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Limit cycles of multi-parameter polynomial dynamical systems

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Abstract. This is an overview of our recent woks on a global bifurcation analysis of multi-parameter polynomial dynamical systems. In particular, using our bifurcation-geometric approach, we study the global dynamics and solve the problem on the maximum number and distribution of limit cycles in a polynomial Euler–Lagrange–Liénard type mechanical system. We consider also a rational endocrine system carrying out a global bifurcation analysis of a reduced planar quartic Topp system which models the dynamics of diabetes. Studying global bifurcations and applying the Wintner–Perko termination principle, we prove that such a system can have at most two limit cycles.

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1. Introduction

We carry out a global bifurcation analysis of planar polynomial dynamical systems and, first of all, we would like to recall some basic facts on their singular points and limit cycles. In particular, the study of singular points of polynomial systems will use two index theorems by H. Poincaré; see [2]. The definition of the Poincaré index is the following [2].

Definition 1.1. *Let S be a simple closed curve in the phase plane not passing through a singular point of the system*

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{1.1}$$

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where $P(x,y)$ and $Q(x,y)$ are continuous functions (for example, polynomials), and M be some point on S. If the point M goes around the curve S in the positive direction (counterclockwise) one time, then the vector coinciding with the direction of a tangent to the trajectory passing through the point M is rotated through the angle $2\pi j$ $(j = 0, \pm 1, \pm 2, \ldots)$. The integer j is called the Poincaré index *of the closed curve S relative to the vector field of system* (1.1) *and has the expression*

$$j = \frac{1}{2\pi} \oint_S \frac{P\,dQ - Q\,dP}{P^2 + Q^2}\,. \tag{1.2}$$

According to this definition, the index of a node or a focus, or a center is equal to $+1$ and the index of a saddle is -1. The following Poincaré index theorems are valid [2].

Theorem 1.1. *The indices of singular points in the plane and at infinity sum to $+1$.*

Theorem 1.2. *If all singular points are simple, then along an isocline without multiple points lying in a Poincaré hemisphere which is obtained by a stereographic projection of the phase plane, the singular points are distributed so that a saddle is followed by a node or a focus, or a center and vice versa. If two points are separated by the equator of the Poincaré sphere, then a saddle will be followed by a saddle again and a node or a focus, or a center will be followed by a node or a focus, or a center.*

Consider a polynomial system in the vector form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}), \tag{1.3}$$

where $\boldsymbol{x} \in \mathbf{R}^2$; $\boldsymbol{\mu} \in \mathbf{R}^n$; $\boldsymbol{f} \in \mathbf{R}^2$ ($\boldsymbol{f}$ is a polynomial vector function).

Recall some basic facts concerning limit cycles of (1.3). Assume that system (1.3) has a limit cycle

$$L_0 : \boldsymbol{x} = \boldsymbol{\varphi}_0(t)$$

of minimal period T_0 at some parameter value $\boldsymbol{\mu} = \boldsymbol{\mu}_0 \in \mathbf{R}^n$.

Let l be the straight line normal to L_0 at the point $\boldsymbol{p}_0 = \boldsymbol{\varphi}_0(0)$ and s be the coordinate along l with s positive exterior to L_0. It then follows from the implicit function theorem that there is a $\delta > 0$ such that the Poincaré map $h(s, \boldsymbol{\mu})$ is defined and analytic for $|s| < \delta$ and $\|\boldsymbol{\mu} - \boldsymbol{\mu}_0\| < \delta$. The displacement function for system (1.3) along the normal line l to L_0 is defined as the function

$$d(s, \boldsymbol{\mu}) = h(s, \boldsymbol{\mu}) - s.$$

We denote derivatives of d with respect to s or components of μ by subscripts, and the m-th derivative of d with respect to s by $d_s^{(m)}$. In terms of the displacement function, a multiple limit cycle can be defined as follows [9].

Definition 1.2. *A limit cycle L_0 of* (1.3) *is a* multiple limit cycle *iff*

$$d(0, \boldsymbol{\mu}_0) = d_s(0, \boldsymbol{\mu}_0) = 0.$$

It is a simple limit cycle *(or hyperbolic limit cycle) if it is not a multiple limit cycle; furthermore, L_0 is a limit cycle of multiplicity m iff*

$$d(0, \boldsymbol{\mu}_0) = d_s(0, \boldsymbol{\mu}_0) = \ldots = d_s^{(m-1)}(0, \boldsymbol{\mu}_0) = 0,$$

$$d_s^{(m)}(0, \boldsymbol{\mu}_0) \neq 0.$$

Note that the multiplicity of L_0 is independent of the point $\boldsymbol{p}_0 \in L_0$ through which we take the normal line l.

Let us write down also the following formulae which have already become classical ones and determine the derivatives of the displacement function in terms of integrals of the vector field $\boldsymbol{f}$ along the periodic orbit $\boldsymbol{\varphi}_0(t)$ [9]:

$$d_s(0, \boldsymbol{\mu}_0) \;=\; \exp \int_0^{T_0} \boldsymbol{\nabla} \cdot \boldsymbol{f}(\boldsymbol{\varphi}_0(t), \boldsymbol{\mu}_0)\, \mathrm{d}t - 1$$

and

$$d_{\mu_j}(0, \boldsymbol{\mu}_0) = \frac{-\omega_0}{\|\boldsymbol{f}(\boldsymbol{\varphi}_0(0), \boldsymbol{\mu}_0)\|} \times$$

$$\int_0^{T_0} \exp\left(-\int_0^t \boldsymbol{\nabla} \cdot \boldsymbol{f}(\boldsymbol{\varphi}_0(\tau), \boldsymbol{\mu}_0)\, \mathrm{d}\tau\right) \times \boldsymbol{f} \wedge \boldsymbol{f}_{\mu_j}(\boldsymbol{\varphi}_0(t), \boldsymbol{\mu}_0)\, \mathrm{d}t$$

for $j = 1, \ldots, n$, where $\omega_0 = \pm 1$ according to whether L_0 is positively or negatively oriented, respectively, and where the wedge product of two vectors $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ in $\mathbf{R}^2$ is defined as

$$\boldsymbol{x} \wedge \boldsymbol{y} = x_1\, y_2 - x_2\, y_1.$$

Similar formulae for $d_{ss}(0, \boldsymbol{\mu}_0)$ and $d_{s\mu_j}(0, \boldsymbol{\mu}_0)$ can be derived in terms of integrals of the vector field $\boldsymbol{f}$ and its first and second partial derivatives along $\boldsymbol{\varphi}_0(t)$.

Now we can formulate the Wintner–Perko termination principle [34] for polynomial system (1.3).

Theorem 1.3. *Any one-parameter family of multiplicity-m limit cycles of relatively prime polynomial system* (1.3) *can be extended in a unique way to a maximal one-parameter family of multiplicity-m limit cycles of* (1.3) *which is either open or cyclic.*

If it is open, then it terminates either as the parameter or the limit cycles become unbounded; or, the family terminates either at a singular point of (1.3)*, which is typically a fine focus of multiplicity m, or on a (compound) separatrix cycle of* (1.3) *which is also typically of multiplicity m.*

The proof of this principle for general polynomial system (1.3) with a vector parameter $\boldsymbol{\mu} \in \mathbf{R}^n$ parallels the proof of the planar termination principle for the system

$$\dot{x} = P(x, y, \lambda), \quad \dot{y} = Q(x, y, \lambda) \tag{1.4}$$

with a single parameter $\lambda \in \mathbf{R}$ (see [9, 34]), since there is no loss of generality in assuming that system (1.3) is parameterized by a single parameter λ; i. e., we can assume that there exists an analytic mapping $\boldsymbol{\mu}(\lambda)$ of $\mathbf{R}$ into $\mathbf{R}^n$ such that (1.3) can be written as (1.4) and then we can repeat everything that had been done for system (1.4) in [34]. In particular, λ is said to be a *field-rotation parameter* if it rotates the vectors of the field in one direction [2, 9, 34]. If λ is a field rotation parameter of (1.4), the following Perko's theorem on monotonic families of limit cycles is valid; see [34].

Theorem 1.4. *If L_0 is a nonsingular multiple limit cycle of* (1.4) *for $\lambda = \lambda_0$, then L_0 belongs to a one-parameter family of limit cycles of* (1.4)*; furthermore:*

1) if the multiplicity of L_0 is odd, then the family either expands or contracts monotonically as λ increases through λ_0;

2) if the multiplicity of L_0 is even, then L_0 bifurcates into a stable and an unstable limit cycle as λ varies from λ_0 in one sense and L_0 disappears as λ varies from λ_0 in the opposite sense; i. e., there is a fold bifurcation at λ_0.

We use these theorems and develop our methods for studying limit cycle bifurcations of polynomial dynamical systems [5], [9]–[23]. In Section 2, applying canonical systems with field rotation parameters and using geometric properties of the spirals filling the interior and exterior domains of limit cycles, we solve the problem on the maximum number and distribution of limit cycles in an Euler–Lagrange–Liénard type mechanical system. In Section 3, we consider an endocrine system model carrying out a global qualitative analysis of a reduced planar quartic Topp system which models the dynamics of diabetes; in particular, studying global bifurcations and applying the Wintner–Perko termination principle, we prove that such a system can have at most two limit cycles. This is related to the solution of Hilbert's sixteenth problem on the maximum number and distribution of limit cycles in planar polynomial dynamical systems [9].

2. The Euler–Lagrange–Liénard polynomial mechanical system

We study an Euler–Lagrange–Liénard type equation [23, 24]

$$\ddot{x} + h(x)\,\dot{x}^2 + f(x)\,\dot{x} + g(x) = 0 \tag{2.1}$$

and the corresponding dynamical system

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)\,y - h(x)\,y^2. \tag{2.2}$$

Equation (2.1) is a composition of two equations. One of them is

$$\alpha(q)\,\ddot{q} + \beta(q)\,\dot{q}^2 + \gamma(q) = 0, \tag{2.3}$$

where $q \in R$; $\alpha(q)$, $\beta(q)$ and $\gamma(q)$ are scalar functions, which represents a generic form of dynamics for an n-degree of freedom Euler–Lagrange system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{Q}}\right) - \frac{\partial L}{\partial Q} = B(Q)\,u, \tag{2.4}$$

where $L(Q, \dot{Q})$ is a Lagrangian, $Q \in R^n$ is a vector of generalized coordinates, $u \in R^{n-1}$ and $B(Q)$ is $n \times (n-1)$ matrix function of full rank for each Q. Equation (2.3) can be used, in particular, for solving the periodic motion problem in mechanical systems; see, e. g., [38] and the references therein.

The other one is the Liénard equation

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0 \tag{2.5}$$

with the corresponding dynamical systems in the form

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)\,y, \tag{2.6}$$

particular cases of which we have considered in [10–17]; see also [7, 8, 27, 31, 32, 35, 39]. There are many examples in the natural sciences and technology in which this and related systems are applied [1, 2, 33, 37].

Such systems are often used to model either mechanical or electrical, or biomedical systems, and in the literature, many systems are transformed into Liénard type to aid in the investigations. They can be used, e. g., in certain mechanical systems, where $f(x)$ represents a coefficient of the damping force and $g(x)$ represents the restoring force or stiffness, when modeling wind rock phenomena and surge in jet engines [1,33]. Such systems can be also used to model resistor-inductor-capacitor circuits with non-linear circuit elements. Recently, e. g., the Liénard system has been shown to describe the operation of an optoelectronics circuit that uses a resonant tunnelling diode to drive a laser diode to make an optoelectronic voltage controlled oscillator [37].

There are also a number of examples of technical systems which are modelled with quadratic damping: a term in the second-order dynamics model, which is quadratic with respect to the velocity state variable. These examples include bearings, floating off-shore structures, vibration isolation and ship roll damping models [6, 28]. In robotics, quadratic damping appears in feed-forward control and in nonlinear impedance devices, such as variable impedance actuators [3]. Variable impedance actuators are of particular interest for collaborative robotics [36].

We suppose that system (2.2), where $g(x)$, $h(x)$ and $f(x)$ are arbitrary polynomials, has an anti-saddle (a node or a focus, or a center) at the origin and write it in the form [23, 24]

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(1 + a_1 x + \ldots + a_{2l}\, x^{2l}) + y(\alpha_0 + \alpha_1 x + \ldots + \alpha_{2k}\, x^{2k}) \\ &\quad + y^2(c_0 + c_1 x + \ldots + c_{2n}\, x^{2n}). \end{aligned} \tag{2.7}$$

Note that for $g(x) \equiv x$ and $h(x) \equiv 0$, by the change of variables $X = x$ and $Y = y + F(x)$, where $F(x) = \int_0^x f(s)\,\mathrm{d}s$, (2.7) is reduced to an equivalent system

$$\dot{X} = Y - F(X), \quad \dot{Y} = -X \tag{2.8}$$

which can be written in the new notation of variables [10]–[13] as follows:

$$\dot{x} = y, \quad \dot{y} = -x + F(y) \tag{2.9}$$

or

$$\dot{x} = y, \quad \dot{y} = -x + \gamma_1\, y + \gamma_2\, y^2 + \gamma_3\, y^3 + \ldots + \gamma_{2k}\, y^{2k} + \gamma_{2k+1}\, y^{2k+1}. \tag{2.10}$$

In [10–13], we have presented a solution of Smale's thirteenth problem [39] proving that the Liénard system (2.10) with a polynomial of degree $2k + 1$ can have at most k limit cycles and we can conclude now that our results [10–13] agree with the conjecture of [31] on the maximum number of limit cycles for the classical Liénard polynomial system (2.10). There were some attempts to construct counterexamples to this conjecture, e. g., in [7, 8]. But that "counterexamples" were completely wrong.

In [14–17], we have studied the general Liénard polynomial system ($h(x) \equiv 0$)

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(1 + a_1\,x + \ldots + a_{2l}\,x^{2l}) \\ &\quad + y(\alpha_0 + \alpha_1\,x + \ldots + \alpha_{2k}\,x^{2k}). \end{aligned} \tag{2.11}$$

In [14–16], under some assumptions on the parameters of (2.11), and in [17], in the general case, we have found the maximum number of limit cycles and their possible distribution for system (2.11).

Consider system (2.7) supposing that $a_1^2 + \ldots + a_{2l}^2 \neq 0$. Its finite singularities are determined by the algebraic system

$$x\,(1 + a_1\,x + \ldots + a_{2l}\,x^{2l}) = 0, \quad y = 0. \tag{2.12}$$

This system always has an anti-saddle at the origin and, in general, can have at most $2l+1$ finite singularities which lie on the x-axis and are distributed so that a saddle (or saddle-node) is followed by a node or a focus, or a center and vice versa [2]. For studying the infinite singularities, the methods applied in [2] for Rayleigh's and van der Pol's equations and also Erugin's two-isocline method developed in [9] can be used; see [10–17].

Following [9], we will study limit cycle bifurcations of (2.7) by means of canonical systems containing field rotation parameters of (2.7) [2, 9].

Theorem 2.1. *The Euler–Lagrange–Liénard polynomial system* (2.7) *with limit cycles can be reduced to one of the canonical forms:*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\,(1 + a_1 x + \ldots + a_{2l} x^{2l}) \\ &\quad + y(\alpha_0 - \beta_1 - \ldots - \beta_{2k-1} + \beta_1 x + \alpha_2 x^2 + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_{2k} x^{2k}) \\ &\quad + y^2(c_0 + c_1\,x + \ldots + c_{2n}\,x^{2n}) \end{aligned} \tag{2.13}$$

or

$$\begin{aligned} \dot{x} &= y \equiv P(x,y), \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1}) \\ &\quad + y(\alpha_0 - \beta_1 - \ldots - \beta_{2k-1} + \beta_1 x + \alpha_2 x^2 + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_{2k} x^{2k}) \\ &\quad + y^2(c_0 + c_1\,x + \ldots + c_{2n}\,x^{2n}) \equiv Q(x,y), \end{aligned} \tag{2.14}$$

where $1+a_1x+\ldots+a_{2l}x^{2l}\neq 0$, α_0, $\alpha_2,\ldots,$ α_{2k} *are field rotation parameters and* β_1, $\beta_3,\ldots,$ β_{2k-1} *are semi-rotation parameters.*

Proof. Let us compare system (2.7) with (2.13) and (2.14). It is easy to see that system (2.13) has the only finite singular point: an anti-saddle at the origin. System (2.14) has at list two singular points including an anti-saddle at the origin and a saddle which, without loss of generality, can be always putted into the point $(1,0)$. Instead of the odd parameters α_1, $\alpha_3,\ldots,$ α_{2k-1} in system (2.7), also without loss of generality, we have introduced new parameters β_1, $\beta_3,\ldots,$ β_{2k-1} into (2.13) and (2.14).

We will study now system (2.14) (system (2.13) can be studied absolutely similarly). Let all of the parameters α_0, $\alpha_2,\ldots,$ α_{2k} and β_1, $\beta_3,\ldots,$ β_{2k-1} vanish in this system,

$$\begin{aligned}\dot{x}&=y,\\ \dot{y}&=x(x-1)(1+b_1x+\ldots+b_{2l-1}x^{2l-1})\\ &\quad+y^2(c_0+c_1\,x+\ldots+c_{2n}\,x^{2n})\end{aligned}\tag{2.15}$$

and consider the corresponding equation

$$\begin{aligned}\frac{dy}{dx}&=\frac{x(x-1)(1+b_1x+\ldots+b_{2l-1}x^{2l-1})+y^2(c_0+c_1\,x+\ldots+c_{2n}\,x^{2n})}{y}\\ &\equiv F(x,y).\end{aligned}\tag{2.16}$$

Since $F(x,-y)=-F(x,y)$, the direction field of (2.16) (and the vector field of (2.15) as well) is symmetric with respect to the x-axis. It follows that for arbitrary values of the parameters $b_1,\ldots,b_{2l-1}$ system (2.15) has centers as anti-saddles and cannot have limit cycles surrounding these points. Therefore, we can fix the parameters $b_1,\ldots,b_{2l-1}$ in system (2.14), fixing the position of its finite singularities on the x-axis.

To prove that the even parameters α_0, $\alpha_2,\ldots,$ α_{2k} rotate the vector field of (2.12), let us calculate the following determinants:

$$\begin{aligned}\Delta_{\alpha_0}&=P\,Q'_{\alpha_0}-QP'_{\alpha_0}=y^2\geq 0,\\ \Delta_{\alpha_2}&=P\,Q'_{\alpha_2}-QP'_{\alpha_2}=x^2y^2\geq 0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \Delta_{\alpha_{2k}}&=P\,Q'_{\alpha_{2k}}-QP'_{\alpha_{2k}}=x^{2k}y^2\geq 0.\end{aligned}$$

By definition of a field rotation parameter [2,9], for increasing each of the parameters α_0, $\alpha_2,\ldots,$ α_{2k}, under the fixed others, the vector field of system (2.14) is rotated in the positive direction (counterclockwise) in the whole phase plane; and, conversely, for decreasing each of these parameters, the vector field of (2.14) is rotated in the negative direction (clockwise).

Calculating the corresponding determinants for the parameters β_1, $\beta_3, \ldots, \beta_{2k-1}$, we can see that

$$\Delta_{\beta_1} = P\,Q'_{\beta_1} - QP'_{\beta_1} = (x-1)\,y^2,$$
$$\Delta_{\beta_3} = P\,Q'_{\beta_3} - QP'_{\beta_3} = (x^3-1)\,y^2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Delta_{\beta_{2k-1}} = P\,Q'_{\beta_{2k-1}} - QP'_{\beta_{2k-1}} = (x^{2k-1}-1)\,y^2.$$

It follows [2,9] that, for increasing each of the parameters $\beta_1, \beta_3, \ldots,$ β_{2k-1}, under the fixed others, the vector field of system (2.14) is rotated in the positive direction (counterclockwise) in the half-plane $x > 1$ and in the negative direction (clockwise) in the half-plane $x < 1$ and vice versa for decreasing each of these parameters. We will call these parameters as semi-rotation ones.

Thus, for studying limit cycle bifurcations of (2.7), it is sufficient to consider the canonical systems (2.13) and (2.14) containing the field rotation parameters $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$ and the semi-rotation parameters $\beta_1, \beta_3, \ldots, \beta_{2k-1}$. The theorem is proved [23, 24]. □

By means of the canonical systems (2.13) and (2.14), we will prove the following theorem [23, 24].

Theorem 2.2. *The Euler–Lagrange–Liénard polynomial system* (2.7) *can have at most $k+l+1$ limit cycles, $k+1$ surrounding the origin and l surrounding one by one the other singularities of* (2.7).

Proof. According to Theorem 2.1, for the study of limit cycle bifurcations of system (2.7), it is sufficient to consider the canonical systems (2.13) and (2.14) containing the field rotation parameters α_0, $\alpha_2, \ldots, \alpha_{2k}$ and the semi-rotation parameters $\beta_1, \beta_3, \ldots, \beta_{2k-1}$. We will work with (2.14) again (system (2.13) can be considered in a similar way).

Vanishing all of the parameters $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$ and $\beta_1, \beta_3, \ldots, \beta_{2k-1}$ in (2.14), we will have system (2.15) which is symmetric with respect to the x-axis and has centers as anti-saddles. Its center domains are bounded by either separatrix loops or digons of the saddles or saddle-nodes of (2.15) lying on the x-axis.

Let us input successively the semi-rotation parameters $\beta_1, \beta_3, \ldots,$ β_{2k-1} into system (2.15) beginning with the parameters at the highest degrees of x and alternating with their signs. So, begin with the parameter β_{2k-1} and let, for definiteness, $\beta_{2k-1}>0$:

$$\begin{gathered}\dot{x}=y,\\ \dot{y}=x(x-1)(1+b_1x+\ldots+b_{2l-1}x^{2l-1})\\ +y(-\beta_{2k-1}+\beta_{2k-1}x^{2k-1})+y^2(c_0+c_1\,x+\ldots+c_{2n}\,x^{2n}).\end{gathered}\tag{2.17}$$

In this case, the vector field of (2.17) is rotated in the negative direction (clockwise) in the half-plane $x<1$ turning the center at the origin into a rough stable focus. All of the other centers lying in the half-plane $x>1$ become rough unstable foci, since the vector field of (2.17) is rotated in the positive direction (counterclockwise) in this half-plane [2, 9].

Fix β_{2k-1} and input the parameter $\beta_{2k-3}<0$ into (2.17):

$$\begin{gathered}\dot{x}=y,\\ \dot{y}=x(x-1)(1+b_1x+\ldots+b_{2l-1}x^{2l-1})\\ +y(-\beta_{2k-3}-\beta_{2k-1}+\beta_{2k-3}x^{2k-3}+\beta_{2k-1}x^{2k-1})\\ +y^2(c_0+c_1\,x+\ldots+c_{2n}\,x^{2n}).\end{gathered}\tag{2.18}$$

Then the vector field of (2.18) is rotated in the opposite directions in each of the half-planes $x<1$ and $x>1$. Under decreasing β_{2k-3}, when $\beta_{2k-3}=-\beta_{2k-1}$, the focus at the origin becomes nonrough (weak), changes the character of its stability and generates a stable limit cycle. All of the other foci in the half-plane $x>1$ will also generate unstable limit cycles for some values of β_{2k-3} after changing the character of their stability. Under further decreasing β_{2k-3}, all of the limit cycles will expand disappearing on separatrix cycles of (2.18) [2, 9].

Denote the limit cycle surrounding the origin by Γ_0, the domain outside the cycle by D_{01}, the domain inside the cycle by D_{02} and consider logical possibilities of the appearance of other (semi-stable) limit cycles from a "trajectory concentration" surrounding this singular point. It is clear that, under decreasing the parameter β_{2k-3}, a semi-stable limit cycle cannot appear in the domain D_{02}, since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation [10–17].

By contradiction, we can also prove that a semi-stable limit cycle cannot appear in the domain D_{01}. Suppose it appears in this domain for some values of the parameters $\beta^*_{2k-1}>0$ and $\beta^*_{2k-3}<0$. Return to system (2.15) and change the inputting order for the semi-rotation parameters. Input first the parameter $\beta_{2k-3}<0$:

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= x(x-1)(1+b_1x+\ldots+b_{2l-1}x^{2l-1})\\ &+ y(-\beta_{2k-3}+\beta_{2k-3}x^{2k-3})+y^2(c_0+c_1\,x+\ldots+c_{2n}\,x^{2n}).\end{aligned} \tag{2.19}$$

Fix it under $\beta_{2k-3}=\beta^*_{2k-3}$. The vector field of (2.19) is rotated counterclockwise and the origin turns into a rough unstable focus. Inputting the parameter $\beta_{2k-1}>0$ into (2.19), we get again system (2.18) the vector field of which is rotated clockwise. Under this rotation, a stable limit cycle Γ_0 will appear from a separatrix cycle for some value of β_{2k-1}. This cycle will contract, the outside spirals winding onto the cycle will untwist and the distance between their coils will increase under increasing β_{2k-1} to the value β^*_{2k-1}. It follows that there are no values of $\beta^*_{2k-3}<0$ and $\beta^*_{2k-1}>0$ for which a semi-stable limit cycle could appear in the domain D_{01}.

This contradiction proves the uniqueness of a limit cycle surrounding the origin in system (2.18) for any values of the parameters β_{2k-3} and β_{2k-1} of different signs. Obviously, if these parameters have the same sign, system (2.18) has no limit cycles surrounding the origin at all. On the same reason, this system cannot have more than l limit cycles surrounding the other singularities (foci or nodes) of (2.18) one by one.

It is clear that inputting the other semi-rotation parameters $\beta_{2k-5},\ldots,\beta_1$ into system (2.18) will not give us more limit cycles, since all of these parameters are rough with respect to the origin and the other anti-saddles lying in the half-plane $x>1$. Therefore, the maximum number of limit cycles for the system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= x(x-1)(1+b_1x+\ldots+b_{2l-1}x^{2l-1})\\ &+y(-\beta_1-\ldots-\beta_{2k-3}-\beta_{2k-1}+\beta_1x+\ldots+\beta_{2k-3}x^{2k-3}+\beta_{2k-1}x^{2k-1})\\ &+y^2(c_0+c_1\,x+\ldots+c_{2n}\,x^{2n})\end{aligned} \tag{2.20}$$

is equal to $l+1$ and they surround the anti-saddles (foci or nodes) of (2.20) one by one.

Suppose that $\beta_1+\ldots+\beta_{2k-3}+\beta_{2k-1}>0$ and input the last rough parameter $\alpha_0>0$ into system (2.20):

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= x(x-1)(1+b_1x+\ldots+b_{2l-1}x^{2l-1})\\ &+ y(\alpha_0-\beta_1-\ldots-\beta_{2k-1}+\beta_1x+\ldots+\beta_{2k-1}x^{2k-1})\\ &+y^2(c_0+c_1\,x+\ldots+c_{2n}\,x^{2n}).\end{aligned} \tag{2.21}$$

This parameter rotating the vector field of (2.21) counterclockwise in the whole phase plane also will not give us more limit cycles, but under increasing α_0, when $\alpha_0 = \beta_1 + \ldots + \beta_{2k-1}$, we can make the focus at the origin nonrough (weak), after the disappearance of the limit cycle Γ_0 in it. Fix this value of the parameter α_0 ($\alpha_0 = \alpha_0^*$):

$$\begin{gathered} \dot{x} = y, \\ \dot{y} = x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1}) \\ + y(\beta_1 x + \ldots + \beta_{2k-1} x^{2k-1}) + y^2(c_0 + c_1\, x + \ldots + c_{2n}\, x^{2n}). \end{gathered} \tag{2.22}$$

Let us input now successively the other field rotation parameters $\alpha_2, \ldots, \alpha_{2k}$ into system (2.22) beginning again with the parameters at the highest degrees of x and alternating with their signs; see [10–17]. So, begin with the parameter α_{2k} and let $\alpha_{2k} < 0$:

$$\begin{gathered} \dot{x} = y, \\ \dot{y} = x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1}) \\ + y(\beta_1 x + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_{2k} x^{2k}) \\ + y^2(c_0 + c_1\, x + \ldots + c_{2n}\, x^{2n}). \end{gathered} \tag{2.23}$$

In this case, the vector field of (2.23) is rotated clockwise in the whole phase plane and the focus at the origin changes the character of its stability generating again a stable limit cycle. The limit cycles surrounding the other singularities of (2.23) can also still exist. Denote the limit cycle surrounding the origin by Γ_1, the domain outside the cycle by D_1 and the domain inside the cycle by D_2. The uniqueness of a limit cycle surrounding the origin (and limit cycles surrounding the other singularities) for system (2.23) can be proved by contradiction like we have done above for (2.18); see also [10–17].

Let system (2.23) have the unique limit cycle Γ_1 surrounding the origin and l limit cycles surrounding the other antisaddles of (2.23). Fix the parameter $\alpha_{2k} < 0$ and input the parameter $\alpha_{2k-2} > 0$ into (2.23):

$$\begin{gathered} \dot{x} = y, \\ \dot{y} = x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1}) \\ + y(\beta_1 x + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_{2k-2} x^{2k-2} + \alpha_{2k} x^{2k}) \\ + y^2(c_0 + c_1\, x + \ldots + c_{2n}\, x^{2n}). \end{gathered} \tag{2.24}$$

Then the vector field of (2.24) is rotated in the opposite direction (counterclockwise) and the focus at the origin immediately changes the character of its stability (since its degree of nonroughness decreases and the sign of the field rotation parameter at the lower degree of x changes) generating the second (unstable) limit cycle Γ_2. The limit cycles surrounding the other singularities of (2.24) can only disappear in the corresponding foci (because of their roughness) under increasing the parameter α_{2k-2}.

Under further increasing α_{2k-2}, the limit cycle Γ_2 will join with Γ_1 forming a semi-stable limit cycle, Γ_{12}, which will disappear in a "trajectory concentration" surrounding the origin. Can another semi-stable limit cycle appear around the origin in addition to Γ_{12}? It is clear that such a limit cycle cannot appear either in the domain D_1 bounded on the inside by the cycle Γ_1 or in the domain D_3 bounded by the origin and Γ_2 because of the increasing distance between the spiral coils filling these domains under increasing the parameter [10–17].

To prove the impossibility of the appearance of a semi-stable limit cycle in the domain D_2 bounded by the cycles Γ_1 and Γ_2 (before their joining), suppose the contrary, i. e., that for some values of these parameters, $\alpha_{2k}^* < 0$ and $\alpha_{2k-2}^* > 0$, such a semi-stable cycle exists. Return to system (2.22) again and input first the parameter $\alpha_{2k-2} > 0$:

$$\begin{gathered}\dot{x} = y, \\ \dot{y} = x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1}) \\ + y(\beta_1 x + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_{2k-2} x^{2k-2}) \\ + y^2 (c_0 + c_1 x + \ldots + c_{2n} x^{2n}).\end{gathered} \tag{2.25}$$

This parameter rotates the vector field of (2.25) counterclockwise preserving the origin as a nonrough stable focus.

Fix this parameter under $\alpha_{2k-2} = \alpha_{2k-2}^*$ and input the parameter $\alpha_{2k} < 0$ into (2.25) getting again system (2.22). Since, by our assumption, this system has two limit cycles surrounding the origin for $\alpha_{2k} > \alpha_{2k}^*$, there exists some value of the parameter, α_{2k}^{12} ($\alpha_{2k}^{12} < \alpha_{2k}^* < 0$), for which a semi-stable limit cycle, Γ_{12}, appears in system (2.24) and then splits into a stable cycle Γ_1 and an unstable cycle Γ_2 under further decreasing α_{2k}. The formed domain D_2 bounded by the limit cycles Γ_1, Γ_2 and filled by the spirals will enlarge since, on the properties of a field rotation parameter, the interior unstable limit cycle Γ_2 will contract and the exterior stable limit cycle Γ_1 will expand under decreasing α_{2k}. The distance between the spirals of the domain D_2 will naturally increase, which will prevent the appearance of a semi-stable limit cycle in this domain for $\alpha_{2k} < \alpha_{2k}^{12}$ [10–17].

Thus, there are no such values of the parameters, $\alpha_{2k}^* < 0$ and $\alpha_{2k-2}^* > 0$, for which system (2.24) would have an additional semi-stable limit cycle surrounding the origin. Obviously, there are no other values of the parameters α_{2k} and α_{2k-2} for which system (2.24) would have more than two limit cycles surrounding this singular point. On the same reason, additional semi-stable limit cycles cannot appear around the other singularities (foci or nodes) of (2.24). Therefore, $l + 2$ is the maximum number of limit cycles in system (2.24).

Suppose that system (2.24) has two limit cycles, Γ_1 and Γ_2, surrounding the origin and l limit cycles surrounding the other antisaddles of (2.24) (this is always possible if $-\alpha_{2k} \gg \alpha_{2k-2} > 0$). Fix the parameters α_{2k}, α_{2k-2} and consider a more general system inputting the third parameter, $\alpha_{2k-4} < 0$, into (2.22):

$$\begin{gathered}\dot{x} = y,\\ \dot{y} = x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1})\\ + y(\beta_1 x + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_{2k-4} x^{2k-4} + \alpha_{2k-2} x^{2k-2} + \alpha_{2k} x^{2k})\\ + y^2 (c_0 + c_1\, x + \ldots + c_{2n}\, x^{2n}).\end{gathered} \tag{2.26}$$

For decreasing α_{2k-4}, the vector field of (2.26) will be rotated clockwise and the focus at the origin will immediately change the character of its stability generating a third (stable) limit cycle, Γ_3. With further decreasing α_{2k-4}, Γ_3 will join with Γ_2 forming a semi-stable limit cycle, Γ_{23}, which will disappear in a "trajectory concentration" surrounding the origin; the cycle Γ_1 will expand disappearing on a separatrix cycle of (2.26).

Let system (2.26) have three limit cycles surrounding the origin: Γ_1, Γ_2, Γ_3. Could an additional semi-stable limit cycle appear with decreasing α_{2k-4} after splitting of which system (2.26) would have five limit cycles around the origin? It is clear that such a limit cycle cannot appear either in the domain D_2 bounded by the cycles Γ_1 and Γ_2 or in the domain D_4 bounded by the origin and Γ_3 because of the increasing distance between the spiral coils filling these domains after decreasing α_{2k-4}. Consider two other domains: D_1 bounded on the inside by the cycle Γ_1 and D_3 bounded by the cycles Γ_2 and Γ_3. As before, we will prove the impossibility of the appearance of a semi-stable limit cycle in these domains by contradiction.

Suppose that for some set of values of the parameters $\alpha^*_{2k} < 0$, $\alpha^*_{2k-2} > 0$ and $\alpha^*_{2k-4} < 0$ such a semi-stable cycle exists. Return to system (2.22) again inputting first the parameters $\alpha_{2k-2} > 0$ and $\alpha_{2k-4} < 0$:

$$\begin{gathered}\dot{x} = y,\\ \dot{y} = x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1})\\ + y(\beta_1 x + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_{2k-4} x^{2k-4} + \alpha_{2k} x^{2k})\\ + y^2 (c_0 + c_1\, x + \ldots + c_{2n}\, x^{2n}).\end{gathered} \tag{2.27}$$

Fix the parameter α_{2k-2} under the value α^*_{2k-2}. With decreasing α_{2k-4}, a separatrix cycle formed around the origin will generate a stable limit cycle Γ_1. Fix α_{2k-4} under the value α^*_{2k-4} and input the parameter $\alpha_{2k} > 0$ into (2.27) getting system (2.26).

Since, by our assumption, (2.26) has three limit cycles for $\alpha_{2k} > \alpha_{2k}^*$, there exists some value of the parameter α_{2k}^{23} ($\alpha_{2k}^{23} < \alpha_{2k}^* < 0$) for which a semi-stable limit cycle, Γ_{23}, appears in this system and then splits into an unstable cycle Γ_2 and a stable cycle Γ_3 with further decreasing α_{2k}. The formed domain D_3 bounded by the limit cycles Γ_2, Γ_3 and also the domain D_1 bounded on the inside by the limit cycle Γ_1 will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there [10–17].

All other combinations of the parameters α_{2k}, α_{2k-2}, and α_{2k-4} are considered in a similar way. It follows that system (2.26) can have at most $l+3$ limit cycles.

If we continue the procedure of successive inputting the field rotation parameters, $\alpha_{2k}, \ldots, \alpha_2$, into system (2.22),

$$\begin{gathered} \dot{x} = y, \\ \dot{y} = x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1}) \\ + y(\beta_1 x + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_2 x^2 + \ldots + \alpha_{2k} x^{2k}) \\ + y^2 (c_0 + c_1 x + \ldots + c_{2n} x^{2n}), \end{gathered} \tag{2.28}$$

it is possible to obtain k limit cycles surrounding the origin and l surrounding one by one the other singularities (foci or nodes) ($-\alpha_{2k} \gg \alpha_{2k-2} \gg -\alpha_{2k-4} \gg \alpha_{2k-6} \gg \ldots$).

Then, by means of the parameter $\alpha_0 \neq \beta_1 + \ldots + \beta_{2k-1}$ ($\alpha_0 > \alpha_0^*$, if $\alpha_2 < 0$, and $\alpha_0 < \alpha_0^*$, if $\alpha_2 > 0$), we will have the canonical system (2.14) with an additional limit cycle surrounding the origin and can conclude that this system (i. e., the Euler–Lagrange–Liénard polynomial system (2.7) as well) has at most $k+l+1$ limit cycles, $k+1$ surrounding the origin and l surrounding one by one the antisaddles (foci or nodes) of (2.14) (and (2.7) as well). The theorem is proved [23, 24]. □

3. The Topp model of diabetes dynamics

In [40], a novel model of coupled β-cell mass, insulin, and glucose dynamics was presented, which is used to investigate the normal behavior of the glucose regulatory system and pathways into diabetes. The behavior of the model is consistent with the observed behavior of the glucose regulatory system in response to changes in blood glucose levels, insulin sensitivity, and β-cell insulin secretion rates.

In the post-absorptive state, glucose is released into the blood by the liver and kidneys, removed from the interstitial fluid by all the cells of the body, and distributed into many physiological compartments, e. g., arterial blood, venous blood, cerebral spinal fluid, interstitial fluid [40].

Since we are primarily concerned with the evolution of fasting blood glucose levels over a time-scale of days to years, glucose dynamics are modeled with a single-compartment mass balance equation

$$\dot{G} = a - (b + cI)G. \tag{3.1}$$

Insulin is secreted by pancreatic β-cells, cleared by the liver, kidneys, and insulin receptors, and distributed into several compartments, e. g., portal vein, peripheral blood, and interstitial fluid. The main concern is the long-time evolution of fasting insulin levels in peripheral blood. Since the dynamics of fasting insulin levels on this time-scale are slow, we use a single-compartment equation given by

$$\dot{I} = \frac{\beta G^2}{1 + G^2} - \alpha I. \tag{3.2}$$

Despite a complex distribution of pancreatic β cells throughout the pancreas, β-cell mass dynamics have been successfully quantified with a single-compartment model

$$\dot{\beta} = (-l + mG - nG^2)\beta. \tag{3.3}$$

Finally, the Topp model (a rational endocrine system) is

$$\begin{aligned} \dot{G} &= a - (b + cI)G, \\ \dot{I} &= \frac{\beta G^2}{1 + G^2} - \alpha I, \\ \dot{\beta} &= (-l + mG - nG^2)\beta \end{aligned} \tag{3.4}$$

with parameters as in [40].

On the short timescale, β is approximately constant and, relabelling the variables, the fast dynamics is a planar system

$$\begin{aligned} \dot{x} &= a - (b + c\,y)x, \\ \dot{y} &= \frac{\beta x^2}{1 + x^2} - \alpha\, y. \end{aligned} \tag{3.5}$$

By rescaling time, this can be written in the form of a quartic dynamical system:

$$\begin{aligned} \dot{x} &= (1 + x^2)(a - (b + c\,y)x) \equiv P, \\ \dot{y} &= \beta x^2 - \alpha\, y(1 + x^2) \equiv Q. \end{aligned} \tag{3.6}$$

Together with (3.6), we will also consider an auxiliary system (see [2, 9, 34])

$$\dot{x} = P - \gamma Q, \qquad \dot{y} = Q + \gamma P, \tag{3.7}$$

applying to these systems new bifurcation methods and geometric approaches developed in [5, 9–22, 24] and carrying out the qualitative analysis of (3.6).

Consider system (3.6). Its finite singularities are determined by the algebraic system

$$\begin{aligned} &(1+x^2)(a-(b+c\,y)x)=0, \\ &\beta x^2-\alpha\,y(1+x^2)=0 \end{aligned} \tag{3.8}$$

which can give us at most three singular points in the first quadrant: a saddle S and two antisaddles (non-saddles), A_1 and A_2, according to the second Poincaré index theorem (Theorem 1.2). Suppose that with respect to the x-axis they have the following sequence: A_1, S, A_2. System (3.6) can also have one singular point (an antisaddle) or two singular points (an antisaddle and a saddle-node) in the first quadrant.

To study singular points of (3.6) at infinity, consider the corresponding differential equation

$$\frac{dy}{dx}=\frac{\beta x^2-\alpha\,y(1+x^2)}{(1+x^2)(a-(b+c\,y)x)}. \tag{3.9}$$

Dividing the numerator and denominator of the right-hand side of (3.9) by $x^4 (x \neq 0)$ and denoting y/x by u (as well as dy/dx), we will get the equation

$$u^2=0, \quad \text{where} \quad u=y/x, \tag{3.10}$$

for all infinite singularities of (3.9) except when $x=0$ (the "ends" of the y-axis); see [2, 9]. For this special case we can divide the numerator and denominator of the right-hand side of (3.9) by y^4 $(y \neq 0)$ denoting x/y by v (as well as dx/dy) and consider the equation

$$v^2=0, \quad \text{where} \quad v=x/y. \tag{3.11}$$

According to the Poincaré index theorems (Theorem 1.1 and Theorem 1.2), the equations (3.10) and (3.11) give us two double singular points (saddle-nodes) at infinity for (3.9): on the "ends" of the x and y axes.

Using the obtained information on singular points and applying geometric approaches developed in [5, 9–22, 24], we can study now the limit cycle bifurcations of system (3.6).

Applying the definition of a field rotation parameter [2, 9, 34], to system (3.6), let us calculate the corresponding determinants for the parameters a, b, c, α, and β, respectively:

$$\Delta_a = PQ'_a - QP'_a = -(1+x^2)(\beta x^2-\alpha\,y(1+x^2)), \tag{3.12}$$

$$\Delta_b = PQ'_b - QP'_b = x(1+x^2)(\beta x^2 - \alpha\, y(1+x^2)), \tag{3.13}$$

$$\Delta_c = PQ'_c - QP'_c = xy(1+x^2)(\beta x^2 - \alpha\, y(1+x^2)), \tag{3.14}$$

$$\Delta_\alpha = PQ'_\alpha - QP'_\alpha = -y(1+x^2)^2(a-(b+c\,y)x), \tag{3.15}$$

$$\Delta_\beta = PQ'_\beta - QP'_\beta = x^2(1+x^2)(a-(b+c\,y)x). \tag{3.16}$$

It follows from (3.12)–(3.14) that in the first quadrant the signs of Δ_a, Δ_b, Δ_c depend on the sign of $\beta x^2 - \alpha\, y(1+x^2)$ and from (3.15) and (3.16) that the signs of Δ_α and Δ_β depend on the sign of $a-(b+c\,y)x$ on increasing (or decreasing) the parameters a, b, c, α, and β, respectively.

Therefore, to study limit cycle bifurcations of system (3.6), it makes sense together with (3.6) to consider also the auxiliary system (3.7) with field-rotation parameter γ:

$$\Delta_\gamma = P^2 + Q^2 \geq 0. \tag{3.17}$$

Using system (3.7) and applying Perko's results, we prove the following theorem [21, 22, 24].

Theorem 3.1. *The reduced Topp system* (3.6) *can have at most two limit cycles.*

Proof. In [4, 5, 30, 41], where a similar quartic system was studied, it was proved that the cyclicity of singular points in such a system is equal to two and that the system can have at least two limit cycles; see also [18, 20, 26, 29] with similar results.

Consider systems (3.6)–(3.7) supposing that the cyclicity of singular points in these systems is equal to two and that the systems can have at least two limit cycles. Let us prove now that these systems have at most two limit cycles. The proof is carried out by contradiction applying Catastrophe Theory; see [9, 34].

We will study more general system (3.7) with three parameters: α, β, and γ (the parameters a, b, and c can be fixed, since they do not generate limit cycles). Suppose that (3.7) has three limit cycles surrounding the singular point A_1, in the first quadrant. Then we get into some domain of the parameters α, β, and γ being restricted by definite conditions on three other parameters, a, b, and c. This domain is bounded by two fold bifurcation surfaces forming a cusp bifurcation surface of multiplicity-three limit cycles in the space of the parameters α, β, and γ.

The corresponding maximal one-parameter family of multiplicity-three limit cycles cannot be cyclic, otherwise there will be at least one point corresponding to the limit cycle of multiplicity four (or even higher) in the parameter space.

Extending the bifurcation curve of multiplicity-four limit cycles through this point and parameterizing the corresponding maximal one-parameter family of multiplicity-four limit cycles by the field rotation parameter, γ, according to Theorem 1.4, we will obtain two monotonic curves of multiplicity-three and one, respectively, which, by the Wintner–Perko termination principle (Theorem 1.3), terminate either at the point A_1 or on a separatrix cycle surrounding this point. Since on our assumption the cyclicity of the singular point is equal to two, we have obtained a contradiction with the termination principle stating that the multiplicity of limit cycles cannot be higher than the multiplicity (cyclicity) of the singular point in which they terminate.

If the maximal one-parameter family of multiplicity-three limit cycles is not cyclic, using the same principle (Theorem 1.3), this again contradicts the cyclicity of A_1 not admitting the multiplicity of limit cycles to be higher than two. This contradiction completes the proof in the case of one singular point in the first quadrant.

Suppose that system (3.7) with three finite singularities, A_1, S, and A_2, has two small limit cycles around, for example, the point A_1 (the case when limit cycles surround the point A_2 is considered in a similar way). Then we get into some domain in the space of the parameters α, β, and γ which is bounded by a fold bifurcation surface of multiplicity-two limit cycles.

The corresponding maximal one-parameter family of multiplicity-two limit cycles cannot be cyclic, otherwise there will be at least one point corresponding to the limit cycle of multiplicity three (or even higher) in the parameter space. Extending the bifurcation curve of multiplicity-three limit cycles through this point and parameterizing the corresponding maximal one-parameter family of multiplicity-three limit cycles by the field rotation parameter, γ, according to Theorem 1.4, we will obtain a monotonic curve which, by the Wintner–Perko termination principle (Theorem 1.3), terminates either at the point A_1 or on some separatrix cycle surrounding this point. Since we know at least the cyclicity of the singular point which on our assumption is equal to one in this case, we have obtained a contradiction with the termination principle.

If the maximal one-parameter family of multiplicity-two limit cycles is not cyclic, using the same principle (Theorem 1.3), this again contradicts the cyclicity of A_1 not admitting the multiplicity of limit cycles higher than one. Moreover, it also follows from the termination principle that either an ordinary (small) separatrix loop or a big loop, or an eight-loop cannot have the multiplicity (cyclicity) higher than one in this case. Therefore, according to the same principle, there are no more than one limit cycle in the exterior domain surrounding all three finite singularities, A_1, S, and A_2.

Thus, taking into account all other possibilities for limit cycle bifurcations (see [4, 5, 30, 41]), we conclude that system (3.7) (and (3.6) as well) cannot have either a multiplicity-three limit cycle or more than two limit cycles in any configuration. The theorem is proved [21, 22, 24]. □

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