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Existence of periodic traveling waves in Fermi–Pasta–Ulam type systems on 2D-lattice with saturable nonlinearities

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Abstract. The article is devoted to the Fermi–Pasta–Ulam type systems with saturable nonlinearities that describes an infinite systems of particles on a two dimensional lattice. The main result concerns the existence of traveling waves solutions with periodic relative displacement profiles. By means of critical point theory, we obtain sufficient conditions for the existence of such solutions.

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1. Introduction

In the present paper we study the Fermi–Pasta–Ulam type systems that describes the dynamics of an infinite systems of nonlinearly coupled particles on a two dimensional lattice. It is assumed that each particle interacts nonlinearly with its four nearest neighbors. The equations of motion of the system considered are of the form

$$\ddot{q}_{n,m} = W_1'(q_{n+1,m} - q_{n,m}) - W_1'(q_{n,m} - q_{n-1,m})$$

$$+W_2'(q_{n,m+1} - q_{n,m}) - W_2'(q_{n,m} - q_{n,m-1}), (n,m) \in \mathbb{Z}^2, \tag{1.1}$$

where $q_{n,m} = q_{n,m}(t)$ is a coordinate of the (n,m)-th particle at time t, W_1 and W_2 are the potentials of interaction. Equations (1.1) form an infinite system of ordinary differential equations.

Notice that this system is a representative of a wide class of systems called lattice dynamical systems extensively studied in last decades. Such

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systems also include the Discrete Sine–Gordon type equations and the Fermi–Pasta–Ulam type systems. Equations of such type are of interest in view of numerous applications in physics [2, 16–18, 22].

Among the solutions of such systems, traveling waves deserve special attention. In papers [1, 21, 23, 24] periodic and solitary traveling waves in Fermi–Pasta–Ulam system on 1D–lattice are studied. While [7], [11] and [13] deal with traveling waves for such systems on 2D–lattice. In papers [6, 14,19,20] traveling waves for infinite systems of linearly coupled oscillators on 2D–lattice are studied, while [26] deal with periodic in time solutions for such systems. Paper [8] is devoted to the existence of solitary traveling waves for such systems. Papers [5, 9, 10] is devoted to the existence of homoclinic and heteroclinic traveling waves for the discrete sine–Gordon type equations on 2D–lattice.

In contrast to the previous results (see [3] and [7]), in this paper we study system (1.1) with saturable nonlinearities which means that at infinity $W_i'(r)$ growth as $const \cdot r$, i.e. $W_i(r)$ are asymptotically quadratic at infinity ($i = 1, 2$). Note that in [12] and [23] such nonlinearities are considered. Important examples of saturable nonlinearities are the following

$$f(u) = \frac{\nu|u|^p}{1+\mu|u|^p}u,\ \mu > 0, \nu > 0, p > 1,$$

and

$$f(u) = \chi(1 - \exp(-a|u|^p)u,\ \chi > 0, a > 0, p > 0.$$

2. Statement of a problem

A traveling wave solution of Eq. (1.1) is a function of the form

$$q_{n,m}(t) = u(n\cos\varphi + m\sin\varphi - ct),$$

where the profile function $u(s)$ of the wave, or simply profile, satisfies the equation

$$c^2u''(s) = W_1'(u(s+\cos\varphi) - u(s)) - W_1'(u(s) - u(s-\cos\varphi))$$
$$+W_2'(u(s+\sin\varphi) - u(s)) - W_2'(u(s) - u(s-\sin\varphi)), \qquad (2.1)$$

where $s = n\cos\varphi + m\sin\varphi - ct$.

In what follows, a solution of Eq. (2.1) is understood as a function $u(s)$ from the space $C^2(\mathbb{R})$ satisfying Eq. (2.1) for all $s \in \mathbb{R}$.

We consider the case of periodic traveling waves. The profile function of such wave satisfies the following periodicity condition

$$u'(s+2k) = u'(s),\ s \in \mathbb{R}, \qquad (2.2)$$

where $k > 0$ is an arbitrary real number. Note that the profile of such wave is not necessarily periodic. But its relative displacement profiles $r_i^{\pm}$ are periodic:

$$r_1^{\pm}(s) = \int\limits_{s}^{s\pm\cos\varphi} u'(\tau)d\tau,\ r_2^{\pm}(s) = \int\limits_{s}^{s\pm\sin\varphi} u'(\tau)d\tau.$$

Therefore, such waves are also called periodic (see [24]).

We always assume that

(*i*) $W_i(r) = \frac{c_i^2}{2}r^2 + f_i(r)$, *where* $c_i \in \mathbb{R}$, $f_i \in C^1(\mathbb{R})$, $f_i(0) = f_i'(0) = 0$ *and* $f_i'(r) = o(r)$ *as* $r \to 0$, $i = 1, 2$*;*

(*ii*) *there exists a finite limit* $\lim\limits_{r\to\pm\infty} \frac{f_i'(r)}{r} = l$, *and the functions* $g_i(r) = f_i'(r) - lr$ *are bounded* $(i = 1, 2)$*;*

(*iii*) $f_i(r) \geq 0$ *for all* $r \in \mathbb{R}$ *and for every* $r_0 > 0$ *there exists* $\delta_0 = \delta_0(r_0) > 0$ *such that*

$$\frac{1}{2}rf_i'(r) - f_i(r) \geq \delta_0$$

for $|r| \geq r_0$ $(i = 1, 2)$*.*

To simplify notation, we denote

$$h_i(r) := f_i'(r) = lr + g_i(r),\ i = 1, 2,$$

and

$$G_i(r) := \int\limits_{0}^{r} g_i(\rho)d\rho,\ i = 1, 2,$$

and additionally assume that one of two conditions is satisfied:

(iv_2) $G_i(r) \to -\infty$ *as* $r \to \pm\infty$ $(i = 1, 2)$*;*

or

(v_2) $c^2\left(\frac{\pi n}{k}\right)^2 - 4(c_1^2 + l)\sin^2\left(\frac{\pi n}{2k}\cos\varphi\right) - 4(c_2^2 + l)\sin^2\left(\frac{\pi n}{2k}\sin\varphi\right) \neq 0$ *for all* $n \in \mathbb{N}$.

Remark 2.1. Assumption (*iii*) implies, in particular, that the functions $f_i(r)$ are increasing for $r \geq 0$ and descending for $r \leq 0$, and $G_i(r) < 0$ for all $r \neq 0$, $i = 1, 2$.

The important role is played by the quantity defined by the equality

$$c_0(\varphi) := \sqrt{c_1^2 \cos^2 \varphi + c_2^2 \sin^2 \varphi}.$$

Let E_k be the Hilbert space defined by

$$E_k = \big\{u \in H^1_{loc}(\mathbb{R}) : \ u'(s+2k) = u'(s), \ u(0) = 0\big\}$$

with the scalar product

$$(u,v)_k = \int\limits_{-k}^{k} u'(s)v'(s)ds$$

and corresponding norm $\|u\|_k = (u,u)^{\frac{1}{2}}$. By the embedding theorem, $E_k \subset C([-k,k])$, where $C([-k,k])$ is the space of continuous functions on $[-k,k]$. The norm in dual space E_k^* is denoted by $\|\cdot\|_{k,*}$. In fact, E_k is 1-codimensional subspace of the Hilbert space

$$\widetilde{E}_k = \{u \in H^1_{loc}(\mathbb{R}) : u'(s+2k) = u'(s)\}$$

with

$$\int_{-k}^{k} u'(s)v'(s)ds + u(0)v(0)$$

as the scalar product.

On $\widetilde{E}_k$ we define operators $\widetilde{E}_k \to \widetilde{E}_k$:

$$(Au)(s) := u(s+\cos\varphi) - u(s) = \int_s^{s+\cos\varphi} u'(\tau)d\tau,$$

$$(Bu)(s) := u(s+\sin\varphi) - u(s) = \int_s^{s+\sin\varphi} u'(\tau)d\tau.$$

These operators are bounded linear operators satisfying the inequalities (see [3], Lemma 6.1)

$$\|Au\|_{L^\infty(-k,k)} \le l_1(k)\cdot\|u\|_k, \ \|Au\|_{L^2(-k,k)} \le |\cos\varphi|\cdot\|u\|_k, \tag{2.3}$$

$$\|Bu\|_{L^\infty(-k,k)} \le l_2(k)\cdot\|u\|_k, \ \|Bu\|_{L^2(-k,k)} \le |\sin\varphi|\cdot\|u\|_k, \tag{2.4}$$

where

$$l_1(k) = \begin{cases} |\cos\varphi|\sqrt{\left[\frac{1}{2k}\right]+1}, & 0 < 2k < 1, \\ |\cos\varphi|, & 2k \ge 1, \end{cases}$$

and

$$l_2(k) = \begin{cases} |\sin\varphi|\sqrt{\left[\frac{1}{2k}\right]+1}, & 0 < 2k < 1, \\ |\sin\varphi|, & 2k \geq 1, \end{cases}$$

where $\left[\frac{1}{2k}\right]$ is the integer part of $\frac{1}{2k}$.

On the space E_k we consider the functional

$$J_k(u) = \int_{-k}^{k} \left[\frac{c^2}{2}(u'(s))^2 - \frac{c_1^2}{2}(Au(s))^2 - \frac{c_2^2}{2}(Bu(s))^2\right.$$
$$\left. - f_1(Au(s)) - f_2(Bu(s))\right] ds.$$

Remark 2.2. It is easily verified that, under the assumptions imposed, the functional J_k is well-defined C^1-functional on E_k, and its derivative is given by the formula

$$\langle J_k'(u), h\rangle = \int_{-k}^{k} \left[c^2 u'(s)h'(s) - c_1^2 Au(s)Ah(s) - c_2^2 Bu(s)Bh(s)\right.$$
$$\left. - f_1'(Au(s))Ah(s) - f_2'(Bu(s))Bh(s)\right] ds$$

for $u, h \in E_k$. Moreover, any critical point of the functional J_k is a solution of Eq. (2.1) satisfying (2.2).

Thus, to establish the existence of solutions to Eq. (2.1) satisfying (2.2), it is suffice to prove the existence of nontrivial critical points of the functional J_k. This requires a special form of the mountain pass theorem (see [24, 25]).

Let $I : H \to \mathbb{R}$ be a C^1-functional on a Hilbert space H with the norm $\|\cdot\|$. We say that I satisfies the *Palais-Smale condition*, if the following condition is satisfied:

(PS) *Let* $\{u_n\} \subset H$ *be a such sequence that* $\{I(u_n)\}$ *is bounded and* $I'(u_n) \to 0$, $n \to \infty$. *Then* $\{u_n\}$ *contains a convergent subsequence.*

If there exist $e \in H$ and $r > 0$ such that $\|e\| > r$ and

$$\beta := \inf_{\|u\|=r} I(u) > I(0) \geq I(e),$$

then we say that the functional I possesses the *mountain pass geometry.*

The following theorem of the mountain pass type can be found in [15] (Theorem 10).

Theorem 2.1. *Suppose that the C^1-functional $I : H \to \mathbb{R}$ satisfies the Palais–Smale condition and possesses the mountain pass geometry. Let $P : H \to H$ be a continuous mapping such that*

$$I(Pu) \le I(u)$$

for all $u \in H$, $P(0) = 0$ and $P(e) = e$. Then there exists a critical point $u \in \overline{PH}$ (the closure of PH) of the functional I with the critical value

$$I(u) = b := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) \ge \beta,$$

where $\Gamma := \{\gamma \in C([0,1], H) : \gamma(0) = 0, \gamma(1) = e\}$.

Let

$$(Pu)(s) := \int_0^s |u'(t)| dt.$$

Remark 2.3. It is easily verified that P is a continuous map from E_k into itself and PE_k consists of a non-decreasing functions.

3. Main result

The main result of this paper is the following theorem that establishes the existence of periodic waves with non-decreasing and non-increasing profiles.

Theorem 3.1. *Assume* (i_2)–(iii_2) *and either* (iv_2) *or* (v_2). *If* $\varphi \in [\pi n, \frac{\pi}{2} + \pi n]$, $n \in \mathbb{Z}$, $k > 0$ *and* $c_0^2 < c^2 < c_0^2 + l$, *then Eq. (2.1) has a non-constant non-decreasing and non-increasing solutions satisfying (2.2).*

Note that from a physical point of view, the increasing waves are *expansion waves*, and the decreasing waves are *compression waves.*

Remark 3.1. Since we consider monotone waves, we may only suppose that the assumptions of Theorem 3.1 hold for $r \ge 0$ (respectively, for $r \le 0$), and obtain non-decreasing (respectively, non-increasing) waves. On the other hand, proving the results we may assume, for instance, that $f_i(r)$ are even functions.

For convenience, we represent the functional J_k in the form

$$J_k(u) = \frac{1}{2} Q_k(u,u) - \int_{-k}^{k} [G_1(Au(s)) + G_2(Bu(s))]\, ds, \qquad (3.1)$$

where

$$Q_k(u,h)=\int_{-k}^{k}\big[c^2u'(s)v'(s)-(c_1^2+l)Au(s)Ah(s)-(c_2^2+l)Bu(s)Bh(s)\big]\,ds.$$

Then the derivative can be written as

$$\langle J_k'(u),h\rangle = Q_k(u,h)-\int_{-k}^{k}\left[g_1(Au(s))Ah(s)+g_2(Bu(s))Bh(s)\right]ds \quad (3.2)$$

for $u,h\in E_k$.

Let

$$\sigma(\xi):=c^2\xi^2-4(c_1^2+l)\sin^2\left(\frac{\xi}{2}\cos\varphi\right)-4(c_2^2+l)\sin^2\left(\frac{\xi}{2}\sin\varphi\right)$$

and $\xi_n=\frac{\pi n}{k}$, where $n=1,2,\dots$. We set

$$e_0(s)=s,\ e_n^{(1)}(s)=\sin(\xi_n s),\ e_n^{(2)}(s)=\cos(\xi_n s)-1,$$

where $n=1,2,\dots$. Then the system of functions

$$\{e_0,e_n^{(1)},e_n^{(2)}:\ n=1,2,\dots\}$$

is a complete orthogonal system in E_k. This system is also orthogonal with respect to the bilinear form Q_k. In addition,

$$Q_k(e_0,e_0)=2k(c^2-c_0^2-l)$$

and

$$Q_k\left(e_n^{(1)},e_n^{(1)}\right)=Q_k\left(e_n^{(2)},e_n^{(2)}\right)=k\sigma(\xi_n),\ n=1,2,\dots .$$

Let

$$E_k^-:=\operatorname{span}\{e_0,e_n^{(1)},e_n^{(2)}:\ \sigma(\xi_n)<0\},$$
$$E_k^0:=\operatorname{span}\{e_n^{(1)},e_n^{(2)}:\ \sigma(\xi_n)=0\}$$

and

$$E_k^+:=\operatorname{span}\{e_n^{(1)},e_n^{(2)}:\ \sigma(\xi_n)>0\}.$$

These subspaces are mutually orthogonal with respect to both the scalar product and the form Q_k, and

$$E_k=E_k^-\oplus E_k^0\oplus E_k^+.$$

The spaces E_k^- and E_k^0 are finite dimensional, and E_k^+ is infinite dimensional. Obviously, the form Q_k is negative definite on E_k^-, positive definite on E_k^+, and zero on E_k^0.

We denote by u^-, u^0 and u^+ the orthogonal projections of $u \in E_k$ on E_k^-, E_k^0 and E_k^+, respectively.

Now we verify the conditions of Theorem 2.1 for the functional J_k.

Lemma 3.1. *Under the assumptions of Theorem 3.1 functional J_k satisfies the Palais–Smale condition.*

Proof. Let $\{u_n\} \subset E_k$ be a Palais-Smale sequence of J_k, i.e. $\{J_k(u_n)\}$ is bounded and $J_k'(u_n) \to 0$, $n \to \infty$. We prove that the sequence $\{u_n\}$ is bounded. Since the form Q_k is positive (respectively, negative) definite on E_k^+ (respectively, on E_k^-), then there exists $\alpha > 0$ such that

$$\pm Q_k(u,u) \geq \alpha \|u\|_k^2$$

for all $u \in E_k^{\pm}$. Since $J_k'(u_n) \to 0$, $n \to \infty$, then

$$\|J_k'(u_n)\|_{k,*} \leq 1$$

for n large enough. Thus, from (3.2) with $u = u_n$ and $h = u_n^{\pm}$ we have that

$$\alpha \|u_n^{\pm}\|_k^2 \leq \|u_n^{\pm}\|_k + \int\limits_{-k}^{k} \left[|g_1(Au_n(s))||Au_n^{\pm}(s)| + |g_2(Bu_n(s))||Bu_n^{\pm}(s)|\right] ds$$

for all n large enough. By assumption (ii) and inequalities (2.3), (2.4) we obtain that

$$\alpha \|u_n^{\pm}\|_k^2 \leq C \|u_n^{\pm}\|_k$$

with some $C > 0$. Hence, the sequences $\{u_n^+\}$ and $\{u_n^-\}$ are bounded.

In case, when (v) is satisfied, we have $E_k^0 = \{0\}$ and, hence, $\{u_n\} = \{u_n^+ + u_n^-\}$ is bounded sequence.

Now we suppose that (iv_2) is satisfied. Since $\{u_n^+ + u_n^-\}$ is bounded sequence, then it remains to show that $\{u_n^0\} \subset E_k^0$ is bounded too. Suppose the opposite. Then, passing to a subsequence, we may assume that $\|u_n^0\|_k \to \infty$. By the description of E_k^0, we can represent u_n^0 in the form

$$u_n^0(s) = \beta_n \sin(\xi^0 s + \varphi_n),$$

where $|\beta_n| \to \infty$, and $\xi^0 \neq 0$ is an integer multiple of $\frac{\pi}{k}$ such that $\sigma(\xi^0) = 0$. Then

$$Au_n^0(s) = 2\beta_n \sin\left(\frac{\xi^0}{2}\cos\varphi\right)\cos\left(\xi^0(s + \frac{1}{2}\cos\varphi) + \varphi_n\right),$$

$$Bu_n^0(s) = 2\beta_n \sin\left(\frac{\xi^0}{2}\sin\varphi\right)\cos\left(\xi^0(s+\frac{1}{2}\sin\varphi)+\varphi_n\right).$$

And this implies that there exist two constants $\delta > 0$, $\gamma > 0$ and a subset $M_n \subset [-k,k]$ of measure δ such that $|Au_n^0(s)| + |Bu_n^0(s)| \geq \gamma|\beta_n|$ on M_n.

Eq. (3.1) implies that

$$J_k(u_n) = \frac{1}{2}\left[Q_k(u_n^+,u_n^+) + Q_k(u_n^-,u_n^-)\right]$$

$$-\int_{-k}^{k}\left[G_1\left(Au_n^+(s)+Au_n^-(s)+Au_n^0(s)\right)\right.$$

$$\left.+G_2\left(Bu_n^+(s)+Bu_n^-(s)+Bu_n^0(s)\right)\right]ds. \tag{3.3}$$

By Remark 2.1, $G_i(r) < 0$ on $\mathbb{R}$, and inequalities (2.3), (2.4) shows that $\{Au_n^+(s)+Au_n^-(s)\}$ and $\{Bu_n^+(s)+Bu_n^-(s)\}$ are bounded sequences in $L^\infty(-k,k)$. Thus,

$$-\int_{-k}^{k}\left[G_1\left(Au_n^+(s)+Au_n^-(s)+Au_n^0(s)\right)\right.$$

$$\left.+G_2\left(Bu_n^+(s)+Bu_n^-(s)+Bu_n^0(s)\right)\right]ds$$

$$\geq -\int_{M_n}\left[G_1\left(Au_n^+(s)+Au_n^-(s)+Au_n^0(s)\right)\right.$$

$$\left.+G_2\left(Bu_n^+(s)+Bu_n^-(s)+Bu_n^0(s)\right)\right]ds$$

$$\to +\infty.$$

Since all other terms in the right hand side of (3.3) are bounded, we have that $J_k(u_n) \to +\infty$. We got a contradiction, which proves that $\{u_n^0\} \subset E_k^0$ is bounded.

Hence, in both cases $\{u_n\}$ is bounded.

Then, up to a subsequence (with the same denotation), $u_n \to u$ weakly in E_k, hence, $Au_n \to Au$ and $Bu_n \to Bu$ weakly in E_k, and strongly in $L^2(-k,k)$ and $C([-k,k])$ (by the compactness of Sobolev embedding). A straightforward calculation shows that

$$\|u_n - u\|_k^2 = \int_{-k}^{k}\left(c^2(u_n'(s)-u'(s))^2 + c^2(u_n(s)-u(s))^2\right)ds$$

$$= \langle J_k'(u_n) - J_k'(u), u_n - u\rangle$$

$$+c_1^2\|Au_n - Au\|^2_{L^2(-k,k)} + c_2^2\|Bu_n - Bu\|^2_{L^2(-k,k)}$$

$$+\int_{-k}^{k} \left(f_1'(Au_n(s)) - f_1'(Au(s))\right)(Au_n(s) - Au(s))\,ds$$

$$+\int_{-k}^{k} \left(f_2'(Bu_n(s)) - f_2'(Bu(s))\right)(Bu_n(s) - Bu(s))\,ds.$$

Obviously that all the terms on the right hand part converge to 0 (first, fourth and fifth by weak convergence, second and third terms converge to 0 by strong convergence). Thus, $\|u_n - u\|_k \to 0$ as $n \to \infty$, and proof is complete. □

Lemma 3.2. *Under the assumptions of Theorem 3.1 functional J_k possesses the mountain pass geometry.*

Proof. Due to (i), for any $\varepsilon > 0$ there exists $r_0 > 0$ such that $|f_i(r)| \le \varepsilon r^2$ as $r \le r_0$ $(i = 1, 2)$. Then, by (2.3) and (2.4), as $\|u\|_k \le r_0$ we have

$$J_k(u) \ge \int_{-k}^{k} \left[\frac{c^2}{2}(u'(s))^2 - \frac{c_1^2}{2}(Au(s))^2 - \frac{c_2^2}{2}(Bu(s))^2 - \varepsilon(Au(s))^2\right.$$

$$\left. - \varepsilon(Bu(s))^2\right] ds \ge \frac{c^2}{2}\|u\|_k^2 - \frac{c_1^2}{2}\cos^2\varphi\|u\|_k^2 - \frac{c_2^2}{2}\sin^2\varphi\|u\|_k^2$$

$$-\varepsilon\cos^2\varphi\|u\|_k^2 - \varepsilon\sin^2\varphi\|u\|_k^2 = \frac{c^2 - c_0^2 - 2\varepsilon}{2}\|u\|_k^2.$$

Choosing ε small enough, we obtain that there is $\beta > 0$ such that $J_k(u) \ge \beta > 0$ as $\|u\|_k = r_0$.

Let us now show that there exists an element $e \in E_k$ such that $J_k(e) < 0$.

Let $e_0(s) = s$ and $\tau > 0$. From (ii) it follows that $|G_i(r| \le C|r|$ with some constant $C > 0$. Then Eq. (3.1) implies that

$$J_k(\tau e_0) \le k(c^2 - c_0^2 - l)\tau^2 + 2k(|\cos\varphi| + |\sin\varphi|)|\tau|.$$

Thus, $J_k(\tau e_0) < 0$ for $|\tau|$ large enough, and hence, there exists τ_0 such that $J_k(\tau_0 e_0) < 0$. Now it remains to take $e = \tau_0 e_0$ and the lemma is proved. □

Proof of Theorem 3.1. Let the conditions of the theorem hold for $r \geq 0$. Lemmas 3.1 and 3.2 show that J_k satisfies almost all conditions of Theorem 2.1. It only remains to verify the inequality $J_k(Pu) \leq J_k(u)$ for all $u \in E_k$.

Let $\varphi \in [2\pi n, \frac{\pi}{2} + 2\pi n]$, $n \in \mathbb{Z}$. Since

$$(APu)(s) = \int_s^{s+\cos\varphi} (Pu)'(\tau)d\tau = \int_s^{s+\cos\varphi} |u'(\tau)|d\tau \geq \left| \int_s^{s+\cos\varphi} u'(\tau)d\tau \right|$$

and

$$(BPu)(s) = \int_s^{s+\sin\varphi} (Pu)'(\tau)d\tau = \int_s^{s+\sin\varphi} |u'(\tau)|d\tau \geq \left| \int_s^{s+\sin\varphi} u'(\tau)d\tau \right|,$$

then

$$(APu)(s) \geq |(APu)(s)| \geq (Au)(s)$$

and

$$(BPu)(s) \geq |(BPu)(s)| \geq (Bu)(s).$$

Since, due to Remark 2.1, the potentials $f_i(r)$ are increasing, we have that

$$J_k(Pu) = \int_{-k}^{k} \left[c^2((Pu)'(s))^2 - c_1^2(APu(s))^2 - c_2^2(BPu(s))^2\right.$$

$$\left. - f_1(APu(s)) - f_2(BPu(s))\right] ds$$

$$= \int_{-k}^{k} \left[c^2(u'(s))^2 - c_1^2(APu(s))^2 - c_2^2(BPu(s))^2\right.$$

$$\left. - f_1(APu(s)) - f_2(BPu(s))\right] ds \leq$$

$$\leq \int_{-k}^{k} \left[c^2(u'(s))^2 - c_1^2(Au(s))^2 - c_2^2(Bu(s))^2\right.$$

$$\left. - f_1(Au(s)) - f_2(Bu(s))\right] ds = J_k(u).$$

Hence, by Theorem 2.1 there exists nontrivial critical point $u \in PE_k$ of the functional J_k such that $J_k(u) \geq \beta$ with $\beta > 0$ from Lemma 3.1. By Remark 2.2, $u \in PE_k \subset E_k$ is a solution of problem (2.1), (2.1). Furthermore, by Remark 2.3, this solution is non-decreasing and non-constant due to the definition of space E_k.

The case $r \leq 0$ is similar (with P replaced by $-P$). In this case, non-increasing solutions are obtained.

It is easy to see that for $\varphi \in [\pi + 2\pi n, \frac{3\pi}{2} + 2\pi n]$, $n \in \mathbb{Z}$, in the case $r \geq 0$ non-increasing solutions are obtained, and in the case $r \leq 0$ non-decreasing solutions are obtained.

The proof is complete. □

Conclusion

Thus, in the present paper we obtain some result on the existence of non-constant monotone traveling waves with periodic relative displacement profiles in Fermi-Pasta-Ulam type systems with saturable nonlinearities on a two-dimensional lattice.

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