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On some approximative properties of Gauss-Weierstrass singular operators

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Abstract. In the paper, we formulate direct approximation theorems for continuous in the neighbourhood of some point x, $-\infty < x < \infty$, functions. Namely, the upper bounds were obtained for approximation of functions by their Gauss-Weierstrass singular operators in terms of a majorant function for the modules of continuity of the first and second orders of the respective functions.

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1. Introduction

Let us consider a boundary value problem in the unit circle (see, e.g., [3]) for the equation

$$\frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{(-1)^{l+1}}{\rho^2}\frac{\partial^{2l} U}{\partial x^{2l}} = 0 \tag{1.1}$$

(here l is a natural number, $0 \leq \rho < 1$, $-\pi \leq x \leq \pi$) in case that the function $U(\rho, x)$ is bounded in the unit circle

$$\Omega = (0 \leq \rho < 1; \quad -\pi \leq x \leq \pi)$$

and

$$U(\rho, x)\big|_{\rho=1} = \varphi(x), \tag{1.2}$$

where $\varphi(x)$ is a summable 2π-periodic function, and the equality holds in the sense of convergence in p-mean, $1 \leq p \leq \infty$.

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According to [3], a solution of the boundary value problem (1.1)–(1.2) is given by the function

$$U_{\rho,l}(\varphi;x)=\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(x+t)\left\{\frac{1}{2}+\sum_{k=1}^{\infty}\rho^{k^l}\cos kt\right\}dt, \tag{1.3}$$

where $0\le\rho<1$, $l\in\mathbb{N}$, which is usually called the Abel–Poisson-type operator [2] or the generalized Abel–Poisson operator [6, 10–13].

If $l=1$, from the relation (1.3) we get an expression for the Poisson operator $U_{\rho,1}(\varphi;x):=P_{\rho}(\varphi;x)$ (see [2, 4, 8]). In the case $l=2$, the formula (1.3) yields

$$U_{\rho,2}(\varphi;x):=W_{\rho}(\varphi;x)=\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(x+t)\left\{\frac{1}{2}+\sum_{k=1}^{\infty}\rho^{k^2}\cos kt\right\}dt, \tag{1.4}$$

which is called the Gauss–Weierstrass singular operator [2, 4].

Then, analogically to that in the paper [1], we put $\delta=(\ln\frac{1}{\rho})^{-1}$ in the formula (1.4). The Gauss–Weierstrass singular operator takes the form

$$W_{\delta}(\varphi;x)=\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(x+t)\left\{\frac{1}{2}+\sum_{k=1}^{\infty}e^{-\frac{k^2}{\delta}}\cos kt\right\}dt. \tag{1.5}$$

Using the methods of the paper [2], we can show that the equality (1.5) can be written as follows

$$W_{\delta}(\varphi;x)=\frac{1}{2}\sqrt{\frac{\delta}{\pi}}\int_{-\infty}^{\infty}\varphi(x+t)e^{-\frac{t^2}{4}\delta}dt. \tag{1.6}$$

In what follows, by $C:=C(-\infty;\infty)$ we denote the space of continuous on $(-\infty;\infty)$ functions with the finite norm

$$\|f\|_C:=\max_{x\in(-\infty;\infty)}|f(x)|,$$

and by $L_p:=L_p(-\infty;\infty)$, $1\le p\le\infty$, the spaces of, respectively, summable with pth power on $(-\infty;\infty)$ functions equipped with the norm

$$\|f\|_{L_p}:=\left(\int_{-\infty}^{\infty}|f(x)|^p dx\right)^{\frac{1}{p}},\quad 1\le p<\infty,$$

and of measurable and essentially bounded on $(-\infty;\infty)$ functions, where the norm is given by

$$\|f\|_{L_\infty}:=\operatorname*{ess\,sup}_{x\in(-\infty;\infty)}|f(x)|.$$

For continuous functions, $\|f\|_{L_\infty} \equiv \|f\|_C$.

It is known [17, § 1.18-1.20], that if $e^{-cx^2}\varphi(x) \in L_1$ for some constant $c > 0$, then in each point x_0 of continuity of the function $\varphi(x)$ there exists a limit

$$\lim_{\substack{x\to x_0\\ \delta\to\infty}} W_\delta(\varphi; x) = \varphi(x_0), \tag{1.7}$$

that does not depend on the way of tending $x \to x_0$, $\delta \to \infty$ [14, Ch. III, § 3]. The relation (1.7) can be interpreted as a convergence in the metric of the corresponding space

$$\lim_{\delta\to\infty} \|W_\delta(\varphi; x) - \varphi(x)\|_{L_p} = 0. \tag{1.8}$$

One of the questions that we deal with in this paper is the rate of deviation of the operator (1.6) as $\delta \to \infty$ from the function $\varphi(x)$, on which it is, actually, constructed. This will specify the equality (1.8).

2. Main results

In the above notation, the following theorem holds.

Theorem 2.1. *If the function $\varphi(x)$ is continuous in the neighbourhood of point x, $-\infty < x < \infty$, and the modulus of continuity of the second order $\omega_2(\varphi; t) \le \omega(t)$, where $\omega(t)$, $t > 0$, is a function of the type of the second order modulus of continuity, then at each point x, $-\infty < x < \infty$, the estimate holds*

$$|W_\delta(\varphi; x) - \varphi(x)| \le \left(\frac{3}{2} + \frac{2}{\sqrt{\pi}}\right)\omega\left(\frac{1}{\sqrt{\delta}}\right), \qquad \text{as } \delta \to \infty. \tag{2.1}$$

Proof. Performing the respective transformations in the right-hand side of (1.6), we write the Gauss–Weierstrass operator in the form

$$W_\delta(\varphi; x) = \frac{1}{2}\sqrt{\frac{\delta}{\pi}} \int_{-\infty}^{\infty} \varphi(t) e^{-\frac{(x-t)^2}{4}\delta} dt. \tag{2.2}$$

Further, we make a change of variables

$$\frac{x-t}{2}\sqrt{\delta} = z, \quad t = x - \frac{2z}{\sqrt{\delta}}, \quad dt = -\frac{2}{\sqrt{\delta}} dz, \tag{2.3}$$

in the right-hand side of (2.2), and get

$$W_\delta(\varphi; x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi\left(x - \frac{2z}{\sqrt{\delta}}\right) e^{-z^2} dz$$
$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} \varphi\left(x - \frac{2z}{\sqrt{\delta}}\right) e^{-z^2} dz + \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \varphi\left(x - \frac{2z}{\sqrt{\delta}}\right) e^{-z^2} dz$$
$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \left(\varphi\left(x + \frac{2z}{\sqrt{\delta}}\right) + \varphi\left(x - \frac{2z}{\sqrt{\delta}}\right)\right) e^{-z^2} dz. \tag{2.4}$$

It is known [5, formula 3.321(3)], that

$$\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-z^2}dz=\frac{1}{2}. \tag{2.5}$$

Hence, combining the relations (2.4) and (2.5) we derive to an integral representation of the quantity $W_\delta(\varphi;x)-\varphi(x)$, namely,

$$\begin{aligned}&W_\delta(\varphi;x)-\varphi(x)\\&=\frac{1}{\sqrt{\pi}}\int_0^\infty\left(\varphi\left(x+\frac{2z}{\sqrt{\delta}}\right)-2\varphi(x)+\varphi\left(x-\frac{2z}{\sqrt{\delta}}\right)\right)e^{-z^2}dz.\end{aligned} \tag{2.6}$$

With respect to the notation (2.3), we have

$$x=t+\frac{2z}{\sqrt{\delta}},\quad x+\frac{2z}{\sqrt{\delta}}=t+\frac{4z}{\sqrt{\delta}},\quad x-\frac{2z}{\sqrt{\delta}}=t,\quad dz=-\frac{\sqrt{\delta}}{2}dt,$$

and therefore (2.6) yields

$$\begin{aligned}&|W_\delta(\varphi;x)-\varphi(x)|\\&\le\frac{1}{2}\sqrt{\frac{\delta}{\pi}}\int_{-\infty}^{x}\left|\varphi\left(t+\frac{4z}{\sqrt{\delta}}\right)-2\varphi\left(t+\frac{2z}{\sqrt{\delta}}\right)+\varphi(t)\right|e^{-\frac{(x-t)^2}{4}\delta}dt.\end{aligned} \tag{2.7}$$

By the definition and the properties of the second order modulus of continuity (see, e.g., [16, p. 17]), it holds

$$\begin{aligned}&\left|\varphi\left(t+\frac{4z}{\sqrt{\delta}}\right)-2\varphi\left(t+\frac{2z}{\sqrt{\delta}}\right)+\varphi(t)\right|\\&\le\omega\left(\frac{2}{\sqrt{\delta}}z\right)\le(1+2|z|)^2\omega\left(\frac{1}{\sqrt{\delta}}\right).\end{aligned} \tag{2.8}$$

Applying (2.8) to the right-hand side of (2.7) and taking into account (2.3), we obtain

$$\begin{aligned}&|W_\delta(\varphi;x)-\varphi(x)|\\&\le\frac{1}{2}\sqrt{\frac{\delta}{\pi}}\int_{-\infty}^{x}(1+|x-t|\sqrt{\delta})^2\omega\left(\frac{1}{\sqrt{\delta}}\right)e^{-\frac{(x-t)^2}{4}\delta}dt\\&=\omega\left(\frac{1}{\sqrt{\delta}}\right)\cdot\left(\frac{1}{2}\sqrt{\frac{\delta}{\pi}}\int_{-\infty}^{x}e^{-\frac{(x-t)^2}{4}\delta}dt+\frac{\delta}{\sqrt{\pi}}\int_{-\infty}^{x}|x-t|e^{-\frac{(x-t)^2}{4}\delta}dt\right.\\&\left.\quad+\frac{1}{2}\frac{\delta\sqrt{\delta}}{\sqrt{\pi}}\int_{-\infty}^{x}(x-t)^2e^{-\frac{(x-t)^2}{4}\delta}dt\right).\end{aligned} \tag{2.9}$$

To calculate the first integral from the right-hand side of (2.9), we use the notation (2.3) and the formula (2.5), and hence get

$$\frac{1}{2}\sqrt{\frac{\delta}{\pi}}\int_{-\infty}^{x} e^{-\frac{(x-t)^2}{4}\delta}dt = \frac{1}{\sqrt{\pi}}\int_{0}^{\infty} e^{-z^2}dz = \frac{1}{2}. \tag{2.10}$$

Similarly, for the second integral from the right-hand side of (2.9), we obtain

$$\begin{aligned}\frac{\delta}{\sqrt{\pi}}\int_{-\infty}^{x}|x-t|e^{-\frac{(x-t)^2}{4}\delta}dt &= \frac{\delta}{\sqrt{\pi}}\cdot\frac{2}{\sqrt{\delta}}\int_{0}^{\infty}\frac{2z}{\sqrt{\delta}}e^{-z^2}dz \\ &= \frac{4}{\sqrt{\pi}}\int_{0}^{\infty} ze^{-z^2}dz = \frac{2}{\sqrt{\pi}}\int_{-\infty}^{0} e^{\tau}d\tau = \frac{2}{\sqrt{\pi}}. \end{aligned} \tag{2.11}$$

Finally, we move to the third integral from the right-hand side of (2.9), where after integrating by parts and taking into account the relation (2.5) get

$$\begin{aligned}&\frac{1}{2}\frac{\delta\sqrt{\delta}}{\sqrt{\pi}}\int_{-\infty}^{x}(x-t)^2 e^{-\frac{(x-t)^2}{4}\delta}dt \\ &= \sqrt{\frac{\delta}{\pi}}\left((x-t)e^{-\frac{(x-t)^2}{4}\delta}\Big|_{t=-\infty}^{t=x} + \int_{-\infty}^{x} e^{-\frac{(x-t)^2}{4}\delta}dt\right) \\ &= \sqrt{\frac{\delta}{\pi}}\left(0 - \lim_{t\to-\infty}(x-t)e^{-\frac{(x-t)^2}{4}\delta} + \sqrt{\frac{\pi}{\delta}}\right) = 1. \end{aligned} \tag{2.12}$$

Putting (2.10), (2.11) and (2.12) in the corresponding parts of the right-hand side of (2.9), we prove the formula (2.1). This yields the statement of Theorem 2.1. □

Analogical to Theorem 2.1 result holds also in terms of a majorant function for the modulus of continuity of the first order.

Theorem 2.2. *If the function $\varphi(x)$ is continuous in the neighbourhood of point x, $-\infty < x < \infty$, and the modulus of continuity $\omega_1(\varphi; t) \le \tilde{\omega}(t)$, where $\tilde{\omega}(t)$, $t > 0$, is a function of the type of modulus of continuity, then at each point x, $-\infty < x < \infty$, the estimate holds*

$$|W_\delta(\varphi; x) - \varphi(x)| \le \left(1 + \frac{2}{\sqrt{\pi}}\right)\tilde{\omega}\left(\frac{1}{\sqrt{\delta}}\right), \qquad \text{as } \delta \to \infty. \tag{2.13}$$

Proof. We will speculate analogically to that in proving Theorem 2.1, and hence from (2.2), (2.3) and (2.5) get

$$\begin{aligned} &W_\delta(\varphi;x)-\varphi(x)\\ &=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\left(\varphi\left(x-\frac{2z}{\sqrt{\delta}}\right)-\varphi(x)\right)e^{-z^2}dz\\ &=\frac{1}{2}\sqrt{\frac{\delta}{\pi}}\int_{-\infty}^{\infty}\left(\varphi(t)-\varphi\left(t+\frac{2z}{\sqrt{\delta}}\right)\right)e^{-\frac{(x-t)^2}{4}\delta}dt. \end{aligned}\tag{2.14}$$

By the properties of the first order modulus of continuity, from (2.14) we can write

$$\begin{aligned} |W_\delta(\varphi;x)-\varphi(x)| &\le \frac{1}{2}\sqrt{\frac{\delta}{\pi}}\int_{-\infty}^{\infty}\tilde{\omega}\left(\frac{2z}{\sqrt{\delta}}\right)e^{-\frac{(x-t)^2}{4}\delta}dt\\ &\le \frac{1}{2}\sqrt{\frac{\delta}{\pi}}\tilde{\omega}\left(\frac{1}{\sqrt{\delta}}\right)\int_{-\infty}^{\infty}(1+|x-t|\sqrt{\delta})e^{-\frac{(x-t)^2}{4}\delta}dt\\ &\le \frac{1}{2}\sqrt{\frac{\delta}{\pi}}\tilde{\omega}\left(\frac{1}{\sqrt{\delta}}\right)\left(\int_{-\infty}^{\infty}e^{-\frac{(x-t)^2}{4}\delta}dt+\sqrt{\delta}\int_{-\infty}^{\infty}|x-t|e^{-\frac{(x-t)^2}{4}\delta}dt\right). \end{aligned}\tag{2.15}$$

Further, taking into account (2.3) and (2.5), the relation (2.15) yields

$$\begin{aligned} &|W_\delta(\varphi;x)-\varphi(x)|\\ &\le \frac{1}{2}\sqrt{\frac{\delta}{\pi}}\tilde{\omega}\left(\frac{1}{\sqrt{\delta}}\right)\left(\frac{2}{\sqrt{\delta}}\int_{-\infty}^{\infty}e^{-z^2}dz+2\sqrt{\delta}\int_{-\infty}^{x}(x-t)e^{-\frac{(x-t)^2}{4}\delta}dt\right)\\ &=\frac{1}{2}\sqrt{\frac{\delta}{\pi}}\tilde{\omega}\left(\frac{1}{\sqrt{\delta}}\right)\left(\frac{2}{\sqrt{\delta}}\sqrt{\pi}+2\sqrt{\delta}\cdot\frac{2}{\delta}\,e^{-\frac{(x-t)^2}{4}\delta}\Big|_{t=-\infty}^{t=x}\right)\\ &=\frac{1}{2}\sqrt{\frac{\delta}{\pi}}\tilde{\omega}\left(\frac{1}{\sqrt{\delta}}\right)\left(2\sqrt{\frac{\pi}{\delta}}+\frac{4}{\sqrt{\delta}}(1-0)\right)=\left(1+\frac{2}{\sqrt{\pi}}\right)\tilde{\omega}\left(\frac{1}{\sqrt{\delta}}\right), \end{aligned}$$

that proves Theorem 2.2. □

Remark 2.1. Similar to (2.1) and (2.13) estimates could be written for the metric of the space L_p, $p \ge 1$.

Remark 2.2. In the paper [9], the direct approximation theorems were obtained for functions $\varphi(x)$, $-\infty < x < \infty$, by the Gauss–Weierstrass singular operators in the spaces C and L_1 under sufficiently big restrictions on the class of functions to which $\varphi(x)$ belongs. Namely, the mentioned above class of functions $\varphi(x)$ was defined [9] by a certain function $\tau(u)$, $u > 0$, such that

1) $\tau(u)$, $u > 0$, is positive and increasing and $\frac{\tau(u)}{u}$ is monotonically decreasing;

2) $|\varphi(x+u) - \varphi(x)| = O(\tau(u))$;

3) $\tau(x \cdot y) = \tau(x) \cdot \tau(y)$;

4) for some fixed value u_0 it holds $\left(\frac{u_0}{\sqrt{z}}\right)^2 e^{-\left(\frac{u_0}{\sqrt{z}}\right)^2} \frac{\tau(u_0)}{\tau(\sqrt{z})} = O(1)$;

5) $\frac{d}{du}(u^2\tau(u)) = O(u\tau(u))$.

Under the conditions 1)–5) on the function $\tau(u)$, $u > 0$, for $\varphi(x) \in L_1$ such that

$$\int_0^u (\varphi(x+t) - 2\varphi(x) + \varphi(x-t))dt = O(u^2\tau(u)),$$

the following estimate was obtained in [9, Theorem 2.1]:

$$|W_\delta(\varphi; x) - \varphi(x)| = O\left(\sqrt{\frac{1}{\delta}}\tau\left(\sqrt{\frac{1}{\delta}}\right)\right), \qquad \text{as } \delta \to \infty. \tag{2.16}$$

When estimating the respective L_1-norm, authors assume [9, Theorem 2.2] that $\varphi(x) \in L_1$ satisfies the relation

$$\int_0^u \int_{-\infty}^{\infty} |\varphi(x+t) - 2\varphi(x) + \varphi(x-t)| dx dt = O(u^2\tau(u)),$$

and then for $\tau(u)$, $u > 0$, satisfying the conditions 1)–5) from above, it holds

$$\|W_\delta(\varphi; x) - \varphi(x)\|_{L_1} = O\left(\sqrt{\frac{1}{\delta}}\tau\left(\sqrt{\frac{1}{\delta}}\right)\right), \qquad \text{as } \delta \to \infty. \tag{2.17}$$

We note, that the functional equation 3), which was considered earlier by A.L. Cauchy, has a unique solution on the classes of continuous (and continuously differentiable) functions, namely $\tau(x) = x^\alpha$, where $\alpha > 0$ is some constant. Such a function $\tau(x)$ for $0 < \alpha \leq 1$ is a sufficiently partial, but very important modulus of continuity, that is used to define the classes $\mathrm{Lip}\,\alpha$ or $\mathrm{Lip}(\alpha, \rho)$, $\rho > 1$ (see [7]). For the classes of functions $\mathrm{Lip}\,\alpha$, the estimates (2.16) and (2.17) give a nice degree of approximation of the order $\delta^{-\frac{1+\alpha}{2}}$.

The following statement generalizes the result from [9] for the space L_p, $p > 1$.

Theorem 2.3. *Let $\varphi(x) \in L_p$, $p > 1$, and the function $\tau(u)$, $u > 0$, is such that the conditions 1)-5) hold and*

$$\int_0^u \Phi(t)dt = O(u^2\tau(u)),$$

where

$$\Phi(t) = \left(\int_{-\infty}^{\infty} |\varphi(x+t) - 2\varphi(x) + \varphi(x-t)|^p \, dx\right)^{\frac{1}{p}}.$$

Then the estimate

$$\|W_\delta(\varphi; x) - \varphi(x)\|_{L_p} = O\left(\sqrt{\frac{1}{\delta}}\tau\left(\sqrt{\frac{1}{\delta}}\right)\right)$$

holds as $\delta \to \infty$.

To prove Theorem 2.3, we use the methods from paper [9] and the generalized Minkowski inequality [15, p. 592], taking into account peculiarities of the integral metric.

References

[1] Abdullayev, F.G., Kharkevych, Yu.I. (2020). Approximation of the classes $C^\psi_\beta H^\alpha$ by biharmonic Poisson integrals. *Ukr. Math. J., 72* (1), 21–38.

[2] Baskakov, V.A. (1975). Some properties of operators of Abel-Poisson type. *Math. Notes of the Academy of Sciences of the USSR, 17*(2), 101–107.

[3] Bugrov, Ja.S. (1963). Inequalities of the type of Bernstein inequalities and their application to the investigation of the differential properties of solutions of differential equations of higher order. *Mathematica (Cluj), 5*(28), 7–25.

[4] Falaleev, L.P. (2001). On approximation of functions by generalized Abel-Poisson operators. *Sib. Math. J., 42*(4), 926–936.

[5] Gradshtein, I.S., Ryzhik, I.M. (1963). *Tables of integrals, sums, series, and products*. Fizmatgiz, Moscow (in Russian).

[6] Gutlyanskiĭ, V., Ryazanov, V., Yakubov, E., Yefimushkin, A. (2021). On the Hilbert boundary-value problem for Beltrami equations with singularities. *J. Math. Sci., 254*(3), 357–374.

[7] Hardy, C.H., Littlewood, J.E. (1928). A convergence criterium for Fourier series. *Math. Leitschrift, 28*, 612–634.

[8] Kal'chuk, I.V., Kharkevych, Yu.I., Pozharska, K.V. (2020). Asymptotics of approximation of functions by conjugate Poisson integrals. *Carpatian Math. Publ., 12*(1), 138–147.

[9] Khan, A., Umar, S. (1981). On the order of approximation to a function by generalized Gauss-Weierstrass singular integrals. *Commun. Fac. Sci. Univ. Ank., Series A1, 30*, 55–62.

[10] Kharkevych, Yu.I. (2017). On approximation of the quasi-smooth functions by their Poisson type integrals. *Journal of Automation and Information Sciences, 49*(10), 74–81.

[11] Kharkevych, Yu.I. (2018). Asymptotic expansions of upper bounds of deviations of functions of class W^r from their generalized Poisson integrals. *Journal of Automation and Information Sciences, 50*(8), 38–39.

[12] Ryazanov, V.I. (2019). Stieltjes integrals in the theory of harmonic functions. *J. Math. Sci., 243*(6), 922–933.

[13] Ryazanov V.I. (2019). On the theory of the boundary behavior of conjugate harmonic functions. *Complex Anal. Oper. Theory, 13*, 2899–2915.

[14] Tikhonov, A.N., Samarskii, A.A. (1963). *Equations of mathematical physics.* Pergamon Press Ltd.

[15] Timan, A.F. (1963). *Theory of approximation of functions of a real variable.* Pergamon Press, Oxford.

[16] Timan, M.F. (2009). *The approximation and the properties of periodic functions.* Naukova Dumka, Kiev (in Russian).

[17] Titchmarsh, E.C. (1948). *Introduction to the theory of Fourier integrals.* Oxford University Press.

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