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Topological properties of closed weakly m-semiconvex sets

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Abstract. The present work considers properties of generally convex sets in the n-dimensional real Euclidean space $\mathbb{R}^n$, $n > 1$, known as weakly m-semiconvex, $m = 1, 2, \ldots, n-1$. For all that, the subclass of not m-semiconvex sets is distinguished from the class of weakly m-semiconvex sets. A set of the space $\mathbb{R}^n$ is called ***m-semiconvex*** if, for any point of the complement of the set to the whole space, there is an m-dimensional half-plane passing through this point and not intersecting the set. An open set of $\mathbb{R}^n$ is called ***weakly m-semiconvex*** if, for any point of the boundary of the set, there exists an m-dimensional half-plane passing through this point and not intersecting the given set. A closed set of $\mathbb{R}^n$ is called ***weakly m-semiconvex*** if it is approximated from the outside by a family of open weakly m-semiconvex sets. An example of a closed set with three connected components of the subclass of weakly 1-semiconvex but not 1-semiconvex sets in the plane is constructed. It is proved that this number of components is minimal for any closed set of the subclass. An example of a closed set of the subclass with a smooth boundary and four components is constructed. It is proved that this number of components is minimal for any closed, bounded set of the subclass having a smooth boundary and a not 1-semiconvex interior. It is also proved that the interior of a closed, weakly 1-semiconvex set with a finite number of components in the plane is weakly 1-semiconvex. Weakly m-semiconvex but not m-semiconvex domains and closed connected sets in $\mathbb{R}^n$ are constructed for any $n \geq 3$ and any $m = 1, 2, \ldots, n-2$.

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1. Introduction

As is known, a set of the multidimensional real Euclidean space $\mathbb{R}^n$ is called *convex* if, together with its two arbitrary points, it contains the

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entire segment connecting the points [2]. Moreover, the intersection of an arbitrary number of convex sets is again a convex set. This property of convex sets makes it possible to determine the minimal convex set that contains an arbitrary given set as follows:

Definition 1.1. ([2]) *The intersection of all convex sets containing a given set $X \subset \mathbb{R}^n$ is called the* ***convex hull*** *of the set X and is denoted by*

$$\operatorname{conv} X = \bigcap_{K \supset X} K, \quad \textit{where sets } K \textit{ are convex.}$$

A class of m - semiconvex sets is one of the classes of generally convex sets. A semiconvexity notion was proposed by Yuriy Zeliskii [7] and it was used in the formulation of a shadow problem generalization. The shadow problem was proposed by Gulmirza Khudaiberganov [4, 5] and is stated as follows: *To find the minimal number of open (closed) balls in the real Euclidean space $\mathbb{R}^n$ that are pairwise disjoint, whose centers are located on a sphere S^{n-1} (see* [1]*), do not contain the sphere center, and such that any straight line passing through the sphere center intersects at least one of the balls.* To formulate the generalized shadow problem, first, let us give the following definitions which we also use in our investigation.

Any m-dimensional affine subspace of the space $\mathbb{R}^n$, $0 \le m < n$, is called an *m-dimensional plane.*

Definition 1.2. *One of two parts of an m-dimensional plane, $m \ge 1$, of the space $\mathbb{R}^n$, $n \ge 2$, into which it is divided by its any of $(m-1)$-dimensional planes (herewith, the points of the $(m-1)$-dimensional plane are included) is said to be an* ***m-dimensional half-plane****.*

For instance, the 1 – dimensional half-plane is a ray, the 2 – dimensional half-plane is a half-plane, etc.

Definition 1.3. ([6]) *A set $E \subset \mathbb{R}^n$ is called* ***m-semiconvex with respect to a point*** *$x \in \mathbb{R}^n \setminus E$, $1 \le m < n$, if there exists an m-dimensional half-plane H such that $x \in H$ and $H \cap E = \varnothing$.*

Definition 1.4. ([6]) *A set $E \subset \mathbb{R}^n$ is called* ***m-semiconvex****, $1 \le m < n$, if it is m-semiconvex with respect to every point $x \in \mathbb{R}^n \setminus E$.*

One can easily see that both definitions satisfy the axiom of convexity: The intersection of each subfamily of these sets also satisfies the definition. Thus, for any set $E \subset \mathbb{R}^n$ we can consider the minimal m-semiconvex set containing E and defined as follows:

Definition 1.5. ([9]) *The intersection of all m-semiconvex sets with fixed m containing a given set $E \subset \mathbb{R}^n$ is called the m-**semiconvex hull** of the set E and is denoted by*

$$\operatorname{conv}_m E = \bigcap_{K \supset E} K, \quad \textit{where sets } K \textit{ are } m\textit{-semiconvex.}$$

The generalized shadow problem is *To find the minimum number of pairwise disjoint closed (open) balls in $\mathbb{R}^n$ (centered on a sphere S^{n-1} and whose radii are smaller than the radius of the sphere) such that any ray starting at the center of the sphere necessarily intersects at least one of these balls.*

In the terms of m-semiconvexity, this problem can be reformulated as follows: *What is the minimum number of pairwise disjoint closed (open) balls in $\mathbb{R}^n$ whose centers are located on a sphere S^{n-1} and the radii are smaller than the radius of this sphere such that the center of the sphere belongs to the 1-semiconvex hull of the family of these balls?*

In the paper [7] the generalized shadow problem is solved as $n = 2$. And only the sufficient number of the balls is indicated as $n = 3$.

We shall use the following standard notations. For a set $G \subset \mathbb{R}^n$ let $\overline{G}$ be its closure, $\operatorname{Int} G$ be its interior, and $\partial G = \overline{G} \setminus \operatorname{Int} G$ be its boundary.

Definition 1.6. ([8]) *An open set $G \subset \mathbb{R}^n$ is called* **weakly m-semiconvex**, *$1 \leq m < n$, if it is m-semiconvex with respect to any point $x \in \partial G$.*

Definition 1.7. ([3]) *They say that a set E* **is approximated from the outside** *by a family of open sets E_k, $k = 1, 2, \ldots$, if $\overline{E_{k+1}}$ is contained in E_k, and $E = \cap_k E_k$.*

It can be proved that any set approximated from the outside by a family of open sets is closed.

Definition 1.8. ([8]) *A closed set $E \subset \mathbb{R}^n$ is called* **weakly m-semiconvex** *if it can be approximated from the outside by a family of open weakly m-semiconvex sets.*

Thus, any weakly m-semiconvex set E is either open or closed. Among closed weakly m-semiconvex sets there are also sets with empty interior:

$$E = \overline{E} = \overline{E} \setminus \operatorname{Int} E = \partial E.$$

Let us denote the classes of m-semiconvex and weakly m-semiconvex sets in $\mathbb{R}^n$, $n \geq 2$, $1 \leq m < n$, by $\mathbf{S^n_m}$ and $\mathbf{WS^n_m}$, respectively. There are weakly 1-semiconvex sets in $\mathbb{R}^2$ which are not 1-semiconvex, i. e., the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ is not empty. Moreover, the following proposition is true:

Lemma 1.1. ([8]) *Let an open set $E \subset \mathbb{R}^2$ belong to the class $\mathbf{WS_1^2} \setminus \mathbf{S_1^2}$. Then E is disconnected.*

The maximal connected subsets E^i, $i = 1, 2, \ldots$, of a nonempty set $E \subset \mathbb{R}^n$ are called *connected components (components)* of the set E. Herewith, $E = \cup_i E^i$.

In [8] the elegant in its simplicity example of an open set of the class $\mathbf{WS_1^2} \setminus \mathbf{S_1^2}$ with three connected components was constructed (see Figure 1).

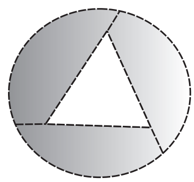

Figure 1.

In addition, the assumption was made that any open set of the class $\mathbf{WS_1^2} \setminus \mathbf{S_1^2}$ consists of not less than three components. This proposition was proved in [9].

Lemma 1.2. ([9]) *Let an open set $E \subset \mathbb{R}^2$ belong to the class $\mathbf{WS_1^2} \setminus \mathbf{S_1^2}$. Then E consists of not less than three components.*

We say that a component G of an open, bounded subset of the plane has *smooth boundary* if ∂G is the image of a C^1-embedding of the unit circle. We say that an open, bounded subset of the plane has smooth boundary if each of its components has smooth boundary.

Lemma 1.3. ([10]) *Let an open, bounded set $E \subset \mathbb{R}^2$ with smooth boundary belong to the class $\mathbf{WS_1^2} \setminus \mathbf{S_1^2}$. Then E consists of not less than four components.*

Definition 1.9. ([10]) *A point $x \in \mathbb{R}^n \setminus E$ is called an m-***nonsemiconvexity point of a set** *$E \subset \mathbb{R}^n$ if any m-dimensional half-plane with x on its boundary intersects E.*

The set of all m-nonsemiconvexity points of a set $E \subset \mathbb{R}^n$, $n \geq 2$, is denoted by $(E)^\diamondsuit_m$, $1 \leq m < n$. Thus, if a set $E \subset \mathbb{R}^n$ is not m-semiconvex, then obviously $(E)^\diamondsuit_m \neq \varnothing$. And let

$$(E)^\diamondsuit_1 := (E)^\diamondsuit, \quad E \subset \mathbb{R}^n, \quad n \geq 2.$$

Definition 1.10. ([10]) *We say that* **a set** $E \subset \mathbb{R}^n$ **is projected from a point** $x \in \mathbb{R}^n$ **on a set** $G \subset \mathbb{R}^n$ *if any ray, starting at the point* x *and intersecting* E*, intersects* G *as well.*

Two interesting lemmas follow directly from Lemmas 1.2, 1.3.

Lemma 1.4. ([10]) *Let an open set* $E \subset \mathbb{R}^2$ *belong to the class* $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ *and consist of three components. Then none of its components is projected on the union of the others from a point of* 1*-nonsemiconvexity of* E*.*

Lemma 1.5. ([10]) *Let an open, bounded set* $E \subset \mathbb{R}^2$ *belong to the class* $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ *and consist of four components with smooth boundary. Then none of its components is projected on the union of the others from a point of* 1*-nonsemiconvexity of* E*.*

The present work proceeds the research of Yu. Zelinskii and his students by investigating the topological properties mainly of closed sets of the classes $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ and $\mathbf{WS^n_m} \setminus \mathbf{S^n_m}$, $n \geq 3$, $1 \leq m < n-1$.

In chapter 2, an example of a closed set of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ with three components and an example of a closed set of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ with smooth boundary and with four and more components are constructed.

In chapter 3, it is proved that, similarly to the case of open sets of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$, the closed sets of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ also consist of not less than three components. Moreover, if they are bounded, have smooth boundary and not 1-semiconvex interior, then they consist of not less than four components. It is also proved that the interior of a closed, weakly 1-semiconvex set with a finite number of components in the plane is weakly 1-semiconvex.

In chapter 4, domains and closed connected sets of the classes $\mathbf{WS^n_m} \setminus \mathbf{S^n_m}$, $n \geq 3$, $1 \leq m < n-1$, are constructed. In conclusion, a list of open problems concerning the topic is proposed.

2. Examples of closed sets of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$

First, give some denotations. The interval between points $x, y \in \mathbb{R}^n$ will be written as xy and the distance between the points will be written as $|x-y|$. Let $U(y,\varepsilon) := \{x \in \mathbb{R}^n : |x-y| < \varepsilon\}$, $\varepsilon > 0$, be a neighborhood of a point $y \in \mathbb{R}^n$.

Lemma 2.1. *The closure of an open set* E *of the class* $\mathbf{WS^n_m} \setminus \mathbf{S^n_m}$*,* $n \geq 2$*, is not* m*-semiconvex,* $1 \leq m < n$*.*

Proof. Since $E \in \mathbf{WS^n_m} \setminus \mathbf{S^n_m}$, there exists an m-nonsemiconvexity point $x \in \mathbb{R}^n \setminus \overline{E}$ of the set E. Since $E \subset \overline{E}$, any m-dimensional half-plane with x on its boundary and intersecting E intersects $\overline{E}$ as well. Thus, x is an m-nonsemiconvexity point of $\overline{E}$. □

In [10] examples of open sets of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ including an open set with smooth boundary were constructed (see Figure 2 a), 3 b)). Here we construct examples of closed sets of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ using the examples from [10]. To do this, first, we consider some accessory open sets of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$.

In the following example open sets of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ consisting of three components are considered.

Example 2.1. Let $\mathbf{E}_1$ be an open rectangle with vertices A_i, $i = \overline{1,4}$, and the sides parallel to the axes. Let $\mathbf{E}_2$ be an open set with vertices B_j, $j = \overline{1,6}$, as on Figure 2 a). Let $B_0 \in B_1O$, where O is the origin. Let us consider the points $B_t \in B_0B_1$, $t \in [0,1]$, defined as $B_t := tB_1 + (1-t)B_0$. Let γ_t, $t \in [0,1]$, be the ray starting at B_t and passing through the point A_1. Let $\widetilde{\gamma}_t$ be the ray symmetric to γ_t with respect to the axis Oy. Then let $\widetilde{\gamma}_t$ start at a point $\widetilde{B}_t$ and $C^t_1 := \gamma_t \cap \widetilde{\gamma}_t$, $t \in [0,1]$. Let the length of A_2A_3 be such that the line η passing through the points A_2, B_6 intersects the triangle $B_0\widetilde{B}_0C^0_1$. Then $C^t_2 := \eta \cap \widetilde{\gamma}_t$ and $C^t_3 := \eta \cap \gamma_t$, $t \in [0,1]$.

Let ξ be a straight parallel to the axis Ox, intersecting the rays γ_t, $\widetilde{\gamma}_t$, $t \in [0,1]$, and not intersecting $\mathbf{E}_1$, $\mathbf{E}_2$. Now we construct open rectangles $\mathbf{E}^t_3$, $t \in [0,1]$, with vertices D^t_i, $i = \overline{1,4}$, laying under the line ξ and such that $D^t_1 \in \widetilde{\gamma}_t$, $D^t_2 \in \gamma_t$ (see Figure 2 a)).

Then, by the construction, any ray starting at a point of the open triangle $C^t_1C^t_2C^t_3$ intersects the set $\mathbf{E}^t := \mathbf{E}_1 \cup \mathbf{E}_2 \cup \mathbf{E}^t_3$, and any ray starting at $\partial\mathbf{E}^t$ does not intersect $\mathbf{E}^t$, $t \in [0,1]$. Thus, each set $\mathbf{E}^t$, $t \in [0,1]$, belongs to the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$.

Now construct the example of a closed set of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$.

Example 2.2. Let $\mathbf{E}^0 := \mathbf{E}_1 \cup \mathbf{E}_2 \cup \mathbf{E}^0_3$ be the set from the previous example. By Lemma 2.1, the closed set $\overline{\mathbf{E}^0}$ is not 1-semiconvex. Show that $\overline{\mathbf{E}^0}$ is approximated from the outside by a family of open, weakly 1-semiconvex sets.

Let $\mathbf{E}^d_1 := A^d_1A^d_2A^d_3A^d_4 \supset \overline{\mathbf{E}_1}$, $d > 0$, be the open rectangle such that $|A^d_1A^d_2| = d + |A_1A_2|$, $|A^d_1A^d_4| = d + |A_1A_4|$ and $\overline{\mathbf{E}^{d_2}_1} \subset \mathbf{E}^{d_1}_1$ for any $0 < d_2 < d_1$, Figure 2 b).

Let $\mathbf{E}^d_2 := B^d_1B^d_2 \ldots B^d_6 \supset \overline{\mathbf{E}_2}$ and $\overline{\mathbf{E}^{d_2}_2} \subset \mathbf{E}^{d_1}_2$ for any $0 < d_2 < d_1$. Let $O^d \in B^d_1B^d_6 \cap Oy$ and B^d_0 be a point of the open interval $B^d_1O^d$. Let us consider the points $B^d_t \in B^d_0B^d_1$, $t \in [0,1]$, defined as

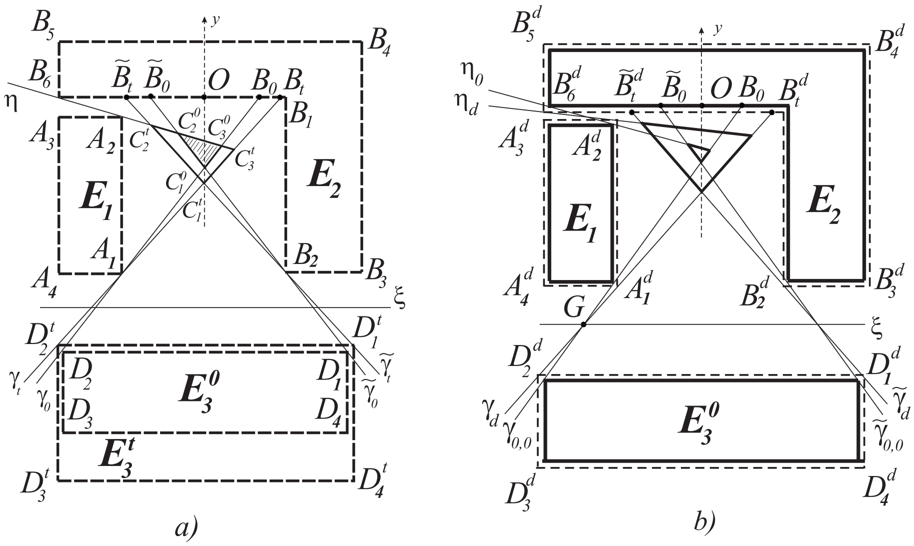

Figure 2.

$B_t^d := tB_1^d + (1-t)B_0^d$. Let $\gamma_{d,t}$, $t \in [0,1]$, be the ray starting at B_t^d and passing through the point A_1^d.

Let d_0 be such that $\mathbf{E}_1^{d_0} \cap \mathbf{E}_2^{d_0} = \varnothing$ and $G = \gamma_{d_0,1} \cap \gamma_{0,0} \neq \varnothing$. For a fixed $d \leq d_0$ among all rays $\gamma_{d,t}$, $t \in [0,1]$, we choose the one $\gamma_d := \gamma_{d,t(d)}$ passing through the point G.

Let $\widetilde{\gamma}_d$ be the ray symmetric to γ_d with respect to the axis Oy, $0 \leq d \leq d_0$.

Let a straight ξ from the previous example be passing through the point G. Construct open rectangles E_3^d, $0 \leq d \leq d_0$, laying under the line ξ, and such that $D_1^d \in \widetilde{\gamma}_d$, $D_2^d \in \gamma_d$ and $\overline{\mathbf{E}_3^{d_2}} \subset \mathbf{E}_3^{d_1}$ for any $d_2 < d_1$.

Consider the family of the open weakly 1-semiconvex sets $\mathbf{E}^{\frac{d_0}{k}} := \bigcup\limits_{j=1}^{3} \mathbf{E}_j^{\frac{d_0}{k}}$, $k = 1, 2, \ldots$. By the constructions, $\overline{\mathbf{E}^{d_2}} \subset \mathbf{E}^{d_1}$ for any $0 < d_2 < d_1 \leq d_0$. Moreover, $\bigcap\limits_k \mathbf{E}^{\frac{d_0}{k}} = \overline{\mathbf{E}^0}$. Thus, the closed set $\overline{\mathbf{E}^0}$ belongs to the class $\mathbf{WS_1^2} \setminus \mathbf{S_1^2}$.

Example 2.3. The following systems of open balls are examples of sets of the class $\mathbf{WS_1^2} \setminus \mathbf{S_1^2}$ with smooth boundary.

Let $B(o_1, r)$, $B(o_2, r)$ be open balls with centers o_1, o_2 placed symmetric with respect to the origin O on the axis Ox (see Figure 3 a)). Let γ be the common tangent line to the balls passing through the origin. Suppose

$x = x(t)$, $y = y(t)$ is one-to-one continuous mapping of the interval $[0, 1]$ onto the arc $\overset{\smile}{P_0P_1} \subset \partial B(o_1, r)$ such that $(x(0), y(0)) \equiv P_0 = \gamma \cap \partial B(o_1, r)$, $(x(1), y(1)) \equiv P_1 = Oo_1 \cap \partial B(o_1, r)$. Let γ_t, $t \in [0, 1]$, be the ray starting at the point $(x(t), y(t)) \in \overset{\smile}{P_0P_1}$ and tangent to the ball $B(o_2, r)$ from the inside, i.e intersecting Ox. Let ξ be a line parallel to the axis Ox and not intersecting the balls but intersecting the rays γ_t. To fix the point $O_1 \in Oy$, let us draw the ball $B(O_1, R_1)$ tangent to the ray γ_1 and the line ξ. Now we consider the balls $B(O_1, R_t)$, $t \in [0, 1]$, with centers at the point O_1 and tangent to the rays γ_t. It is clear that $R_{t^1} < R_{t^2}$ for any $t^1, t^2 \in [0, 1]$ such that $t^1 < t^2$. And let $B(O_2, R_t)$, $t \in [0, 1]$, be the balls symmetric to the corresponding balls $B(O_1, R_t)$ with respect to the origin. Then each system of four open balls $B_t := \{B(o_i, r), B(O_i, R_t), i = 1, 2\}$, $t \in (0, 1]$, is a weakly 1-semiconvex and not 1-semiconvex set. Indeed, by the constructions, for any boundary point of B_t there exists a ray starting at this point and not intersecting the set, whereas any ray starting at a point of the open rhombus $A_tD_tC_tF_t$ generated by the intersection of the ray γ_t, $t \in (0, 1]$, the ray symmetric to it with respect to the axis Ox, and the rays symmetric to them with respect to the axis Oy intersects the set.

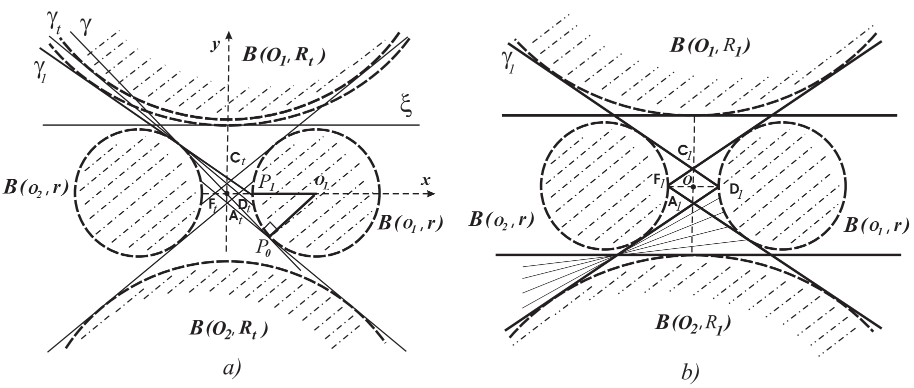

Figure 3.

Let us construct an example of a closed set of the class $\mathbf{WS_1^2} \setminus \mathbf{S_1^2}$ with smooth boundary.

Example 2.4. Let $B(o_1, r)$, $B(o_2, r)$ be the open balls from the previous example and let $|o_1o_2| = 2d$. Let Δr_1 be a number such that $0 < \Delta r_1 < \min\left\{1, \dfrac{r(d-r)}{d+r}\right\}$ and let us draw a system of concentric balls $\{B(o_i, r + \Delta r),\ 0 < \Delta r \leq \Delta r_1,\ i = 1, 2\}$. Let us fix Δr and for every ball

$B(o_i, r+\Delta r)$ construct the rays $\gamma_{t,\Delta r}$ as previews. Since $\Delta r_1 < \dfrac{r(d-r)}{d+r}$, the rays $\gamma_{1,\Delta r_1}$ and $\gamma_{0,0}$ are intersected at a point A_1. Let us fix Δr and among all rays $\gamma_{t,\Delta r}$, $t \in [0,1]$, chose the one $\gamma_{t(\Delta r),\Delta r}$ that is passing through the point A_1. Let us fix the point $O_1 \in Oy$ by constructing the ball $B(O_1, R+\Delta R(\Delta r_1))$ tangent to the ray $\gamma_{1,\Delta r_1}$ and the line ξ passing through the point A_1 (see Figure 4).

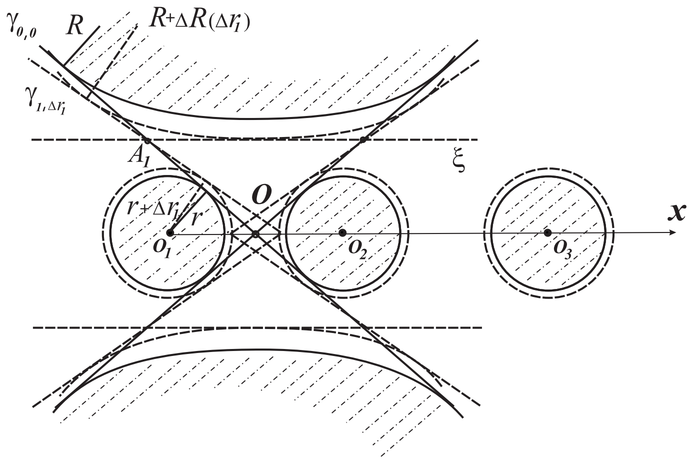

Figure 4.

Now we can construct the system of concentric circles $\{B(O_1, R+\Delta R(\Delta r))\}$ that are tangent to the respective rays $\gamma_{t(\Delta r),\Delta r}$ and the system $\{B(O_2, R+\Delta R(\Delta r))\}$ symmetric to the first one with respect to the origin. It is easy to see that $\Delta R(\Delta r^1) < \Delta R(\Delta r^2)$ for any $\Delta r^1, \Delta r^2 \in (0, \Delta r_1]$ such that $\Delta r^1 < \Delta r^2$.

By the construction, every set

$$B_{\Delta r} := \{B(o_j, r+\Delta r), B(O_j, R+\Delta R(\Delta r)),\ j = 1,2\}, \qquad 0 < \Delta r \le \Delta r_1, \quad (2.1)$$

is weakly 1-semiconvex. Moreover, the set $(B_{\Delta r})^{\diamondsuit}$ of 1-nonsemiconvexity points of the set $B_{\Delta r}$ is the open rhombus $A_{\Delta r}D_{\Delta r}C_{\Delta r}F_{\Delta r}$, $0 < \Delta r \le \Delta r_1$, generated by the intersection of the ray $\gamma_{t(\Delta r),\Delta r}$, the ray symmetric to it with respect to the axis Ox, and the rays symmetric to them with respect to the axis Oy. Thus, each open set $B_{\Delta r}$, $0 < \Delta r \le \Delta r_1$, belongs to the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$. Also note here that, by the construction, $(B_{\Delta r^1})^{\diamondsuit} \subset (B_{\Delta r^2})^{\diamondsuit}$ for any $0 < \Delta r^1 < \Delta r^2 \le \Delta r_1$.

In addition, every set $\overline{B_{\Delta r}}$, $0 < \Delta r < \Delta r_1$, belongs to the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$. Indeed, the sets $\overline{B_{\Delta r}}$ are not 1-semiconvex by Lemma 2.1.

Moreover, $\overline{B_{\Delta r}}$ is approximated from the outside by the family of sets

$$B_{\Delta r,k} := \left\{ B\left(o_j, r+\Delta r+\frac{\Delta r_1-\Delta r}{k}\right), \right.$$
$$\left. B\left(O_j, R+\Delta R\left(\Delta r+\frac{\Delta r_1-\Delta r}{k}\right)\right), j=1,2 \right\},$$
$$0<\Delta r<\Delta r_1, \quad k=1,2,\ldots. \tag{2.2}$$

Now place the center o_3 of the concentric open disks with radii $r+\Delta r$, $0<\Delta r\le \Delta r_1$, at the point $(R+\Delta R(\Delta r_1)+r+\Delta r_1, 0)$. Then the union of five closed disks

$$\left\{\overline{B(o_i, r+\Delta r)}, \overline{B(O_j, R+\Delta R(\Delta r))}, i=1,2,3, j=1,2\right\}, 0<\Delta r<\Delta r_1,$$

belongs to the class $\mathbf{WS^2_1}\setminus\mathbf{S^2_1}$. Adding more closed, nonoverlapping disks of the radius $r+\Delta r$ and centers on the axis Ox in the positive direction, we will get a closed set of the class $\mathbf{WS^2_1}\setminus\mathbf{S^2_1}$ with smooth boundary consisting of any finite or even countable number of components.

3. Topological properties of closed sets of the class $\mathbf{WS^2_1}\setminus\mathbf{S^2_1}$

Before presenting the main results of this chapter, first, let us provide two lemmas and one theorem, given that their statements are extended to all $n\ge 2$, $1\le m<n$.

Lemma 3.1. *Let a closed, weakly m-semiconvex set $E\subset\mathbb{R}^n$, $n\ge 2$, $1\le m<n$, with the number of components N be given. Then E is approximated from the outside by a family of open, weakly m-semiconvex sets E^k, $k=1,2,\ldots$, such that the number of components of each set E^k is not greater than N.*

Proof. Since E is weakly m-semiconvex, there exists a family of open weakly m-semiconvex sets G^k, $k=1,2,\ldots$, approximating E from the outside. Let every set E^k, $k=1,2,\ldots$, consist of only the components of G^k containing points of E. Consider a point $y_k\in\partial E^k$. Then $y_k\in\partial G^k$. Since G^k is open and weakly m-semiconvex, there exists an m-dimensional half-plane L_{y_k} passing through y_k and such that $L_{y_k}\cap G^k\ne\varnothing$. Since $G^k\supset E^k$, then $L_{y_k}\cap E^k\ne\varnothing$. Thus, any set E^k, $k=1,2,\ldots$, is open, weakly m-semiconvex, and consists of components the number of which is not greater than N.

Since $G^k\supset\overline{G^{k+1}}\supset\overline{E^{k+1}}$ and $\overline{E^{k+1}}$ is contained only in those components of G^k which contain points of E, then $E^k\supset\overline{E^{k+1}}$. Let us prove that $E=\cap_k E_k$.

Suppose $x \in \cap_k E_k$, then $x \in E_k$ for any $k = 1, 2, \ldots$. Since $E_k \subset G_k$, then $x \in G_k$ for any $k = 1, 2, \ldots$. Therefore, $x \in \cap_k G_k = E$. Now let $x \in E$. Since $G_k \supset E$, $k = 1, 2, \ldots$, the point x belongs to some component G_k^0 of G_k for any $k = 1, 2, \ldots$. Then $x \in G_k^0 \subset E_k$, $k = 1, 2, \ldots$, which gives $x \in \cap_k E_k$.

Thus, E is approximated from the outside by the family of open sets E_k, $k = 1, 2, \ldots$, by Definition 1.7. □

Lemma 3.2. *Let a closed set $E \subset \mathbb{R}^n$, $n \geq 2$, belong to the class $\mathbf{WS^n_m} \setminus \mathbf{S^n_m}$, $1 \leq m < n$. Then for any family of open, weakly m-semiconvex sets E^k, $k = 1, 2, \ldots$, approximating the set E from the outside, there exists an index $k_0 \in \mathbb{N}$ such that every set E^k, $k = k_0, k_0 + 1, \ldots$, of the family is not m-semiconvex.*

Proof. Since E is not m-semiconvex, it has a point of m-nonsemiconvexity $x \in \mathbb{R}^n \setminus E$. For a family of sets E^k, $k = 1, 2, \ldots$, there exists $k_0 \in \mathbb{N}$ such that every set E^k, $k = k_0, k_0 + 1, \ldots$, does not contain the point x. Any m-dimensional half-plane with x on its boundary intersects $E \subset E^k$, $k = k_0, k_0 + 1, \ldots$, and therefore intersects E^k. Thus, $x \in \mathbb{R}^n \setminus E^k$, $k = k_0, k_0 + 1, \ldots$, is a point of m-nonsemiconvexity of E^k, $k = k_0, k_0 + 1, \ldots$, which means that each set E^k is not m-semiconvex. □

Theorem 3.1. *Let a closed set $E \subset \mathbb{R}^n$, $n \geq 2$, of the class $\mathbf{WS^n_m} \setminus \mathbf{S^n_m}$, $1 \leq m < n$, with the number of components N be given. Then E is approximated from the outside by a family of open sets E^k, $k = 1, 2, \ldots$, of the class $\mathbf{WS^n_m} \setminus \mathbf{S^n_m}$ such that the number of components of each set E^k is not greater than N.*

Proof. By Lemma 3.1, E is approximated from the outside by a family of open, weakly m-semiconvex sets G^k, $k = 1, 2, \ldots$, such that the number of components of each set G^k is not greater than N. By Lemma 3.2, there exists an index k_0 such that every set G^k, $k = k_0, k_0 + 1, \ldots$, belongs to the class $\mathbf{WS^n_m} \setminus \mathbf{S^n_m}$. Thus, E is approximated from the outside by the family of sets

$$E^k := G^{k_0 + (k-1)}, \quad k = 1, 2, \ldots,$$

satisfying the lemma conditions. □

Theorem 3.2. *Let a closed set $E \subset \mathbb{R}^2$ belong to the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$. Then E consists of not less than three components.*

Proof. Suppose E is connected. Then, by Theorem 3.1, it can be approximated from the outside by a family of domains E^k, $k = 1, 2, \ldots$, of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$. But this contradicts Lemma 1.1. Thus, E is disconnected.

Suppose E consists of two components. By Theorem 3.1, it can be approximated from the outside by a family of open sets E^k, $k = 1, 2, \ldots$, of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ consisting of one or two components. This contradicts Lemma 1.2. Thus, E consists of more than two components. Examples 2.2 and 2.4 complete the proof. □

Definition 3.1. *The set of all points of the rays starting at a point $x \in \mathbb{R}^n \setminus A$ and passing through a set $A \subset \mathbb{R}^n$ is called the **cone of the set** A **with respect to the point** x and is denoted by $\mathrm{S}_x A$. We suppose that $x \notin \mathrm{S}_x A$ whenever A is open and $x \in \mathrm{S}_x A$ otherwise.*

It is not difficult to prove that $\mathrm{S}_x A$ is open whenever A is open.

Theorem 3.3. *Let $E \subset \mathbb{R}^2$ be a closed set with a finite number of components and such that $\operatorname{Int} E \neq \varnothing$. If E is weakly 1-semiconvex, then $\operatorname{Int} E$ is weakly 1-semiconvex.*

Proof. Suppose $\operatorname{Int} E$ is not weakly 1-semiconvex. Then there exists a 1-nonsemiconvexity point $y \in \partial E$ of the set $\operatorname{Int} E$.

Without loss of generality, suppose E_i, $i = 1, \ldots, k$, are the components of $\operatorname{Int} E$ such that their cones $\mathrm{S}_y E_i$, $i = 1, \ldots, p$, $p \leq k$, are the angles of values less than 2π and the other $\mathrm{S}_y E_i$, $i = p+1, \ldots, k$, are the angles of the value 2π. Let $\mathrm{S}_{i,j} = \mathrm{S}_y E_i \cap \mathrm{S}_y E_j$, $i, j = 1, \ldots, k$ (see Figure 5 a)). Since y is a 1-nonsemiconvexity point of $\operatorname{Int} E$, then for any fixed index $i \in \{1, \ldots, k\}$ there exist the indices $j(i) \in \{1, \ldots, k\}$ such that $\mathrm{S}_{i,j(i)} \neq \varnothing$. Since the cones $\mathrm{S}_y E_i$, $i = 1, \ldots, p$, are open, we can reduce them so that the intersections of the reduced cones remain non empty. Denote the reduced cones by $\widetilde{\mathrm{S}}_y E_i$, $i = 1, \ldots, p$. Then $\overline{\widetilde{\mathrm{S}}_y E_i} \subset \mathrm{S}_y E_i$. So, the boundary of $\widetilde{\mathrm{S}}_y E_i$ consists of two rays starting at y. Denote them by $\gamma_i^1(y)$, $\gamma_i^2(y)$. Moreover, $\gamma_i^1(y), \gamma_i^2(y) \subset \mathrm{S}_y E_i$. Thus, $\gamma_i^1(y) \cap E_i \neq \varnothing$, $\gamma_i^2(y) \cap E_i \neq \varnothing$ by Definition 3.1.

Let

$$x_i^1 \in \gamma_i^1(y) \cap E_i, \quad x_i^2 \in \gamma_i^2(y) \cap E_i, \; i = 1, \ldots, p.$$

Construct curves $\lambda_i \subset E_i$, $i = \overline{1, p}$, connecting the points x_i^1, x_i^2. Let also x_i^1, x_i^2, $i = p+1, \ldots, k$, be points of E_i, $i = p+1, \ldots, k$, such that a curve $\lambda_i \subset E_i$ connecting the points has the cone $\mathrm{S}_y \lambda_i = \mathrm{S}_y E_i$. Then for any ray $\gamma(y)$ starting at the point y, there exists $i \in \{1, \ldots, k\}$ such that $\gamma(y) \cap \lambda_i \neq \varnothing$.

Consider the function

$$d_j(x) = \inf_{x^0 \in \partial E_j} |x - x^0|, \quad x \in E_j, \quad j = \overline{1, k}.$$

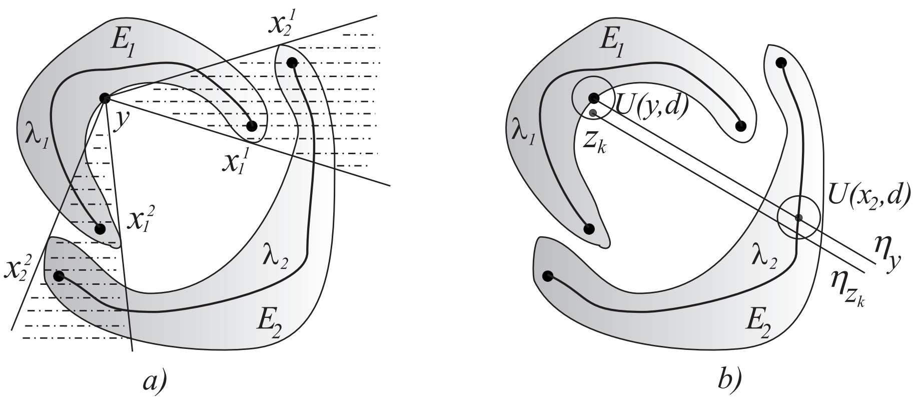

Figure 5.

It is continuous in the domain E_j, $j = \overline{1,k}$. Then its restriction on the compact λ_j, $j = \overline{1,k}$, reaches its minimum $d_j > 0$ on this compact, i. e.,

$$d_j = \min_{x \in \lambda_j} d_j(x), \quad j = \overline{1,k}.$$

Since E has the finite number of components, there exists

$$d = \min_{j=\overline{1,k}} d_j > 0.$$

Then for any point $x \in \lambda_j$, $j = \overline{1,k}$, its neighborhood $U(x,d) \subset E$. Consider the neighborhood $U(y,d)$ of the point y (see Figure 5 b)). Since E is weakly 1-semiconvex, there exists a family of open, weakly 1-semiconvex sets G_k, $k = 1, 2, \ldots$, approximating E from the outside. This gives that starting from some index k_0, $\partial G_k \cap U(y,d) \neq \varnothing$, $k \geq k_0$. Let $z_k \in \partial G_k \cap U(y,d)$, $k = k_0, k_0+1, \ldots$. Draw an arbitrary ray η_{z_k} starting at z_k. The ray η_y parallel to η_{z_k} and starting at y intersects some curve λ_q, $q \in \{1, \ldots, k\}$, at a point x_q. Since $U(x_q,d) \subset E$ and $\eta_{z_k} \cap U(x_q,d) \neq \varnothing$, then $\eta_{z_k} \cap E \neq \varnothing$. Since $G_k \supset E$, $k = 1, 2, \ldots$, then $\eta_{z_k} \cap G_k \neq \varnothing$, $k \geq k_0$.

Since we choose the ray η_{z_k} arbitrarily, the point $z_k \in \partial G_k$ is a 1-nonsemiconvexity point of G_k, $k = k_0, k_0+1, \ldots$, and we have now reached a contradiction. Thus, the assumption is wrong, and the theorem is proved. □

The converse statement is not always true. An example of a closed, not weakly 1-semiconvex set such that its interior is weakly 1-semiconvex is as follows. Consider an open, connected, weakly 1-semiconvex set

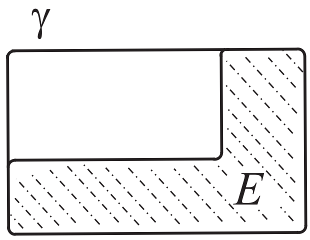

Figure 6.

$E \subset \mathbb{R}^2$ such that $\operatorname{Int} \overline{E} = E$ and connect any two of its boundary points by a curve $\gamma \subset \mathbb{R}^2 \setminus \overline{E}$ (see Figure 6). Then the closed set $\gamma \cup \overline{E}$ is not weakly 1-semiconvex and its interior is weakly 1-semiconvex, since $\operatorname{Int}(\gamma \cup \overline{E}) = E$.

Theorem 3.4. *Let a closed, bounded set $E \subset \mathbb{R}^2$ belong to the class $\mathbf{WS_1^2} \setminus \mathbf{S_1^2}$, have smooth boundary, $\operatorname{Int} E \neq \varnothing$, and $\operatorname{Int} E$ is not 1-semiconvex. Then E consists of not less than four components.*

Proof. Suppose E consists of less than four components. Then $\operatorname{Int} E$ belongs to the class $\mathbf{WS_1^2} \setminus \mathbf{S_1^2}$, has smooth boundary, and consists of less than four components by Theorem 3.3. But this contradicts Lemma 1.3. Example 2.4 completes the proof. □

The condition of Theorem 3.4 that $\operatorname{Int} E$ has 1-nonsemiconvexity points is not unnecessary, since there exists a closed set of the class $\mathbf{WS_1^2} \setminus \mathbf{S_1^2}$ with smooth boundary such that its interior is 1-semiconvex, which the following example shows.

Consider the set B^0 that is the union of four discs from Example 2.4:

$$B^0 = \{B(o_j, r), B(O_j, R),\ j = 1, 2\}.$$

Its closure $\overline{B^0}$ is approximated from the outside by the family of open, weakly 1-semiconvex sets

$$\{B(o_j, r + \Delta r_1/k), B(O_j, R + \Delta R(\Delta r_1/k)),\ j = 1, 2\},\ k = 1, 2, \dots.$$

Moreover, any ray starting at the origin $O \in \mathbb{R}^2 \setminus \overline{B^0}$ intersects $\overline{B^0}$. Thus, $\overline{B^0}$ belongs to the class $\mathbf{WS_1^2} \setminus \mathbf{S_1^2}$. In addition, $\operatorname{Int} \overline{B^0} = B^0$ is 1-semiconvex (see Figure 4).

4. Connected sets of the class $\mathbf{WS^n_m} \setminus \mathbf{S^n_m}$, $n \geq 3$, $1 \leq m < n-1$

It turns out that for the sets of the class $\mathbf{WS^n_m} \setminus \mathbf{S^n_m}$, $n \geq 3$, $1 \leq m < n-1$, the estimate of the number of components is not the same as for the sets of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$. To show this, first, prove the following

Theorem 4.1. *Let $E^p \subset \mathbb{R}^p$, $p \geq 2$, be an open or a closed set of the class $\mathbf{WS^p_1} \setminus \mathbf{S^p_1}$. Then the set $E := E^p \times \mathbb{R}^{n-p} \subset \mathbb{R}^n$, $n \geq 3$, belongs to the class $\mathbf{WS^n_{n-p+1}} \setminus \mathbf{S^n_{n-p+1}}$.*

Proof. First, consider the case when the set E^p and, therefore, the set E are open. Prove that E is weakly $(n-p+1)$-semiconvex. For any point $x = (x_1, \dots, x_p, x_{p+1} \dots, x_n) \in \partial E$, it is true that $x^p = (x_1, \dots, x_p) \in \partial E^p$. By the theorem conditions, there exists a closed ray $\gamma_{x^p} \subset \mathbb{R}^p$, starting at the point x^p and such that $\gamma_{x^p} \cap E^p = \varnothing$. Then the closed, $(n-p+1)$-dimensional half-plane $\gamma_{x^p} \times \mathbb{R}^{n-p}$ passes through the point x and does not intersect E.

Now suppose the set E^p is closed. Then, by the theorem conditions, it is approximated from the outside by a family of open weakly 1-semiconvex sets $E^p_k \subset \mathbb{R}^p$, $p \geq 2$, $k = 1, 2, \dots$. The set E is also closed and is approximated from the outside by the family of open sets $E^p_k \times \mathbb{R}^{n-p} \subset \mathbb{R}^n$, $n \geq 3$, $k = 1, 2, \dots$, which are weakly $(n-p+1)$-semiconvex, as it was proved above. Thus, E is weakly $(n-p+1)$-semiconvex.

Prove that the open or closed set E is not $(n-p+1)$-semiconvex. Consider the point $y = (y_1, \dots, y_p, y_{p+1} \dots, y_n) \in \mathbb{R}^n \setminus E$, where $(y_1, \dots, y_p)$ is a point of 1-nonsemiconvexity of the set E^p. Draw the p-dimensional plane $L^p(y)$ passing through the point y and parallel to the p-dimensional plane containing the set E^p. The set $E^p(y) := L^p(y) \cap E$ obviously is not 1-semiconvex with respect to its affine hull. Then any ray starting at y and laying in the p-dimensional plane $L^p(y)$ intersects E.

Let $H^{n-p+1}(y)$ be an arbitrary $(n-p+1)$-dimensional half-plane with the point y on its boundary that is an $(n-p)$-dimensional plane $L^{n-p}(y)$ and let $L^{n-p+1}(y)$ be the $(n-p+1)$-dimensional plane generated by $H^{n-p+1}(y)$ and its complementary $(n-p+1)$-dimensional half-plane. The intersection $L^{n-p+1}(y) \cap L^p(y)$ is an l-dimensional plane, $l \geq 1$, contained in $L^p(y)$, and $L^{n-p}(y) \cap L^p(y)$ is a k-dimensional plane, $k \geq 0$, also contained in $L^p(y)$. Then $H^{n-p+1}(y) \cap L^p(y)$ contains at least one ray starting at y and intersecting $E^p(y)$, which gives $H^{n-p+1}(y) \cap E \neq \varnothing$. Thus, y is an $(n-p+1)$-nonsemiconvexity point of E. The theorem is proved. □

Theorem 4.2. *There exist domains and closed connected sets in the space* $\mathbb{R}^n$*,* $n \geq 3$*, of the class* $\mathbf{WS^n_m} \setminus \mathbf{S^n_m}$*,* $1 \leq m < n-1$*.*

Proof. Prove the theorem by constructing examples of appropriate sets. First construct the domains in the space $\mathbb{R}^3$ of the class $\mathbf{WS^3_1} \setminus \mathbf{S^3_1}$ approximating from the outside a closed connected set.

Consider the following open sets

$$B_0 := B_{\Delta r_1/2} = \\ = \left\{ B\left(o_j, r + \frac{\Delta r_1}{2}\right), B\left(O_j, R + \Delta R\left(\frac{\Delta r_1}{2}\right)\right), \, j = 1,2 \right\},$$

$$B_k := B_{\Delta r_1/2,\,k} = \left\{ B\left(o_j, r + \frac{\Delta r_1}{2} + \frac{\Delta r_1}{2k}\right), \right. \\ \left. B\left(O_j, R + \Delta R\left(\frac{\Delta r_1}{2} + \frac{\Delta r_1}{2k}\right)\right), j = 1,2 \right\}, \quad k = 1,2,\dots,$$

of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ constructed in Example 2.4, see (2.1), (2.2) as $\Delta r = \Delta r_1/2$. Then the closed set $\overline{B_0} \in \mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ is approximated from the outside by the family of the sets B_k, $k = 1,2,\dots$. And, as it was noticed in Example 2.4, each set $(B_k)^\diamondsuit$, $k = 0,1,2,\dots$, is an open rhombus, moreover, $(B_0)^\diamondsuit \subset (B_{k+1})^\diamondsuit \subset (B_k)^\diamondsuit \subset (B_{\Delta r_1})^\diamondsuit = A_{\Delta r_1} D_{\Delta r_1} C_{\Delta r_1} F_{\Delta r_1}$ for $k = 1,2,\dots$.

Let

$$\Delta r^0 := 0, \quad \Delta r^k := \frac{\Delta r_1}{2k}, \quad k = 1,2,\dots.$$

Consider the sets

$$\tilde{\tilde{E}}{}^3_k := B_k \times [\Delta r^k - s, s - \Delta r^k], \quad s > \Delta r_1, \quad k = 0,1,2,\dots. \tag{4.1}$$

Let $P^2_k \subset \mathbb{R}^2$ be the convex hull of the set B_k, $k = 0,1,2,\dots$. Construct the following prisms:

$$Pl^3_k := P^2_k \times \left[-\Delta r^k - 1 - s, \Delta r^k - s\right], \\ Pr^3_k := P^2_k \times \left[s - \Delta r^k, s + 1 + \Delta r^k\right], \quad k = 0,1,\dots.$$

Now consider the sets

$$\tilde{E}^3_k := \operatorname{Int}(Pl^3_k \cup \tilde{\tilde{E}}{}^3_k \cup Pr^3_k), \quad k = 0,1,\dots.$$

They are 1-semiconvex with respect to any point of $\partial\tilde{E}^3_k$ except the points of the respective rhombuses:

$$\widetilde{Rl}{}^2_k := \{(x_1,x_2,x_3) \in \partial\tilde{E}^3_k : (x_1,x_2) \in (B_k)^\diamondsuit, x_3 = \Delta r^k - s\}, \\ \widetilde{Rr}{}^2_k := \{(x_1,x_2,x_3) \in \partial\tilde{E}^3_k : (x_1,x_2) \in (B_k)^\diamondsuit, x_3 = s - \Delta r^k\}.$$

Moreover,

$$(\tilde{E}_k^3)^{\diamondsuit} = (B_k)^{\diamondsuit} \times [\Delta r^k - s, s - \Delta r^k].$$

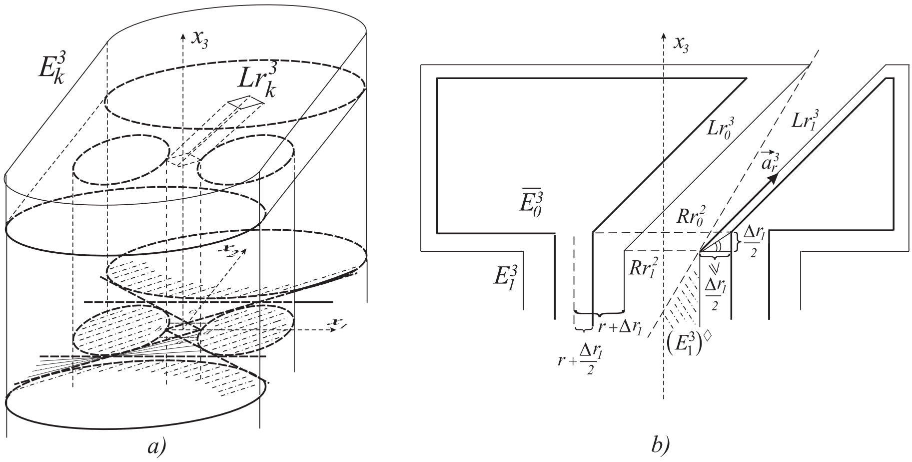

Figure 7.

Let $A'_k D'_k C'_k F'_k$ be the open rhombus the sides of which are parallel to the respective sides of the rhombus $A_{\Delta r_1} D_{\Delta r_1} C_{\Delta r_1} F_{\Delta r_1}$ and let the distance between the sides of $A'_k D'_k C'_k F'_k$ and the respective sides of $A_{\Delta r_1} D_{\Delta r_1} C_{\Delta r_1} F_{\Delta r_1}$ be equal to $\Delta r_1/2 - \Delta r^k$. Then $A_{\Delta r_1} D_{\Delta r_1} C_{\Delta r_1} F_{\Delta r_1} = A'_1 D'_1 C'_1 F'_1$ and

$$A'_0 D'_0 C'_0 F'_0 \supset A'_{k+1} D'_{k+1} C'_{k+1} F'_{k+1} \supset A'_k D'_k C'_k F'_k \supset (B_k)^{\diamondsuit} \supset (B_0)^{\diamondsuit}, \quad k = 1, 2, \dots \quad (4.2)$$

Consider the rhombuses

$$Rl_k^2 := \{(x_1, x_2, x_3) \in \partial \tilde{E}_k^3 : (x_1, x_2) \in A'_k D'_k C'_k F'_k, x_3 = \Delta r^k - s\},$$
$$Rr_k^2 := \{(x_1, x_2, x_3) \in \partial \tilde{E}_k^3 : (x_1, x_2) \in A'_k D'_k C'_k F'_k, x_3 = s - \Delta r^k\},$$

$k = 0, 1, \dots$, and some vectors $\overrightarrow{a_l^3}$, $\overrightarrow{a_r^3}$ such that the angle between $\overrightarrow{a_l^3}$ and the negative direction of the axis Ox_3 and the angle between $\overrightarrow{a_r^3}$ and the positive direction of the axis Ox_3 are greater than 0 and less than $\frac{\pi}{2}$. This provides that two oblique prisms $Ll_k^3 \subset Pl_k^3$, $Lr_k^3 \subset Pr_k^3$ with respective bases Rl_k^2, Rr_k^2 and generating rays parallel to the vectors $\overrightarrow{a_l^3}$, $\overrightarrow{a_r^3}$ are such that $Ll_0^3 \supset Ll_{k+1}^3 \supset Ll_k^3$, $Lr_0^3 \supset Lr_{k+1}^3 \supset Lr_k^3$, $k = 1, 2, \dots$ (see Figure 7 b)).

Remove the closures of the prisms Ll_k^3, Lr_k^3 from the set $\tilde{E}_k^3$, $k = 0, 1, \dots$ (see Figure 7 a)). Then, considering (4.2), the sets

$$E_k^3 := \tilde{E}_k^3 \setminus (\overline{Ll_k^3} \cup \overline{Lr_k^3}), \quad k = 0, 1, \dots,$$

are weakly 1-semiconvex domains. Moreover, choose s form (4.1) large enough so that the prisms

$$L_k^3 := A_k' D_k' C_k' F_k' \times [\Delta r^k - s, s - \Delta r^k], \quad k = 0, 1, \dots,$$

contain the points of 1-nonsemiconvexity of the respective sets E_k^3, i. e.,

$$L_k^3 \supset (\tilde{E}_k^3)^{\diamondsuit} \supset (E_k^3)^{\diamondsuit}, \quad k = 0, 1, \dots. \tag{4.3}$$

Thus, the domains $E_k^3 \subset \mathbb{R}^3$, $k = 0, 1, \dots$, belong to the class $\mathbf{WS_1^3} \setminus \mathbf{S_1^3}$. And the closure $\overline{E_0^3}$ of the set E_0^3 is approximated from the outside by the family of the domains E_k^3, $k = 1, 2, \dots$ (see Figure 7 b)). Moreover, $\overline{E_0^3}$ is not 1-semiconvex by Lemma 2.1. Thus, the closed and connected set $\overline{E_0^3}$ belongs to the class $\mathbf{WS_1^3} \setminus \mathbf{S_1^3}$.

Construct domains in the space $\mathbb{R}^4$ of the class $\mathbf{WS_1^4} \setminus \mathbf{S_1^4}$ approximating from the outside a closed connected set.

Consider the sets

$$\tilde{\tilde{E}}_k^4 := E_k^3 \times [\Delta r^k - s, s - \Delta r^k], \quad k = 0, 1, 2, \dots.$$

Let $P_k^3 \subset \mathbb{R}^3$ be the convex hull of the set E_k^3, $k = 0, 1, 2, \dots$. Construct the following prisms:

$$Pl_k^4 := P_k^3 \times \left[-\Delta r^k - 1 - s, \Delta r^k - s\right],$$
$$Pr_k^4 := P_k^3 \times \left[s - \Delta r^k, s + 1 + \Delta r^k\right], \quad k = 0, 1, \dots.$$

Now consider the sets

$$\tilde{E}_k^4 := \operatorname{Int}(Pl_k^4 \cup \tilde{\tilde{E}}_k^4 \cup Pr_k^4), \quad k = 0, 1, \dots.$$

They are 1-semiconvex with respect to any point of $\partial \tilde{E}_k^4$ except the points of the sets

$$\widetilde{Rl}_k^3 := \{(x_1, x_2, x_3, x_4) \in \partial \tilde{E}_k^4 : (x_1, x_2, x_3) \in (E_k^3)^{\diamondsuit}, x_4 = \Delta r^k - s\},$$
$$\widetilde{Rr}_k^3 := \{(x_1, x_2, x_3, x_4) \in \partial \tilde{E}_k^4 : (x_1, x_2, x_3) \in (E_k^3)^{\diamondsuit}, x_4 = s - \Delta r^k\}.$$

Moreover,

$$(\tilde{E}_k^4)^{\diamondsuit} = (E_k^3)^{\diamondsuit} \times [\Delta r^k - s, s - \Delta r^k], \quad k = 0, 1, \dots.$$

Now consider the following sets:

$$Rl_k^3 := \{(x_1, x_2, x_3, x_4) \in \partial\tilde{E}_k^4 : (x_1, x_2, x_3) \in L_k^3, x_4 = \Delta r^k - s\},$$
$$Rr_k^3 := \{(x_1, x_2, x_3, x_4) \in \partial\tilde{E}_k^4 : (x_1, x_2, x_3) \in L_k^3, x_4 = s - \Delta r^k\},$$

$k = 0, 1, \ldots$. Since $L_{k+1}^3 \supset L_k^3$, then $Rl_{k+1}^3 \supset Rl_k^3$ and $Rr_{k+1}^3 \supset Rr_k^3$. Moreover, considering (4.3),

$$Rl_k^3 \supset \widetilde{Rl}_k^3, \quad Rr_k^3 \supset \widetilde{Rr}_k^3. \tag{4.4}$$

Consider some vectors $\overrightarrow{a_l^4}$, $\overrightarrow{a_r^4}$ such that the angle between $\overrightarrow{a_l^4}$ and the negative direction of the axis Ox_4 and the angle between $\overrightarrow{a_r^4}$ and the positive direction of the axis Ox_4 are greater than 0 and less than $\frac{\pi}{2}$. Remove the closures of two oblique prisms $Ll_k^4 \subset Pl_k^4$, $Lr_k^4 \subset Pr_k^4$ with respective bases Rl_k^3, Rr_k^3 and generating rays parallel to the vectors $\overrightarrow{a_l^4}$, $\overrightarrow{a_r^4}$ from the set $\tilde{E}_k^4$, $k = 0, 1, \ldots$. Then, considering (4.4), the sets

$$E_k^4 := \tilde{E}_k^4 \setminus (\overline{Ll_k^4} \cup \overline{Lr_k^4}), \quad k = 0, 1, \ldots,$$

are weakly 1-semiconvex domains. Moreover, choose s form (4.1) large enough so that the prisms

$$L_k^4 := L_k^3 \times [\Delta r^k - s, s - \Delta r^k], \quad k = 0, 1, \ldots,$$

contain points of 1-nonsemiconvexity of the respective sets E_k^4, i. e., $L_k^4 \supset (\tilde{E}_k^4)^\diamondsuit \supset (E_k^4)^\diamondsuit$, $k = 0, 1, \ldots$.

Thus, the domains $E_k^4 \subset \mathbb{R}^4$, $k = 0, 1, \ldots$, belong to the class $\mathbf{WS_1^4} \setminus \mathbf{S_1^4}$. And the closure $\overline{E_0^4}$ of the set E_0^4 is approximated from the outside by the family of the domains E_k^4, $k = 1, 2, \ldots$. Moreover, $\overline{E_0^4}$ is not 1-semiconvex by Lemma 2.1. Thus, the closed and connected set $\overline{E_0^4}$ belongs to the class $\mathbf{WS_1^4} \setminus \mathbf{S_1^4}$.

Extending the process of constructing the sets E_k^n, $k = 1, 2, \ldots$, and $\overline{E_0^n}$ to the spaces $\mathbb{R}^n$, $n > 4$, using the sets E_k^{n-1}, $\overline{E_0^{n-1}}$ by the induction, we obtain domains and closed connected sets of the class $\mathbf{WS_1^n} \setminus \mathbf{S_1^n}$ for any $n \geq 3$. Then, by Theorem 4.1, the domains

$$E_k^{n-m+1} \times \mathbb{R}^{m-1} \subset \mathbb{R}^n, \quad n \geq 3,\ 1 \leq m < n-1,\ k = 1, 2, \ldots,$$

and the closed connected sets

$$\overline{E_0^{n-m+1}} \times \mathbb{R}^{m-1} \subset \mathbb{R}^n, \quad n \geq 3,\ 1 \leq m < n-1,$$

belong to the class $\mathbf{WS_m^n} \setminus \mathbf{S_m^n}$. The theorem is proved. □

Conclusion

In conclusion, we list some open problems arising in this work:

1. *Is Lemma 1.3 valid for an arbitrary unbounded, open set of the class* $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ *with smooth boundary?*

2. *Is Theorem 3.4 valid for an arbitrary closed set of the class* $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ *with smooth boundary?*

3. *Is the interior of a closed, weakly m-semiconvex set of* $\mathbb{R}^n$, $n \geq 2$, *weakly m-semiconvex for any* $m = 1, 2, \ldots, n-1$*?*

4. *What is the minimal number of the components of a set of the class* $\mathbf{WS^n_{n-1}} \setminus \mathbf{S^n_{n-1}}$, $n \geq 3$*?*

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