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Approximate Solution of the System of Linear Volterra–Stieltjes Integral Equations of the Second Kind

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Abstract. A numerical solution of the system of linear Volterra–Stieltjes integral equations of the second kind is established and investigated using the so-called Generalized trapezoid rule. The conditions for estimating the error are also determined and justified. The solution of the example is given applying the proposed method.

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1. Introduction

The theory of integral equations is highly developed and forms a very important and applicable branch of nonlinear analysis. The survey of various types of integral equations and their applications can be found in [3, 4, 9, 11, 14, 15, 18–20], for example. Some practical and theoretical investigations were made in paper [1] for non-classical Volterra integral equations of the first kind. Also the approximate solution is obtained for the considered integral equation. In the study [10], various inverse problems including Volterra operator equations were studied. Some properties for Volterra–Stieltjes integral operators, solvability of nonlinear integral Volterra–Stieltjes equations were given in [7]. In paper [8] quadratic integral equations of Urysohn–Stieltjes type and their applications were investigated. Various numerical solution methods for integral equations were presented in the studies [6,12,13,16,17,19,21]. The notion of derivative of a function by means of a strictly increasing function was given in [2] by Asanov. In the study [5], the generalized trapezoid rule was proposed

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to evaluate the Stieltjes integral approximately by employing the notion of derivative of a function by means of a strictly increasing function.

In the Volterra and Volterra–Stieltjes integral equations, the kernels may not be differentiable by one or both arguments in the considered domain. Therefore, the known approximate methods cannot be used for such equations and their systems. In this paper, a numerical solution is found for the system of linear Volterra–Stieltjes integral equations of the second kind using the notion of derivative of a function by means of a strictly increasing function and the generalized trapezoid rule. The numerical solution of the linear Volterra–Stieltjes integral equations of the second kind is obtained in [6] using the generalized trapezoid rule.

2. Preliminaries

Consider the system of linear integral equations of the second kind

$$u(x) = \int_a^x K(x,s)u(s)d\varphi(s) + f(x),\ x \in [a,b], \tag{2.1}$$

where $K(x,s)$ is a given $m \times m$ matrix-valued continuous function on $G = [(x,s) : a \le s \le x \le b]$, $f(x)$ is a given continuous vector-function on $[a,b]$, $\varphi(x)$ is a given strictly increasing continuous function on $[a,b]$ and $u(x)$ is the sought vector-function on $[a,b]$.

We denote by $\|A\|$ and $\|u\|$ the norms of $m \times m$ matrix $A = (a_{ij})$ and m-dimensional vector $u = (u_i)$ respectively, i.e.

$$\|A\| = \max_i \sum_{j=1}^m |a_{ij}|, \quad \|u\| = \max_i |u_i|.$$

We shall define as $C_m[a,b]$, $m \in N$ the space of m-dimensional vector-functions with elements from $C[a,b]$. In the space $C_m[a,b]$ we define the norm as follows:

$$\|u(t)\|_C = \sup_{t\in[a,b]} \|u(t)\|.$$

We shall define as $C_{mm}(G)$ the space of $m \times m$-matrix-functions with elements from $C(G)$. In the space $C_{mm}(G)$ we define the norm as follows:

$$\|K(x,s)\|_C = \sup_{(x,s)\in G} \|K(x,s)\|.$$

We need the following definition which is given in [2].

Definition 2.1. *The derivative of a function $f(x)$ with respect to $\varphi(x)$ is the function $f'_\varphi(x)$, whose value at $x \in (a,b)$ is the number*

$$f'_\varphi(x) = \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x)}{\varphi(x+\Delta) - \varphi(x)}, \tag{2.2}$$

where $\varphi(x)$ is a given strictly increasing continuous function in (a,b).

If the limit in (2.2) exists, we say that $f(x)$ has a derivative (is differentiable) with respect to $\varphi(x)$. The first derivative $f'_\varphi(x)$ may also be a differentiable function with respect to $\varphi(x)$ at every point $x \in (a,b)$. Then, its derivative

$$f''_\varphi(x) = \left(f'_\varphi(x)\right)'_\varphi$$

is called the second derivative of $f(x)$ with respect to $\varphi(x)$. Consequently, the n-th derivative of $f(x)$ with respect to $\varphi(x)$ is defined by

$$f^{(n)}_\varphi(x) = \left(f^{(n-1)}_\varphi(x)\right)'_\varphi.$$

We need the following theorem from [5].

Theorem 2.1. *Let $\varphi(x)$ and $\psi(x)$ be two strictly increasing continuous functions on $[a,b]$ and $f''_\varphi(x),\ f''_\psi(x) \in C[a,b]$. Then,*

$$|I - A_n| \le \frac{M_0}{12}\left(\varphi(b) - \varphi(a)\right)\left(\omega_\varphi(h)\right)^2 + \frac{M_0'}{12}\left(\psi(b) - \psi(a)\right)\left(\omega_\psi(h)\right)^2,$$

where

$$I = \int_a^b f(x)d\varphi(x) - \int_a^b f(x)d\psi(x),$$

$$M_0 = \left\|f''_\varphi(x)\right\|_C = \sup_{x\in[a,b]} \left|f''_\varphi(x)\right|,$$

$$M_0' = \left\|f''_\psi(x)\right\|_C = \sup_{x\in[a,b]} \left|f''_\psi(x)\right|,$$

$$\omega_\varphi(h) = \sup_{|x-y|\le h} |\varphi(x) - \varphi(y)|,$$

$$\omega_\psi(h) = \sup_{|x-y|\le h} |\psi(x) - \psi(y)|,$$

$$A_n = \frac{1}{2}\sum_{i=1}^n \left[f(x_i) + f(x_{i-1})\right]\left[\varphi(x_i) - \varphi(x_{i-1})\right] - \frac{1}{2}\sum_{i=1}^n \left[f(x_i) + f(x_{i-1})\right]\left[\psi(x_i) - \psi(x_{i-1})\right]$$

and $x_i = a + ih,\ i = 0,\ 1,\ ...,\ n,\ h = (b-a)/n,\ n \in N$ (N denotes the set of natural numbers).

Corollary 2.1. *Let $\varphi(x)$ be a strictly increasing continuous function on $[a,b]$, $\psi(x)=0$ for all $x\in[a,b]$ and $f''_{\varphi}(x)\in C[a,b]$. Then,*

$$|I-A_n|\leq \frac{M_1}{12}\left(\varphi(b)-\varphi(a)\right)\left(\omega_{\varphi}(h)\right)^2,$$

where

$$I=\int_a^b f(x)d\varphi(x)$$

$$A_n=\frac{1}{2}\sum_{i=1}^{n}\left[f(x_i)+f(x_{i-1})\right]\left[\varphi(x_i)-\varphi(x_{i-1})\right]$$

$$M_1=\left\|f''_{\varphi}(x)\right\|_C=\sup_{x\in[a,b]}\left|f''_{\varphi}(x)\right|.$$

Theorem 2.2. *Let $\varphi(x)$ be a strictly increasing continuous function on $[a,b]$, $\|K(x,s)\|\in C(G)$ and $f(x)\in C_m[a,b]$. Then the system of integral equation* (2.1) *has a unique solution $u(x)\in C_m[a,b]$ and*

$$\|u(x)\|_C\leq c_1\|f(x)\|_C,$$

where $c_1=\exp\left\{K_0\left(\varphi(b)-\varphi(a)\right)\right\}$ and

$$K_0=\|K(x,s)\|_C=\sup_{(x,s)\in G}\|K(x,s)\|.$$

Then, we will need the following theorem from [3]:

Theorem 2.3. *Let $F(x,s), F'_{\varphi(x)}(x,s)\in C(G), \varphi(x)$ be a strictly increasing continuous functions on $[a,b]$ and $P(x)=\int_a^x F(x,s)d\varphi(s)$, $x\in[a,b]$. Then*

$$P'_{\varphi(x)}(x)=F(x,x)+\int_a^x F'_{\varphi(x)}(x,s)d\varphi(s),\ x\in[a,b],$$

where

$$F'_{\varphi(x)}(x,s)=\lim_{\Delta x\to 0}\frac{F(x+\Delta x,s)-F(x,s)}{\varphi(x+\Delta x)-\varphi(x)},$$

$$(x,s)\in\left\{(x,s):a<s<x<b\right\},$$

$$P'_{\varphi(x)}(a)=\lim_{\Delta x\to 0+}\frac{P(a+\Delta x)-P(a)}{\varphi(a+\Delta x)-\varphi(a)},$$

$$P'_{\varphi(x)}(b)=\lim_{\Delta x\to 0-}\frac{P(b+\Delta x)-P(b)}{\varphi(b+\Delta x)-\varphi(b)}.$$

Corollary 2.2. *Let $u(x) \in C_m[a,b]$ be a solution of the system of integral equations* (2.1), $\left\|K'_{\varphi(x)}(x,s)\right\| \in C(G)$ *and* $f'_{\varphi(x)} \in C_m[a,b]$. *Then* $u'_{\varphi(x)}(x) \in C_m[a,b]$ *and*

$$u'_{\varphi(x)}(x) = K(x,x)u(x) + \int_a^x K'_{\varphi(x)}(x,s)u(s)d\varphi(s) + f'_{\varphi(x)}(x),$$

where $x \in [a,b]$.

Corollary 2.3. *Let $u(x) \in C_m[a,b]$ be a solution of the system of integral equations* (2.1), $\left\|K''_{\varphi(x)}(x,s)\right\| \in C(G), K'_{\varphi(x)}(x,x) \in C_{mm}[a,b]$ *and* $f''_{\varphi(x)} \in C_m[a,b]$. *Then* $u''_{\varphi(x)}(x) \in C_m[a,b]$ *and*

$$u''_{\varphi(x)}(x) = K(x,x)u'_{\varphi(x)}(x) + \left[(K(x,x))'_{\varphi(x)} + K'_{\varphi(x)}(x,s)\Big|_{s=x}\right]u(x)$$
$$+ \int_a^x K''_{\varphi(x)}(x,s)u(s)d\varphi(s) + f''_{\varphi(x)}(x),\ x \in [a,b].$$

In this paper we assume that $\|K(x,x)\| \in C[a,b]$, $\left\|K''_{\varphi(x)}(x,s)\right\|$, $\left\|K''_{\varphi(s)}(x,s)\right\| \in C(G)$ and $f''_{\varphi(x)}(x) \in C_m[a,b]$. Then, using Theorems 2.2 and 2.3 (and Corollaries 2.2 and 2.3), we show that the number M defines as

$$M = \sup_{(x,s)\in G}\left\|[K(x,s)u(s)]''_{\varphi(s)}\right\|$$

can be determined in terms of quantities $\|f(x)\|_C$, $\left\|f'_{\varphi}(x)\right\|_C$, $\left\|f''_{\varphi}(x)\right\|_C$, $\left\|(K(x,x))'_{\varphi(x)}\right\|_C$, $\left\|K'_{\varphi(x)}(x,s)\right\|_C$, $\left\|K'_{\varphi(s)}(x,s)\right\|_C$, $\left\|K''_{\varphi(x)}(x,s)\right\|_C$ and $\left\|K''_{\varphi(s)}(x,s)\right\|_C$. Under these circumstances, using Theorem 2.1, the integral

$$\int_a^x K(x,s)u(s)d\varphi(s)$$

can be evaluated numerically by employing the generalized trapezoid rule.

3. Numerical Solution

In order to obtain the approximate solution of the system (2.1), we employ the generalized trapezoid rule given in [5] to the integral in (2.1). Let $n \in N$,

$$h = \frac{b-a}{n},$$
$$x_k = a + kh,$$

where $k = 0, 1, ..., n$. Let us substitute $x = x_k$ in the system of integral equations (2.1) and examine the following system of equations:

$$\begin{aligned} &u(x_0) = f(x_0),\ x_0 = a, \\ &u(x_k) = \int_a^{x_k} K(x_k, s)u(s)d\varphi(s) + f(x_k),\ k = 1, 2, ..., n. \end{aligned} \tag{3.1}$$

To evaluate the integral term in (3.1), we employ the generalized trapezoid rule at the nodes $x_0,\ x_1,\ , ...,\ x_k$. So we get

$$\int_a^{x_k} K(x_k, s)u(s)d\varphi(s) = \sum_{j=1}^{k} \frac{1}{2} \left[K(x_k, x_{j-1})u(x_{j-1})\right.$$
$$\left. + K(x_k, x_j)u(x_j)\right] \cdot \left[\varphi(x_j) - \varphi(x_{j-1})\right] + \sum_{j=1}^{k} R_j^{(n)}(u), \tag{3.2}$$

where

$$\left\| R_j^{(n)}(u) \right\| \le \frac{M}{12} \left[\varphi(x_j) - \varphi(x_{j-1})\right]^3, \tag{3.3}$$

$$\begin{aligned} M &= \sup_{(x,s)\in G} \left\| [K(x, s)u(s)]''_{\varphi(s)} \right\| \\ &= \sup_{(x,s)\in G} \left\| K(x, s)u''_{\varphi(s)}(s) + 2K'_{\varphi(s)}(x, s)u'_{\varphi(s)}(s) + K''_{\varphi(s)}(x, s)u(s) \right\|. \end{aligned} \tag{3.4}$$

Substituting (3.2) into (3.1), we get

$$\begin{aligned} &u(x_0) = f(x_0),\ x_0 = a, \\ &u(x_k) = \sum_{j=1}^{k} \frac{1}{2} \left[K(x_k, x_{j-1})u(x_{j-1}) + K(x_k, x_j)u(x_j)\right] \times \left[\varphi(x_j) - \varphi(x_{j-1})\right] \\ &\qquad + \sum_{j=1}^{k} R_j^{(n)}(u) + f(x_k), \end{aligned} \tag{3.5}$$

where $k = 1, 2, ..., n$.

Omitting the terms $\sum_{j=1}^{k} R_j^{(n)}(u)$ appearing in each equation of system (3.5) and $u_k \approx u(x_k)$, we obtain

$$\begin{aligned} &u_0 = f(x_0),\ x_0 = a, \\ &u_k = \sum_{j=1}^{k} \frac{1}{2} \left[K(x_k, x_{j-1})u_{j-1} + K(x_k, x_j)u_j\right] \left[\varphi(x_j) - \varphi(x_{j-1})\right] + f(x_k), \end{aligned} \tag{3.6}$$

where $k = 1, 2, ..., n$.

Let us assume that

$$\frac{1}{2} \sup_{x\in[a,b]} \|K(x,x)\| \, \omega_{\varphi}(h) < 1, \tag{3.7}$$

where $\omega_{\varphi}(h)$ denotes the modulus of continuity of the function φ; that is

$$\omega_{\varphi}(h) = \sup_{|x-y|\le h} |\varphi(x) - \varphi(y)| \, .$$

Under condition (3.7), the system of (3.6) has a unique solution which is given by the formulas

$$\begin{aligned} u_0 &= f(a), \\ u_1 &= \left[E_m - \frac{1}{2}K(x_1,x_1)\left(\varphi(x_1) - \varphi(x_0)\right)\right]^{-1} \\ &\times \left[\frac{1}{2}K(x_1,x_0)\left(\varphi(x_1)-\varphi(x_0)\right)u_0 + f(x_1)\right], \\ u_k &= \left[E_m - \frac{1}{2}K(x_k,x_k)\left(\varphi(x_k) - \varphi(x_{k-1})\right)\right]^{-1} \\ &\times \left[\frac{1}{2}\sum_{j=1}^{k-1} K(x_k,x_j)\left(\varphi(x_{j+1}) - \varphi(x_{j-1})\right)u_j \right. \\ &\left. + \frac{1}{2}K(x_k,x_0)\left(\varphi(x_1)-\varphi(x_0)\right)u_0 + f(x_k)\right], \; k = 2,3,...,n, \end{aligned}$$

where E_m is $m \times m$ unit matrix,

$$A_k = E_m - \frac{1}{2}K(x_k,x_k)\left(\varphi(x_k) - \varphi(x_{k-1})\right) \quad (k = 1,2,...,n)$$

is $m \times m$ square matrix, A_k^{-1} is the inverse of A_k.

We give a concrete example below.

Example 3.1. Let us take the system of integral equations (2.1) for $a = 0$, $b = 1$, $m = 2$ and with

$$\varphi(x) = \sqrt{x} \; for \; x \in [0,1],$$

$$K(x,s) = \begin{pmatrix} \sqrt{xs} & \sqrt{s} - \sqrt{x} \\ 3\left(\sqrt{x} - \sqrt{s}\right) & 3 \end{pmatrix}, \; f(x) = \begin{pmatrix} 1 - \frac{x\sqrt{x}}{3} \\ \sqrt{x} - 3x \end{pmatrix}.$$

It is easily seen that $u(x) = \begin{pmatrix} 1 \\ \sqrt{x} \end{pmatrix}$, $x \in [0,1]$ is the unique solution of the system of integral equations (2.1) and the conditions $f''_{\varphi(x)} \in C_2[0,1]$,

$(K(x,x))'_{\varphi(x)} \in C_{2,2}[0,1]$, $K''_{\varphi(x)}(x,s)$ and $K''_{\varphi(s)}(x,s) \in C_{2,2}(G)$ are hold, where $G = \{(x,s) : 0 \le s \le x \le 1\}$.

Using the proposed method of this study we get the following results: here, 20 nodes are selected, that is $n = 20$. In Table 1, we give the values of the approximate solution obtained by the proposed method of this study and the error in absolute values at the given nodes.

Table 1.1: The values of analytical solution, approximate solution and the error at the nodes.

The nodes x_k	Real value at x_k, $u(x_k) = \begin{bmatrix} u_1(x_k) \\ u_2(x_k) \end{bmatrix}$	Approx. value at x_k, $u_k = \begin{bmatrix} u_1^k \\ u_2^k \end{bmatrix}$	The error at x_k, $\lvert u(x_k) - u_k \rvert = \begin{bmatrix} \lvert u_1(x_k) - u_1^k \rvert \\ \lvert u_2(x_k) - u_2^k \rvert \end{bmatrix}$
0	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
0.05	$\begin{bmatrix} 1 \\ 0.2236067977 \end{bmatrix}$	$\begin{bmatrix} 1.001873865 \\ 0.2236067976 \end{bmatrix}$	$\begin{bmatrix} 0.001873865 \\ 0.0000000001 \end{bmatrix}$
0.1	$\begin{bmatrix} 1 \\ 0.3162277660 \end{bmatrix}$	$\begin{bmatrix} 1.002026151 \\ 0.3163233752 \end{bmatrix}$	$\begin{bmatrix} 0.0020261510 \\ 0.0000956092 \end{bmatrix}$
0.15	$\begin{bmatrix} 1 \\ 0.3872983346 \end{bmatrix}$	$\begin{bmatrix} 1.002112319 \\ 0.3875270474 \end{bmatrix}$	$\begin{bmatrix} 0.0021123190 \\ 0.0002287128 \end{bmatrix}$
0.2	$\begin{bmatrix} 1 \\ 0.4472135954 \end{bmatrix}$	$\begin{bmatrix} 1.002179674 \\ 0.4476060648 \end{bmatrix}$	$\begin{bmatrix} 0.002179674 \\ 0.0003924694 \end{bmatrix}$
0.25	$\begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$	$\begin{bmatrix} 1.002240096 \\ 0.5005843530 \end{bmatrix}$	$\begin{bmatrix} 0.002240096 \\ 0.0005843530 \end{bmatrix}$
0.3	$\begin{bmatrix} 1 \\ 0.5477225575 \end{bmatrix}$	$\begin{bmatrix} 1.002298007 \\ 0.5485259162 \end{bmatrix}$	$\begin{bmatrix} 0.002298007 \\ 0.0008033587 \end{bmatrix}$
0.35	$\begin{bmatrix} 1 \\ 0.5916079783 \end{bmatrix}$	$\begin{bmatrix} 1.002355407 \\ 0.5926571918 \end{bmatrix}$	$\begin{bmatrix} 0.002355407 \\ 0.0010492135 \end{bmatrix}$
0.4	$\begin{bmatrix} 1 \\ 0.6324555320 \end{bmatrix}$	$\begin{bmatrix} 1.002413312 \\ 0.6337775902 \end{bmatrix}$	$\begin{bmatrix} 0.002413312 \\ 0.0013220582 \end{bmatrix}$
0.45	$\begin{bmatrix} 1 \\ 0.6708203931 \end{bmatrix}$	$\begin{bmatrix} 1.002472277 \\ 0.6724426958 \end{bmatrix}$	$\begin{bmatrix} 0.002472277 \\ 0.0016223027 \end{bmatrix}$
0.5	$\begin{bmatrix} 1 \\ 0.7071067810 \end{bmatrix}$	$\begin{bmatrix} 1.002532618 \\ 0.7090573150 \end{bmatrix}$	$\begin{bmatrix} 0.002532618 \\ 0.0019505340 \end{bmatrix}$
0.55	$\begin{bmatrix} 1 \\ 0.7416198487 \end{bmatrix}$	$\begin{bmatrix} 1.002594522 \\ 0.7439273254 \end{bmatrix}$	$\begin{bmatrix} 0.002594522 \\ 0.0023074767 \end{bmatrix}$
0.6	$\begin{bmatrix} 1 \\ 0.7745966692 \end{bmatrix}$	$\begin{bmatrix} 1.002658090 \\ 0.7772906374 \end{bmatrix}$	$\begin{bmatrix} 0.002658090 \\ 0.0026939682 \end{bmatrix}$
0.65	$\begin{bmatrix} 1 \\ 0.8062257748 \end{bmatrix}$	$\begin{bmatrix} 1.002723381 \\ 0.8093367044 \end{bmatrix}$	$\begin{bmatrix} 0.002723381 \\ 0.0031109296 \end{bmatrix}$
0.7	$\begin{bmatrix} 1 \\ 0.8366600265 \end{bmatrix}$	$\begin{bmatrix} 1.002790419 \\ 0.8402193920 \end{bmatrix}$	$\begin{bmatrix} 0.002790419 \\ 0.0035593655 \end{bmatrix}$
0.75	$\begin{bmatrix} 1 \\ 0.8660254040 \end{bmatrix}$	$\begin{bmatrix} 1.002859207 \\ 0.8700657420 \end{bmatrix}$	$\begin{bmatrix} 0.002859207 \\ 0.0040403380 \end{bmatrix}$
0.8	$\begin{bmatrix} 1 \\ 0.8944271908 \end{bmatrix}$	$\begin{bmatrix} 1.002929738 \\ 0.8989821696 \end{bmatrix}$	$\begin{bmatrix} 0.002929738 \\ 0.0045549788 \end{bmatrix}$
0.85	$\begin{bmatrix} 1 \\ 0.9219544457 \end{bmatrix}$	$\begin{bmatrix} 1.003001989 \\ 0.9270589266 \end{bmatrix}$	$\begin{bmatrix} 0.003001989 \\ 0.0051044809 \end{bmatrix}$
0.9	$\begin{bmatrix} 1 \\ 0.9486832980 \end{bmatrix}$	$\begin{bmatrix} 1.003075929 \\ 0.9543733822 \end{bmatrix}$	$\begin{bmatrix} 0.003075929 \\ 0.0056900842 \end{bmatrix}$
0.95	$\begin{bmatrix} 1 \\ 0.9746794345 \end{bmatrix}$	$\begin{bmatrix} 1.003151527 \\ 0.9809925224 \end{bmatrix}$	$\begin{bmatrix} 0.003151527 \\ 0.0063130879 \end{bmatrix}$
1	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1.003228737 \\ 1.006974842 \end{bmatrix}$	$\begin{bmatrix} 0.003228737 \\ 0.006974842 \end{bmatrix}$

4. Estimation of the Error

In this section we investigate the problem of convergence of the approximate solution u_k to the solution of the system of integral equations (2.1) at the nodes as $n \to \infty$.

Theorem 4.1. *Let $\varphi(x)$ be a strictly increasing continuous function on $[a, b]$ and for all $x,\ y \in [a, b]$ the following inequality holds:*

$$|\varphi(x) - \varphi(y)| \leq L\,|x - y|\,,$$

where $L > 0$ and L is independent of the variables x and y. Then, the inequality

$$\|u(x_k) - u_k\| \leq \frac{ML^2\,[\varphi(b) - \varphi(a)]}{12(1-\alpha)} \exp\left\{\frac{K_0 L(b-a)}{1-\alpha}\right\} h^2,$$

$k = 1, 2, ..., n$, holds in which $K_0 = \|K(x, s)\|_C$, $\alpha = \frac{1}{2}\|K(x, x)\|_C\, Lh < 1$ and the number M is determined by (3.4)*.*

Proof. Let the error be denoted by $v_k = u(x_k) - u_k$ for $k = 0, 1, ..., n$. Taking into account (3.5) and (3.6) we have the following system of equations

$$\begin{aligned}
&v_0 = 0,\\
&v_k = \sum_{j=1}^{k} \frac{1}{2}\left[K(x_k, x_{j-1})v_{j-1} + K(x_k, x_j)v_j\right](\varphi(x_j) - \varphi(x_{j-1}))\\
&+ \sum_{j=1}^{k} R_j^{(n)}(u),\ k = 1, 2, ..., n.
\end{aligned}$$

Rearranging the above system of equations, we get

$$\begin{aligned}
&\left(E_m - \frac{1}{2}K(x_1, x_1)\left[\varphi(x_1) - \varphi(x_0)\right]\right) v_1 = R_1^{(n)}(u),\\
&\left(E_m - \frac{1}{2}K(x_k, x_k)\left[\varphi(x_k) - \varphi(x_{k-1})\right]\right) v_k\\
&= \frac{1}{2}\sum_{j=1}^{k-1} K(x_k, x_j)\left[\varphi(x_{j+1}) - \varphi(x_{j-1})\right] v_j + \sum_{j=1}^{k} R_j^{(n)}(u),
\end{aligned} \tag{4.1}$$

where $k = 2, ..., n$. Along with the inequality $\omega_\varphi(h) \leq Lh$, using conditions (3.3) and (3.7), we get the following inequality for v_k from (4.1):

$$\begin{aligned} \|v_1\| &\leq \frac{1}{1-\alpha} R(h), \\ \|v_k\| &\leq \frac{1}{1-\alpha}\left[R(h) + K_0 Lh \sum_{j=1}^{k-1} \|v_j\|\right], \end{aligned} \tag{4.2}$$

where $k = 2, ..., n$, $R(h) = (M/12)L^2h^2\,[\varphi(b) - \varphi(a)]$.

Let the term ϵ_k for $k = 2, ..., n$ be determined by

$$\epsilon_k = \frac{1}{1-\alpha}\left[R(h) + K_0 Lh \sum_{j=1}^{k-1} \epsilon_j\right], \tag{4.3}$$

and $\epsilon_1 = R(h)/(1-\alpha)$ as an initial condition.

It is easily seen that $\|v_k\| \leq \epsilon_k$ for $k = 1, 2, ..., n$. This can be verified by mathematical induction as follows: for $k = 1$ it is trivial. Let $v_j \leq \epsilon_j$ for $j = 1, 2, ..., k-1$. Then, using inequality (4.2), we get

$$\|v_k\| \leq \frac{1}{1-\alpha}\left[R(h) + K_0 Lh \sum_{j=1}^{k-1} \epsilon_j\right] = \epsilon_k.$$

Let us show that

$$\epsilon_j = \frac{R(h)}{1-\alpha}\left(1 + \frac{K_0 Lh}{1-\alpha}\right)^{j-1}, \ j = 1, 2, ..., n \tag{4.4}$$

are the solution of the system of (4.3). Taking into account (4.4), we get

$$\begin{aligned} &\frac{1}{1-\alpha}\left[R(h) + K_0 Lh \sum_{j=1}^{k-1} \epsilon_j\right] \\ &= \frac{R(h)}{1-\alpha}\left\{1 + \frac{K_0 Lh}{1-\alpha}\sum_{j=1}^{k-1}\left(1 + \frac{K_0 Lh}{1-\alpha}\right)^{j-1}\right\} \\ &= \frac{R(h)}{1-\alpha}\left\{1 + \left[\left(1 + \frac{K_0 Lh}{1-\alpha}\right)^{j-1} - 1\right]\right\} = \epsilon_k, \quad k \geq 2. \end{aligned}$$

Here, we use the equality

$$(1+\gamma)^{k-1} - 1 = \gamma \sum_{j=1}^{k-1} (1+\gamma)^{j-1}, \quad k \geq 2,$$

where $\gamma = K_0 Lh/(1-\alpha)$. Consequently, we get the following estimate for the error v_k for all values $k = 1, 2, ..., n$:

$$\|v_k\| \le \frac{R(h)}{1-\alpha}\left(1 + \frac{K_0 Lh}{1-\alpha}\right)^{k-1}.$$

Using the fact that $(1+t)^{1/t}$ is decreasing when $t \to 0+$ and approaches to the number e, we get the following chain of inequalities

$$\left(1 + \frac{K_0 Lh}{1-\alpha}\right)^{k-1} \le \left(1 + \frac{K_0 Lh}{1-\alpha}\right)^{(b-a)/h}$$
$$= \left[\left(1 + \frac{K_0 Lh}{1-\alpha}\right)^{(1-\alpha)/K_0 Lh}\right]^{K_0 L(b-a)/(1-\alpha)} \le e^{K_0 L(b-a)/(1-\alpha)}$$

for $k \le n = (b-a)/h$. Hence, the proof is obtained. □

Remark 4.1. The function

$$\varphi(x) = \begin{pmatrix} x, & 0 \le x \le 1, \\ 2x-1, & 1 < x \le 2, \\ 3x-3, & 2 \le x \le 3 \end{pmatrix}$$

is a strictly increasing continuous function on $[0,3]$, $\varphi'(x) \notin C[0,3]$. But, for all $x, y \in [0,3]$ the following inequality holds:

$$|\varphi(x) - \varphi(y)| \le 4\,|x-y|\,.$$

Theorem 4.2. *Let $\varphi(x)$ be a strictly increasing continuous function on $[a,b]$ and*

$$\beta = K_0\,[\varphi(b) - \varphi(a)] < 1. \tag{4.5}$$

Then, the inequality

$$\|u(x_k) - u_k\| \le \frac{M}{12(1-\beta)}(\omega_\varphi(h))^2\,[\varphi(b) - \varphi(a)]\,, k = 1, 2, ..., n, \tag{4.6}$$

holds in which $K_0 = \|K(x,s)\|_C$.

Proof. Let the error be denoted by $v_k = u(x_k) - u_k$ and set up the system of equations

$$v_1 = \frac{1}{2}K(x_1, x_1)\,[\varphi(x_1) - \varphi(x_0)]\,v_1 + R_1^{(n)}(u),$$
$$v_k = \sum_{j=1}^{k-1} \frac{1}{2}K(x_k, x_j)\,[\varphi(x_{j+1}) - \varphi(x_{j-1})]\,v_j$$
$$+ \frac{1}{2}K(x_k, x_k)\,[\varphi(x_k) - \varphi(x_{k-1})]\,v_k + \sum_{j=1}^{k} R_j^{(n)}(u)$$

for $k=2,3,...,n$. From this system of equations we get

$$\begin{aligned}\|v_k\| \le &\frac{1}{2}K_0 \sup_{j=1,2,...,k}\|v_j\| \\ &\times \left\{\sum_{j=1}^{k-1}\left[\varphi(x_{j+1})-\varphi(x_j)+\varphi(x_j)-\varphi(x_{j-1})\right]+\left[\varphi(x_k)-\varphi(x_{k-1})\right]\right\} \\ &+\frac{M}{12}(\omega_\varphi(h))^2\left[\varphi(b)-\varphi(a)\right]=\frac{1}{2}K_0 \sup_{j=1,2,...,k}\|v_j\| \\ &\times\left[\varphi(x_k)-\varphi(x_1)+\varphi(x_{k-1})-\varphi(x_0)+\varphi(x_k)-\varphi(x_{k-1})\right] \\ &+\frac{M}{12}(\omega_\varphi(h))^2\left[\varphi(b)-\varphi(a)\right]\le K_0\sup_j\|v_j\|\left[\varphi(b)-\varphi(a)\right] \\ &+\frac{M}{12}(\omega_\varphi(h))^2\left[\varphi(b)-\varphi(a)\right],\ k=2,3,...,n.\end{aligned}$$

Using the condition (4.5) we get inequality (4.6). Therefore, Theorem 4.2 is proven. □

References

[1] Apartsyn, A.S. (2003). *Nonclassical Linear Volterra Equations of the First Kind.* VSP, Utrecht.

[2] Asanov, A. (2001). The derivative of a function by means of an increasing function. *Manas Journal of Engineering, 1*, 18–64.

[3] Asanov, A. (2002). Volterra-Stieltjes integral equations of the second kind and the first kind. *Manas Journal of Engineering, 2*, 79–95.

[4] Asanov, A. (2003). The system of Volterra-Stieltjes integral equations. *Manas Journal of Engineering, 1*(4), 65–78.

[5] Asanov, A., Chelik, M.H., Chalish, A.N. (2011). Approximating the Stieltjes integral by using the generalized trapezoid rule. *Le Mathematiche, 66*(2), 13–21.

[6] Asanov, A., Hazar, E., Eroz, M., Matanova, K., Abdyldaeva, E. (2016). Approximate Solution of Volterra-Stieltjes Linear Integral Equations of the Second Kind with the Generalized Trapezoid Rule. *Adv. Math. Phys.*, 1–6.

[7] Banas, J., O'Regan, D. (2005). Volterra–Stieltjes integral operators. *Mathematical and Computer Modelling, 1*, 335–344.

[8] Banas, J., Rodrigues, J.R., Sadarangani, K. (2000). On a class of Urysohn–Stieltjes quadratic integral equations and their applications. *J. Comput. Appl. Math., 113*, 35–50.

[9] Banas, J., Dubiel, A. (2017). Solvability of a Volterra–Stieltjes integral equation in the class of functions having limits at infinity. *Electron. J. Qual. Theory Differ. Equ., 53*, 1–17.

[10] Bukhgeim, A.L. (1999). *Volterra Equations and Inverse Problems.* VSP, Utrecht.

[11] Corduneanu, C. (1991). *Integral Equations and Applications.* Cambridge University Press, New York.

[12] Delves, L.M., Walsh, J. (1974). *Numerical Solution of Integral Equations.* Oxford University Press.

[13] Diogo, T., Ford, N.J., Lima, P., Valtchev, S. (2006). Numerical methods for a Volterra integral equation with non-smooth solutions. *J. Comput. Appl. Math., 189*, 412–423.

[14] Federson, M., Bianconi, R. (2001). Linear Volterrra-Stieltjes integral equations in the sense of the Kurzweil-Henstock integral. *Archivum Mathematicum, 37*(4), 307–328.

[15] Federson, M., Bianconi, R., Barbanti, L. (2002). Linear Volterra integral equations. *Acta Mathematicae Applicatae Sinica, 18*, 553–560.

[16] Isaacson, S.A., Kirby, R.M. (2011). Numerical solution of linear Volterra integral equations of the second kind with sharp gradients. *J. Comput. Appl. Math., 235*, 4283–4301.

[17] Maleknejad, K., Aghazadeh, N. (2005). Numerical solution of Volterra integral equations of the second kind with convolution kernel by using Taylor-series expansion. *Appl. Math. Comput., 161*(3), 915–922.

[18] Pogorzelski, W. (1966). *Integral equations and their applications.* Pergamon Press, Oxford-New York-Frankfurt; PWN-Polish Scientific Publishers, Warsaw.

[19] Polyanin, A.D., Manzhirov, A.V. (2008). *Handbook of Integral Equations.* Chapman & Hall— CRC Press, Boca Raton.

[20] Tricomi, F.G. (1985) *Integral equations.* Dover Publications, New York.

[21] Zhang, S., Lin, Y., Rao, M. (2000). Numerical solutions for second-kind Voltera integral equations by Galerkin methods. *Applications of Mathematics, 45*(1), 19–39.

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