2\right)\right) H_n^{(1,3)}(x) - x^3\beta^3 H_n^{(0,3)}(x)\\
L_n^{\beta}(e_4;x) &= x^4 + \left(6x^3 - x^4\left(6+4\beta\right)\right) H_n^{(3,4)}(x)\\
&\quad + \left(7x^2 - 18x^3 - x^4\left(6\beta^2-11\right)\right) H_n^{(2,4)}(x)\\
&\quad + \left(x - 7x^2 + 12x^3 - x^4\left(6+4\beta^3\right)\right) H_n^{(1,4)}(x) - x^4\beta^4 H_n^{(0,4)}(x).
\end{aligned}$$

With the help of Corollaries 2.1 and 2.2 , the following formulas for the central moments of the SMBS operators are easily obtained.

Corollary 2.3. *For $x \in [0,1]$ and $n \in \mathbb{N}$, we have*

$$L_n^{\alpha,\beta}((e_1-x);x) = (\alpha-\beta x)H_n^{(0,1)}(x)$$
$$L_n^{\alpha,\beta}((e_1-x)^2;x) = x(1-x)H_n^{(1,2)}(x) + (\alpha-\beta x)^2H_n^{(0,2)}(x)$$
$$L_n^{\alpha,\beta}((e_1-x)^3;x) = x(1-x)(3(\alpha-\beta x)+1-2x)H_n^{(1,3)}(x) + (\alpha-\beta x)^3H_n^{(0,3)}(x)$$
$$L_n^{\alpha,\beta}((e_1-x)^4;x) = 3x^2(1-x)^2H_n^{(2,4)}(x) + (\alpha-\beta x)^4H_n^{(0,4)}(x) + x(1-x)\left(x^2\left(6\beta^2+8\beta+6\right) - x(4\beta(1+3\alpha)+6+8\alpha) +6\alpha^2+4\alpha+1\right)H_n^{(1,4)}(x).$$

Corollary 2.4. *Let $\alpha = 0$. For $x \in [0,1]$ and $n \in \mathbb{N}$, we have*

$$L_n^{\beta}((e_1-x);x) = -\beta x H_n^{(0,1)}(x)$$
$$L_n^{\beta}((e_1-x)^2;x) = x(1-x)H_n^{(1,2)}(x) + (\beta x)^2H_n^{(0,2)}(x)$$
$$L_n^{\beta}((e_1-x)^3;x) = x(1-x)(-3\beta x+1-2x)H_n^{(1,3)}(x) - (\beta x)^3H_n^{(0,3)}(x)$$
$$L_n^{\beta}((e_1-x)^4;x) = 3x^2(1-x)^2H_n^{(2,4)}(x) + x(1-x)\left(x^2\left(6\beta^2+8\beta+6\right) -x(4\beta+6)+1\right)H_n^{(1,4)}(x) + (\beta x)^4H_n^{(0,4)}(x).$$

Lemma 2.3. *We have*

$$L_n^{\alpha,\beta}\left((e_1-x)^4;x\right) \leq c_n(x)\frac{2}{(n\alpha_n)^2} \tag{2.7}$$

for every $x \in (0,1]$ and each $n \in \mathbb{N}$, where c_n is a function defined on $(0,1]$ and satisfies

$$\lim_{n\to\infty} c_n(x) = 3(1-x)^2 \tag{2.8}$$

for all $x \in (0,1]$.

Proof. Let $x \in (0,1]$ and $n \in \mathbb{N}$. By using the inequality (2.1) for $i = 4$ and $k = 0,2,3$ in the last equation in Corollary 2.3 we get followings:

$$\begin{aligned} L_n^{\alpha,\beta}\left((e_1-x)^4;x\right) &\leq 3x^2(1-x)^2\frac{1}{2}\left(\frac{2}{n\alpha_n x}\right)^2 + (\alpha-\beta x)^4\frac{1}{2}\left(\frac{2}{n\alpha_n x}\right)^4 \\ &\quad + x(1-x)\left|x^2\left(6\beta^2+8\beta+6\right) - x(4\beta(1+3\alpha)+6+8\alpha) +6\alpha^2+4\alpha+1\right|\frac{1}{2}\left(\frac{2}{n\alpha_n x}\right)^3 \\ &= \frac{1}{2}\left(\frac{2}{n\alpha_n}\right)^2\left(3(1-x)^2 + O\left(\frac{1}{n\alpha_n x^4}\right)\right) \\ &= \frac{2}{(n\alpha_n)^2}c_n(x). \end{aligned}$$

□

3. Main Results and Proofs

In this section, we give some uniform approximation theorems using Korovkin's Theorem under some restrictions. Also, we obtain some results for approximation rates in terms of modulus of continuity and Voronovkaya type theorems.

Theorem 3.1. *If $f \in C[0,1]$, then for all $\epsilon \in (0,1)$, the sequence $\left(L_n^{(\alpha,\beta)}(f)\right)$ converges uniformly to f on $[\epsilon, 1]$.*

Proof. Let $\epsilon \in (0,1)$ be given. To prove the theorem, it will be sufficient to show that the sequence $\left(L_n^{(\alpha,\beta)}(f)\right)$ satisfies the conditions of the Korovkin's Theorem on $[\epsilon, 1]$. From the equations in Corollary 2.1, we have

$$\left|L_n^{\alpha,\beta}(e_0;x) - e_0(x)\right| = 0 =: \gamma_n^0;$$

and using the inequality (2.1) for $i = 1,\ k = 0$, we get for $x \in [\epsilon, 1]$

$$\left|L_n^{\alpha,\beta}(e_1;x) - e_1(x)\right| \leq |\alpha - \beta x|\, H_n^{(0,1)}(x) \leq \frac{|\alpha - \beta x|}{n\alpha_n x} \leq \frac{C_1}{n\alpha_n \varepsilon} =: \gamma_n^1,$$

where $C_1 = \max\{\alpha, \beta - \alpha\}$; and finally, using the inequality (2.1) for $i = 2,\ k = 0, 1$, we get

$$\begin{aligned}
\left|L_n^{\alpha,\beta}(e_2;x) - e_2(x)\right| &\leq \left|x(1+2\alpha) - x^2(1+2\beta)\right| H_n^{(1,2)}(x) \\
&\quad + \left|\alpha^2 - x^2\beta^2\right| H_n^{(0,2)}(x) \\
&\leq \frac{|1+2\alpha - x(1+2\beta)|}{n\alpha_n} + \frac{2\left|\alpha^2 - x^2\beta^2\right|}{(n\alpha_n x)^2} \\
&\leq \frac{C_2}{n\alpha_n} + \frac{C_3}{(n\alpha_n)^2} =: \gamma_n^2
\end{aligned}$$

where $C_2 = \max_{x\in[\varepsilon,1]} |2\alpha + 1 - x(1+2\beta)|$ and $C_3 = 2\max_{x\in[\varepsilon,1]} \left|\alpha/x^2 - \beta^2\right|$. Consequently, since $\lim_{n\to\infty}\gamma_n^i = 0$ for each $i = 0, 1, 2$, then the sequence $\left(L_n^{\alpha,\beta}(f)\right)$ converges uniformly to the function f on $[\varepsilon, 1]$. □

In Theorem 3.1, it is seen that uniform convergence is possible only in compact subintervals of $(0,1]$ if α is nonzero. If α is equal to zero, this problem disappears. We can see this from the next theorem.

Theorem 3.2. *If $f \in C[0,1]$, then the sequence $\left(L_n^{\beta}(f)\right)$ converges uniformly to f on $[0,1]$.*

Proof. Let $f \in C[0,1]$ and $x \in [0,1]$, from the equations in Corollary 2.2, we have

$$\left| L_n^{\beta}(e_0;x) - e_0(x) \right| = 0 =: \lambda_n^0$$

and using the inequality (2.1) for $i = 1,\ k = 0$, we get

$$\left| L_n^{\beta}(e_1;x) - e_1(x) \right| \leq \beta x H_n^{(0,1)}(x) \leq \frac{\beta}{n\alpha_n} =: \lambda_n^1$$

and finally, using the inequality (2.1) for $i = 2,\ k = 0, 1$, we get

$$\begin{aligned} \left| L_n^{\alpha,\beta}(e_2;x) - e_2(x) \right| &\leq \left| x - x^2(1+2\beta) \right| H_n^{(1,2)}(x) + x^2\beta^2 H_n^{(0,2)}(x) \\ &\leq \frac{|1 - x(1+2\beta)|}{n\alpha_n} + \frac{2\beta^2}{(n\alpha_n)^2} \\ &\leq \frac{2+2\beta}{n\alpha_n} + \frac{2\beta^2}{(n\alpha_n)^2} =: \lambda_n^2. \end{aligned}$$

Since $\lim_{n\to\infty} \lambda_n^i = 0$ for each $i = 0, 1, 2$, then the sequence $\left(L_n^{\beta}(f) \right)$ converges to the function f uniformly on $[0,1]$. □

Theorem 3.3. *If $f \in C[0,1]$, then, for all $\varepsilon \in (0,1)$, we have*

$$\left\| L_n^{\alpha,\beta}(f) - f \right\|_{[\varepsilon,1]} \leq 2\omega\left(f; \sqrt{\frac{C}{n\alpha_n}} \right), \tag{3.1}$$

where C is a constant independent of n, $\omega(f;.)$ is the modulus of contiu-nity of f and $\|.\|_{[a,b]}$ is the uniform norm defined on the space $C[a,b]$.

Proof. Since $L_n^{\alpha,\beta}$ is a positive and linear operator for each $n \in \mathbb{N}$, the inequality

$$\begin{aligned} \left| L_n^{\alpha,\beta}(f;x) - f(x) \right| &\leq L_n^{\alpha,\beta}\left(|f - f(x)|; x \right) \\ &\leq \left(1 + \frac{1}{\delta} L_n^{\alpha,\beta}\left(|e_1 - x|; x \right) \right) \omega(f;\delta) \end{aligned}$$

holds for each $\delta > 0$. By using Cauchy–Schwarz inequality, we get

$$\left| L_n^{\alpha,\beta}(f;x) - f(x) \right| \leq \left(1 + \frac{1}{\delta} \sqrt{L_n^{\alpha,\beta}\left((e_1 - x)^2; x \right)} \right) \omega(f;\delta). \tag{3.2}$$

From Corollary 2.3 and the inequality (2.1) for $i = 2$ and $k = 0, 1$, we get

$$\begin{aligned} L_n^{\alpha,\beta}((e_1-x)^2;x) &= x(1-x)H_n^{(1,2)}(x) + (\alpha-\beta x)^2 H_n^{(0,2)}(x) \\ &\leq \frac{1-x}{n\alpha_n} + \frac{2(\alpha-\beta x)^2}{(n\alpha_n x)^2} \\ &\leq \frac{1}{n\alpha_n}\left(1+2\left(\frac{\alpha+\beta}{\varepsilon}\right)^2\right) =: \frac{C}{n\alpha_n} \end{aligned}$$

for each $x \in [\varepsilon, 1]$. If we take $\delta = \sqrt{C/(n\alpha_n)}$ in (3.2), then the desired inequality is achieved. □

For $\alpha = 0$, we can extend the inequality (3.1) to the interval $[0, 1]$ by the following theorem.

Theorem 3.4. *For any $f \in C[0,1]$, we have*

$$\left\| L_n^{\beta}(f) - f \right\|_{[0,1]} \leq 2\omega\left(f; \sqrt{\frac{1}{n\alpha_n} + \frac{2\beta^2}{(n\alpha_n)^2}}\right).$$

Proof. Since L_n^{β} is a positive and linear operator for each $n \in \mathbb{N}$, the inequality

$$\begin{aligned} \left| L_n^{\beta}(f;x) - f(x) \right| &\leq L_n^{\beta}\left(|f - f(x)|; x\right) \\ &\leq \left(1 + \frac{1}{\delta} L_n^{\beta}\left(|e_1 - x|; x\right)\right)\omega\left(f;\delta\right) \end{aligned}$$

holds for each $\delta > 0$. By using Cauchy-Schwarz inequality, we get

$$\left| L_n^{\beta}(f;x) - f(x) \right| \leq \left(1 + \frac{1}{\delta}\sqrt{L_n^{\beta}\left((e_1-x)^2;x\right)}\right)\omega\left(f;\delta\right). \tag{3.3}$$

From Corollary 2.4 and the inequality (2.1) for $i = 2$ and $k = 0, 1$, we get

$$\begin{aligned} L_n^{\beta}((e_1-x)^2;x) &= x(1-x)H_n^{(1,2)}(x) + (\beta x)^2 H_n^{(0,2)}(x) \\ &\leq \frac{1-x}{n\alpha_n} + \frac{2\beta^2}{(n\alpha_n)^2} \\ &\leq \frac{1}{n\alpha_n} + \frac{2\beta^2}{(n\alpha_n)^2}. \end{aligned}$$

Considering that $L_n^{\beta}((e_1-x)^2;x)=0$ for $x=0$, for every $x\in[0,1]$, it is seen that the inequality ;

$$L_n^{\beta}((e_1-x)^2;x)\leq \frac{1}{n\alpha_n}+\frac{2\beta^2}{(n\alpha_n)^2}$$

is achieved. This inequality is written in place in (3.3) and if take $\delta^2=\frac{1}{n\alpha_n}+\frac{2\beta^2}{(n\alpha_n)^2}$ we reach the desired inequality . □

The following is a Voronovskaya-type theorem for SMBS operators.

Theorem 3.5. *If $f\in C^2[0,1]$, then*

$$\lim_{n\to\infty} n\alpha_n\left[L_n^{\alpha,\beta}(f;x)-f(x)\right]=\frac{(\alpha-\beta x)}{x}f'(x)+\frac{(1-x)}{2}f''(x) \quad (3.4)$$

for every $x\in(0,1]$. If $\alpha_n\to\infty$, then the following equation is true for $x=0$:

$$\lim_{n\to\infty}\alpha_n\left[L_n^{\alpha,\beta}(f;0)-f(0)\right]=\alpha f'(x)\ . \quad (3.5)$$

Proof. Let $x\in(0,1]$ be fixed. The Taylor formula of the function f at the point x can be written as

$$f(t)=f(x)+(t-x)f'(x)+\frac{1}{2}(t-x)^2f''(x)+(t-x)^2r(t;x)\,, \quad (3.6)$$

where $r(\cdot\,;x)$ is a continuous function at point $t=x$ and $\lim_{t\to x} r(t;x)=0$. If SMBS operators are applied to both sides of equation (3.6), we get

$$\begin{aligned}L_n^{\alpha,\beta}(f;x)=&f(x)L_n^{\alpha,\beta}(e_0;x)+f'(x)L_n^{\alpha,\beta}((e_1-x);x)\\&+\frac{f''(x)}{2}L_n^{\alpha,\beta}\left((e_1-x)^2;x\right)+L_n^{\alpha,\beta}\left(r(\cdot\,;x)(e_1-x)^2;x\right).\end{aligned}$$

Considering the first two equations in Corollary 2.3, we obtain

$$\begin{aligned}L_n^{\alpha,\beta}(f;x)-f(x)=&\ f'(x)(\alpha-\beta x)H_n^{(0,1)}(x)\\&+\frac{1}{2}f''(x)\left(x(1-x)H_n^{(1,2)}(x)+(\alpha-\beta x)^2H_n^{(0,2)}(x)\right)\\&+L_n^{\alpha,\beta}\left(r(\cdot\,;x)(e_1-x)^2;x\right).\end{aligned}$$

As a result of the necessary arrangements, if $x<1$, then we have

$$\begin{aligned}&\left|n\alpha_n\left(L_n^{\alpha,\beta}(f;x)-f(x)\right)-\frac{(\alpha-\beta x)f'(x)}{x}-\frac{(1-x)f''(x)}{2}\right|\\&\le\left|n\alpha_n xH_n^{(0,1)}(x)-1\right|\left|f'(x)\right|\\&\quad+\left(\left|n\alpha_n xH_n^{(1,2)}(x)-1\right|+\frac{(\alpha-\beta x)^2}{1-x}n\alpha_n H_n^{(0,2)}(x)\right)\left|f''(x)\right|\\&\quad+n\alpha_n L_n^{\alpha,\beta}\left(r(\cdot\,;x)(e_1-x)^2;x\right).\qquad(3.7)\end{aligned}$$

If $x=1$, then the inequality

$$\begin{aligned}&\left|n\alpha_n\left(L_n^{\alpha,\beta}(f;1)-f(1)\right)-(\alpha-\beta)f'(1)\right|\le\left|n\alpha_n H_n^{(0,1)}(1)-1\right|\left|f'(1)\right|\\&+\frac{n\alpha_n}{2}(\alpha-\beta)^2H_n^{(0,2)}(1)\left|f''(1)\right|+L_n^{\alpha,\beta}\left(r(\cdot\,;1)(e_1-1)^2;1\right)\qquad(3.8)\end{aligned}$$

is obtained. From the inequalities (2.4), we get

$$\lim_{n\to\infty}n\alpha_n xH_n^{(0,1)}(x)=\lim_{n\to\infty}n\alpha_n xH_n^{(1,2)}(x)=1$$

and if the inequality (2.1) is used we obtain

$$0\le\lim_{n\to\infty}n\alpha_n H_n^{(0,2)}(x)\le\lim_{n\to\infty}n\alpha_n\frac{2}{(n\alpha_n x)^2}=\lim_{n\to\infty}\frac{2}{n\alpha_n x^2}=0.$$

Thus the right hands of the inequalities (3.7) and (3.8) vanish when $n\to\infty$.

It is sufficient to show that $\lim_{n\to\infty}n\alpha_n L_n^{\alpha,\beta}\left(r(\cdot\,;x)(e_1-x)^2;x\right)=0$ to complete the proof. If we apply the Cauchy–Schwarz inequality and then use the inequality (2.7) we obtain

$$\begin{aligned}&n\alpha_n L_n^{\alpha,\beta}\left(r(\cdot\,;x)(e_1-x)^2;x\right)\\&\quad\le\sqrt{n^2\alpha_n{}^2L_n^{\alpha,\beta}\left((e_1-x)^4;x\right)}.\sqrt{L_n^{\alpha,\beta}\left((r(\cdot\,;x))^2;x\right)}\\&\quad\le\sqrt{n^2\alpha_n{}^2\frac{2}{(n\alpha_n)^2}c_n(x)}.\sqrt{L_n^{\alpha,\beta}\left((r(\cdot\,;x))^2;x\right)}\\&\quad=\sqrt{2c_n(x)}.\sqrt{L_n^{\alpha,\beta}\left((r(\cdot\,;x))^2;x\right)}.\end{aligned}$$

From the equation (2.8), the first multiplier goes to $\sqrt{6(1-x)}$ when $n\to\infty$. Since $r(x;x)=0$ for the second multiplier, according to the Korovkin Theorem, we obtain

$$\lim_{n\to\infty}L_n^{\alpha,\beta}\left((r(\cdot\,;x))^2;x\right)=(r(x;x))^2=0.$$

This ends the proof of (3.4). Now, let $x = 0$. Since the Taylor formula of the function f is $L_n^{\alpha,\beta}(f;0) = f\left(\frac{\alpha}{\alpha_n+\beta}\right)$, considering the Lagrange form of the remainder, we have

$$\alpha_n\left(L_n^{\alpha,\beta}(f;0) - f(0)\right) = \alpha_n\left(f\left(\frac{\alpha}{\alpha_n+\beta}\right) - f(0)\right)$$
$$= \frac{\alpha\alpha_n}{\alpha_n+\beta}f'(0) + \frac{\alpha^2\alpha_n}{(\alpha_n+\beta)^2}f''(\xi_n),$$

where $\xi_n \in (0, \frac{\alpha}{\alpha_n+\beta})$. If the limit of both sides is taken when $n \to \infty$, the equation (3.5) is obtained, since $f'' \in C[0,1]$ and $\alpha_n \to \infty$ when $n \to \infty$. Thus, the proof of the theorem is now complete. □

Corollary 3.1. *If $f \in C^2[0,1]$, then we have*

$$\lim_{n\to\infty} n\alpha_n\left[L_n^{\beta}(f;x) - f(x)\right] = -\beta f'(x) + \frac{(1-x)}{2}f''(x) \tag{3.9}$$

for every $x \in (0,1]$.

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