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On the degenerate Beltrami equation and hydrodynamic normalization

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Abstract. We study linear Beltrami equation on the Riemann sphere under the assumption that its measurable complex-valued coefficient $\mu(z)$ has compact support in $\mathbb{C}$ and $\|\mu\|_\infty = 1$. Sufficient conditions for the existence of regular homeomorphic $W^{1,1}_{\rm loc}$ solutions to the Beltrami equation with hydrodynamic normalization at infinity are given, in particular, provided that either the dilatation K_μ has the BMO dominant or the so–called tangent dilatations K^T_μ satisfy the integral divergence conditions of the Lehto type. Moreover, the corresponding applications to the degenerate $A-$harmonic equation associated with the Beltrami equation are formulated, too.

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1. Introduction

The analytic theory of quasiconformal mappings f in the complex plane $\mathbb{C}$ is based on the Beltrami partial differential equation

$$f_{\bar z} = \mu(z)\, f_z \quad \text{a.e.} \tag{1.1}$$

with the complex-valued measurable coefficient μ satisfying the uniform ellipticity assumption $\|\mu\|_\infty < 1$. The measurable Riemann mapping theorem, see e.g. [1, 4, 22] and [50], states that, given a measurable function μ in the plane $\mathbb{C}$ with $\|\mu\|_\infty < 1$, there is a quasiconformal homeomorphism $f : \mathbb{C} \to \mathbb{C}$ in $W^{1,2}_{\rm loc}(\mathbb{C})$ satisfying (1.1).

In the case $|\mu(z)| < 1$ a.e. in $\mathbb{C}$ and $\|\mu\|_\infty = 1$, equation (1.1) is called a degenerate Beltrami equation and the structure of the solutions heavily

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depends on the degeneration of μ. Contrary to the case of the uniform ellipticity assumption on μ, the degenerate Beltrami equation does not need to have a homeomorphic solution at all, see e.g. [19]. Therefore, in order to obtain some existence results, extra constraints must be imposed on μ.

The existence problem for degenerate Beltrami equation is currently an active area of research, see, e.g., [5–7, 12, 15, 36–41, 13, 47] and the recent book [14] which contains, in particular, the extended bibliography on the topic.

The main goal of this paper is to study the existence problem for the degenerate Beltrami equation in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ when $\mu(z)$ has a compact support and the homeomorphic solution f, which we are looking for, has hydrodynamic normalization at the infinity, i.e., $f(z) = z + o(1)$ as $z \to \infty$.

Let us fix some definitions and notations. The degeneration of μ is usually expressed in terms of the pointwise maximal dilatation function

$$K_\mu(z) \ := \ \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{1.2}$$

and some mentioned above existence results for the degenerate Beltrami equation have been formulated in terms of corresponding restrictions on K_μ. Note that the degeneration of μ now is equivalent to the fact that $\operatorname{ess\,sup} K_\mu(z) = \infty$. We see that the function $K_\mu(z)$ takes into account the absolute value of μ only. In order to obtain some sharper existence results the argument of $\mu(z)$ should also be considered. It can be done, in particular, replacing K_μ with a more flexible dilatation function

$$K_\mu^T(z, z_0) \ := \ \frac{\left|1 - \frac{\overline{z - z_0}}{z - z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2} \tag{1.3}$$

that is called the *tangent dilatation quotient* of the Beltrami equation with respect to the point $z_0 \in \mathbb{C}$, see e.g. [2, 5, 6, 12, 21] and [36–41]. Note that

$$K_\mu^{-1}(z) \leqslant K_\mu^T(z, z_0) \leqslant K_\mu(z) \qquad \forall\ z\,,\ z_0 \in \mathbb{C}\,. \tag{1.4}$$

The geometric sense of the quantity K_μ^T can be found in the monographs [14] and [24].

A function f in the Sobolev class $W^{1,1}_{\mathrm{loc}}$ is called a *regular solution* of the Beltrami equation (1.1) if f satisfies (1.1) a.e. and its Jacobian $J_f(z) = |f_z|^2 - |f_{\bar z}|^2 > 0$ a.e. in $\mathbb{C}$.

Let D be a domain in $\mathbb{C}$. Suppose that a function $\varphi : D \to \mathbb{R}$ is locally integrable in some neighborhood of a point $z_0 \in D$. Following [17], we say that φ has a *finite mean oscillation* at $z_0 \in D$, write $\varphi \in \mathrm{FMO}(z_0)$, if the relation

$$\limsup_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} \int\limits_{B(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon| \, dm(z) \, < \, \infty$$

holds, where the element $dm(z)$ corresponds to a Lebesgue measure in $\mathbb{C}$ and

$$\overline{\varphi}_\varepsilon \, := \, \frac{1}{\pi\varepsilon^2} \int\limits_{B(x_0,\varepsilon)} \varphi(z) \, dm(z) \ ,$$

see also [33] and [39, section 2]. We say that φ has a finite mean oscillation in D, write $\varphi \in \mathrm{FMO}(D)$, if $\varphi \in \mathrm{FMO}(x_0)$ for any $x_0 \in D$.

Here and later on, given $z_0 \in \mathbb{C}$, $r > 0$ and $0 < r_1 < r_2$, we set

$$B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}\,, \quad S(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}\,,$$

$$A = A(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}\,.$$

The following 2 statements are principal results of our paper.

Theorem 1.1. *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a measurable function with compact support S and $|\mu(z)| < 1$ a.e. Suppose that $K^T_\mu(z, z_0) \leqslant Q_{z_0}(z)$ a.e. in U_{z_0} for every point $z_0 \in S$, a neighborhood U_{z_0} of z_0 and a function $Q_{z_0} : U_{z_0} \to [0,\infty]$ in the class* $\mathrm{FMO}(z_0)$.

Then the Beltrami equation (1.1) *has regular homeomorphic solutions f with the hydrodynamic normalization $f(z) = z + o(1)$ as $z \to \infty$.*

In particular, the conclusion holds if K_μ has a majorant in the well-known class BMO (bounded mean oscillation) by John–Nirenberg in $\mathbb{C}$, see the corresponding discussions in Section 3.

Theorem 1.2. *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a measurable function with a compact support S and $|\mu(z)| < 1$ a.e. Suppose that $K_\mu \in L^1(S)$ and*

$$\int\limits_0^{\delta(z_0)} \frac{dr}{r k^T_\mu(z_0, r)} \, = \, \infty \qquad \forall \ z_0 \in \mathbb{C} \tag{1.5}$$

where $k^T_\mu(z_0, r)$ is the average of $K^T_\mu(z, z_0)$ over the circle $S(z_0, r)$.

Then the Beltrami equation (1.1) *has regular homeomorphic solutions f with the hydrodynamic normalization $f(z) = z + o(1)$ as $z \to \infty$.*

The structure of the present paper is the following. Section 2 contains a general lemma on the existence of regular homeomorphic solutions of the Beltrami equation with hydrodynamic normalization at infinity. On its basis, we derive then the principal results formulated above and some consequences in Section 3. Moreover, we give a series of other integral criteria for it in Section 4 that have an independent interest. Finally, Section 5 contains the corresponding applications to the hydromechanics.

2. The main lemma

Here we give a direct proof of a general lemma on the existence of regular homeomorphic solutions of the Beltrami equation with hydrodynamic normalization at infinity. For this goal, let us start from the following useful statement.

Proposition 2.1. *Let K be a compact set in $\mathbb{C}$, and let $\mu : \mathbb{C} \to \mathbb{C}$ be a measurable function such that $\mu(z) = 0$ for $z \in \mathbb{C} \setminus K$ and $|\mu(z)| < 1$ a.e. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a homeomorphic ACL solution of the equation* (1.1) *with $f(z) \to \infty$ as $z \to \infty$. Then there exists $c_0 \in \mathbb{C}$ such that $f(z) = c_1 z + c_0 + o(1)$ as $z \to \infty$ with $c_1 \neq 0$.*

Proof. Since μ has a compact support, we have by a lemma of the Weil type, see e.g. Corollaries II.B.1 in [1], that f is conformal in a neighborhood of infinity, i.e., outside of a closed disk $\{z \in \mathbb{C} : |z| \le R_0\}$, $R_0 > 0$. Without loss of generality, we may assume that $f(z) \neq 0$ for $|z| > R_0$ because f is a homeomorphism.

Then the mapping $F(\zeta) := 1/f(1/\zeta)$ is a conformal mapping in the punctured disk $\mathbb{D}_0 := \{z \in \mathbb{C} : 0 < |\zeta| < r_0\}$ where $r_0 = 1/R_0$ and, moreover, $F(\zeta) \to 0$ as $\zeta \to 0$. Thus, by the Riemann extension theorem, see e.g. Proposition II.3.7 in [10], the extended function $\tilde{F}$ is conformal in the disk $\mathbb{D}_0^* := \{z \in \mathbb{C} : |\zeta| < r_0\}$. Consequently, by the Rouche theorem $\tilde{F}'(0) \neq 0$, see e.g. Theorem 63 in [48], and the function $\tilde{F}$ has the expansion of the form $a_1\zeta + a_2\zeta^2 + \ldots$ in the disk $\mathbb{D}_0^*$ with $a_1 \neq 0$. Hence, along the set $\{z \in \mathbb{C} : |z| > R_0\}$,

$$f(z) \;=\; \frac{1}{F\left(\frac{1}{z}\right)} \;=\frac{1}{a_1 z^{-1} + a_2 z^{-2} + \ldots}$$

$$=\; \frac{z}{a_1}\left(1 + \frac{a_2}{a_1} z^{-1} + \ldots\right)^{-1} \;=\; a_1^{-1} z \;-\; a_1^{-2} a_2 \;+\; o(1)\;,$$

i.e., the conclusion holds with $c_1 = 1/a_1 \neq 0$ and $c_0 = -a_2/a_1^2$. □

Remark 2.1. The hypothesis that $f(z) \to \infty$ as $z \to \infty$ in Proposition 2.1 can be removed. But for this goal, it would be necessary to give more refined topological arguments. Namely, to prove that 0 is not essential singular point of $g(\zeta) := f(1/\zeta)$ in the punctured disk $\mathbb{D}_0$, we may apply the Casorati–Weierstrass theorem, see e.g. Proposition II.6.3 in [10], because f is a homeomorphism. On this basis, to prove that 0 is a pole of $g(\zeta)$ we are able to apply the Brouwer theorem on the invariance of domain under homeomorphisms, see e.g. Theorem 4.8.16 in [46], and the connectivity of $\overline{\mathbb{C}}$, see e.g. Proposition I.1.1 in [10]. However, we do not need to use it further in the proof of the main lemma at all.

The following lemma is key in the proof of the principal results of the present article, cf. e.g. [40, Lemma 3] and [45, Lemma 1]).

Lemma 2.1. *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a measurable function with a compact support S and $|\mu(z)| < 1$ a.e. Suppose that $K_\mu \in L^1(S)$ and, for every $z_0 \in S$, there exist $\varepsilon_0 = \varepsilon(z_0) > 0$ and a family of measurable functions $\psi_{z_0,\varepsilon} : (0, \varepsilon_0) \to [0, \infty)$ such that*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z, z_0) \cdot \psi^2_{z_0,\varepsilon}(|z - z_0|)\, dm(z) = o(I^2_{z_0}(\varepsilon)) \quad \text{as } \varepsilon \to 0 \;, \tag{2.1}$$

where

$$0 < I_{z_0}(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\, dt < \infty \qquad \forall\, \varepsilon \in (0, \varepsilon_0) \; . \tag{2.2}$$

Then the Beltrami equation (1.1) *has regular homeomorphic solutions f with the hydrodynamic normalization $f(z) = z + o(1)$ as $z \to \infty$.*

In the proof of Lemma 2.1 further, we denote by M the conformal modulus (or 2−modulus) of a family of paths in $\mathbb{C}$, see e.g. [49]. Moreover, given sets E and F and a domain D in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, we denote by $\Gamma(E, F, D)$ the family of all paths $\gamma : [0, 1] \to \overline{\mathbb{C}}$ joining E and F in D, that is, $\gamma(0) \in E$, $\gamma(1) \in F$ and $\gamma(t) \in D$ for all $t \in (0, 1)$.

Let $Q : \mathbb{C} \to (0, \infty)$ be a Lebesgue measurable function. A mapping $f : D \to \overline{\mathbb{C}}$ is called a *ring Q-mapping at a point $z_0 \in D \setminus \{\infty\}$*, if

$$M(f(\Gamma(S(z_0, r_1),\, S(z_0, r_2), D))) \leqslant \int\limits_A Q(z) \cdot \eta^2(|z - z_0|)\, dm(z) \tag{2.3}$$

for each ring $A = A(z_0, r_1, r_2)$ with arbitrary $0 < r_1 < r_2 < d_0 :=$ dist $(z_0, \partial D)$ and all Lebesgue measurable functions $\eta : (r_1, r_2) \to [0, \infty]$

such that

$$\int_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\,. \tag{2.4}$$

In the case $\infty \in D$, a mapping $f : D \to \overline{\mathbb{C}}$ is called a *ring Q-mapping at infinity*, if $f(1/\zeta)$ is a ring $\tilde{Q}-$mapping at 0 with $\tilde{Q}(\zeta) := Q(1/\zeta)$.

Later on, in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, we also use the so-called *spherical (chordal) metric h* defined by the equalities

$$h(z,\zeta) = \frac{|z-\zeta|}{\sqrt{1+|z|^2}\sqrt{1+|\zeta|^2}}\,, \quad z \neq \infty \neq \zeta\,, \quad h(z,\infty) = \frac{1}{\sqrt{1+|z|^2}}\,, \tag{2.5}$$

see e.g. [49, Definition 12.1]. For a given set $E \subset \overline{\mathbb{C}}$, we set

$$h(E) := \sup_{z,\zeta \in E} h(z,\zeta)\,. \tag{2.6}$$

Proof. For every $n \in \mathbb{N}$, we define

$$\mu_n(z) = \begin{cases} \mu(z), & \text{if } \mu(z) \leqslant 1 - 1/n\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

Then by [1, Theorem V.B.3] there is a unique homeomorphic ACL (absolutely continuous on lines) solution $f_n : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Beltrami equation $[f_n]_{\bar{z}} = \mu_n(z) \cdot [f_n]_z$ with $f_n(0) = 0$, $f_n(1) = 1$ and $f_n(\infty) = \infty$.

Note that by [37, Theorem 2.17] f_n is a ring Q-homeomorphism with $Q(z) = K^T_\mu(z, z_0)$ at every point $z_0 \in \mathbb{C}$ and $Q(z) = K^T_\mu(z, 0)$ at ∞. Note also that $K^T_\mu(z, 0) \equiv 1$ outside of S. Thus, by [35, Lemma 4.9], in view of hypothesis (2.1), $\{f_n\}_{n=1}^\infty$ is equicontinuous in $\overline{\mathbb{C}}$ with respect to the chordal metric h. Consequently, f_n is a normal family by the Ascoli theorem, see e.g. [49, Theorem 20.4].

Hence there is a subsequence f_{n_k} of f_n, $k = 1, 2, \ldots$, and a continuous mapping $f_* : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that f_{n_k} uniformly converges to f_* in $\overline{\mathbb{C}}$ with respect to the chordal metric h. In particular, $f_*(\infty) = \infty$. Hence by Proposition 2.1 $f_*(z) = c_1 z + c_0 + o(1)$ as $z \to \infty$ with $c_1 \neq 0$. Moreover, by [42, Corollary 3.1] f_* is a homeomorphic $W^{1,1}_{\rm loc}(\mathbb{C})$ solution of Beltrami equation (1.1). Consequently, the function $f(z) := c_1^{-1} f_*(z) - c_0/c_1$ gives a homeomorphic $W^{1,1}_{\rm loc}(\mathbb{C})$ solution of the Beltrami equation with the hydrodynamic normalization $f(z) = z + o(1)$ as $z \to \infty$.

Thus, it remains to prove that f is regular, i.e., that its Jacobian $J(z) = |f_z|^2 - |f_{\bar{z}}|^2 > 0$ a.e. in $\mathbb{C}$. For this goal, let us first prove that $f^{-1} \in W^{1,2}_{\rm loc}(\mathbb{C})$. Indeed, by [34, Lemma 3.1] $g_k := h_k^{-1} \to g := f^{-1}$

uniformly in $\overline{\mathbb{C}}$ as $k \to \infty$, where $h_k := c_1^{-1} f_{n_k} - c_0/c_1$. Note that h_k and $g_k \in W^{1,2}_{\rm loc}$, $k = 1, 2, \ldots$, because they are quasiconformal mappings. Consequently, these homeomorphisms are locally absolutely continuous, see e.g. [22, Theorem III.6.1]. Observe also that $\mu_k^* := (g_k)_{\bar{w}}/(g_k)_w = -\mu_{n_k} \circ g_k$, see e.g. [1, 4.C.I]). Thus, replacing variables in the integrals, see e.g. [22, Lemma III.2.1]), we obtain that

$$\int\limits_B |\partial g_k(w)|^2 \, dm(w) \ = \ \int\limits_{g_k(B)} \frac{dm(z)}{1 - |\mu_k^*(z)|^2} \ \leqslant \ \int\limits_{B^*} K_\mu(z)\, dm(z) \ < \ \infty$$

for sufficiently large k, where B and B^* are arbitrary domains in $\mathbb{C}$ with compact closures in $\mathbb{C}$ such that $g(\overline{B}) \subset B^*$. It follows from the latter that the sequence g_k is bounded in the space $W^{1,2}(B)$ in each domain B in $\mathbb{C}$ with a compact closure. Hence by [32, Lemma III.3.5] $f^{-1} \in W^{1,2}_{\rm loc}$.

Finally, let us prove that f is regular as we promised. Indeed, f^{-1} is locally absolutely continuous and preserves nulls sets, because $f^{-1} \in {\rm W}^{1,2}_{loc}$, see e.g. Lemma III.3.3 and Theorem III.6.1 in [22]. Moreover, as an ACL mapping f has a.e. partial derivatives and hence by [11] and [25], see also Lemma II.B.1 in [1] or Theorem III.3.1 in [22], it has a total differential a.e.

Let E denote the set of points in $\mathbb{C}$ where f is differentiable and $J_f(z) = 0$. Let us assume that $|E| > 0$. Then $|f(E)| > 0$, too, because $E = f^{-1}(f(E))$ and f^{-1} preserves null sets. However, it is clear that f^{-1} is not differentiable at every point of $f(E)$, contradicting the fact that f^{-1} is differentiable a.e. The contradiction disproves the above assumption and, thus, the proof is complete. □

Taking into account the relations (1.4), let us point out the following important partial case of Lemma 2.1.

Corollary 2.1. *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a function with a compact support S and $|\mu(z)| < 1$ a.e. Suppose that $K_\mu \in L^1(S)$ and that, for all $z_0 \in S$,*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu(z) \cdot \psi^2(|z - z_0|) \ dm(z) \ = \ O\left(\int\limits_{\varepsilon}^{\varepsilon_0} \psi(t) \ dt \right) \tag{2.7}$$

as $\varepsilon \to 0$ for a measurable function $\psi : (0, \varepsilon_0) \to (0, \infty)$, $\varepsilon_0 > 0$, such that

$$\int\limits_0^{\varepsilon_0} \psi(t) \ dt = \infty \,, \qquad \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t) \ dt < \infty \qquad \forall \ \varepsilon \in (0, \varepsilon_0) \ . \tag{2.8}$$

Then the Beltrami equation (1.1) *has regular homeomorphic solutions f with the hydrodynamic normalization $f(z) = z + o(1)$ as $z \to \infty$.*

3. Proof of principal results and some consequences

Versions of the next lemma has been first proved for the class BMO in [36]. For the FMO case, see the papers [17,33,38,39] and the monographs [14] and [24].

Lemma 3.1. *Let D be a domain in $\mathbb{C}$ and let $\varphi : D \to \mathbb{R}$ be a non-negative function of the class* FMO(z_0) *for some $z_0 \in D$. Then*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{\varphi(z)\, dm(z)}{\left(|z-z_0| \log \frac{1}{|z-z_0|}\right)^2} = O\left(\log\log\frac{1}{\varepsilon}\right) \qquad as \quad \varepsilon \to 0 \quad (3.1)$$

for some $\varepsilon_0 \in (0, \delta_0)$, where $\delta_0 = \min(e^{-e}, d_0)$ and $d_0 = \operatorname{dist}(0, \partial D)$.

Proof of Theorem 1.1. Let $z_0 \in \mathbb{C}$, $0 < \varepsilon_0 < e^{-1}$, and let $Q = K^T_\mu(z, z_0)$. Set $\psi(t) := \frac{1}{t\,\log\frac{1}{t}}$, $t \in (0, \varepsilon_0]$. By Lemma 3.1

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} Q(z)\cdot\psi^2(|z-z_0|)\; dm(z) \;=\; O\left(\log\log\frac{1}{\varepsilon}\right).$$

Observe also, that $I(\varepsilon, \varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt = \log \frac{\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon_0}}$. Now, the statement of the Theorem 1.1 follows from Lemma 2.1 by the setting $\psi_{z_0,\varepsilon}(t) := \psi(t) = \frac{1}{t\,\log\frac{1}{t}}$. □

Similarly, choosing in Lemma 2.1 the function $\psi(t) = 1/t$, we come to the next statement on the controlled divergence of the singular integrals of the Calderon–Zygmund type.

Theorem 3.1. *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a function with a compact support S and $|\mu(z)| < 1$ a.e. Suppose that $K_\mu \in L^1(S)$ and*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z, z_0)\, \frac{dm(z)}{|z-z_0|^2} = o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \qquad (3.2)$$

as $\varepsilon \to 0$ for all $z_0 \in S$ and for some $\varepsilon_0 = \varepsilon(z_0) > 0$. Then the Beltrami equation (1.1) *has a regular homeomorphic solution f with the hydrodynamic normalization $f(z) = z + o(1)$ as $z \to \infty$.*

Remark 3.1. Choosing in Lemma 2.1 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (3.2) by

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{K^T_\mu(z, z_0)\, dm(z)}{\left(|z-z_0| \log \frac{1}{|z-z_0|}\right)^2} = o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right) \qquad (3.3)$$

In general, we are able to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t\log 1/t\cdot\log\log 1/t\cdot \ldots\cdot\log\ldots\log 1/t)$.

The following statement is obvious by the triangle inequality.

Proposition 3.1. *If, for a collection of numbers* $\varphi_\varepsilon\in\mathbb{R}$, $\varepsilon\in(0,\varepsilon_0]$,

$$\overline{\lim_{\varepsilon\to 0}}\ \frac{1}{\pi\varepsilon^2}\int\limits_{B(z_0,\varepsilon)}|\varphi(z)-\varphi_\varepsilon|\,dm(z)<\infty\,, \tag{3.4}$$

then φ *is of finite mean oscillation at* z_0.

In particular, choosing here $\varphi_\varepsilon\equiv 0$, $\varepsilon\in(0,\varepsilon_0]$ in Proposition 3.1, we obtain the following.

Corollary 3.1. *If, for a point* $z_0\in D$,

$$\overline{\lim_{\varepsilon\to 0}}\ \frac{1}{\pi\varepsilon^2}\int\limits_{B(z_0,\varepsilon)}|\varphi(z)|\,dm(z)<\infty\,, \tag{3.5}$$

then φ *has finite mean oscillation at* z_0.

Recall that a point $z_0\in D$ is called a *Lebesgue point* of a function $\varphi:D\to\mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon\to 0}\ \frac{1}{\pi\varepsilon^2}\int\limits_{B(z_0,\varepsilon)}|\varphi(z)-\varphi(z_0)|\,dm(z)=0\,. \tag{3.6}$$

It is known that, almost every point in D is a Lebesgue point for every function $\varphi\in L^1(D)$. Thus, we have by Proposition 3.1 the next corollary.

Corollary 3.2. *Every locally integrable function* $\varphi:D\to\mathbb{R}$ *has a finite mean oscillation at almost every point in* D.

Remark 3.2. In particular, by Proposition 3.1 the conclusion of Theorem 1.1 holds if every point $z_0\in S$ is the Lebesgue point of the function Q_{z_0}.

By Corollary 3.1 we obtain the following nice consequence of Theorem 1.1.

Corollary 3.3. *Let* $\mu:\mathbb{C}\to\mathbb{C}$ *be a function with a compact support* S *and* $|\mu(z)|<1$ *a.e. Suppose that* $K_\mu\in L^1(S)$ *and that, for every* $z_0\in S$,

$$\overline{\lim_{\varepsilon\to 0}}\ \frac{1}{\pi\varepsilon^2}\int\limits_{B(z_0,\varepsilon)}K_\mu^T(z,z_0)\,dm(z)<\infty\,. \tag{3.7}$$

Then the Beltrami equation (1.1) *has a regular homeomorphic solution* f *with the hydrodynamic normalization* $f(z)=z+o(1)$ *as* $z\to\infty$.

Recall that a real-valued function u in a domain D in $\mathbb{C}$ is said to be of *bounded mean oscillation* in D, abbr. $u \in \mathrm{BMO}(D)$, if $u \in L^1_{\mathrm{loc}}(D)$ and

$$\|u\|_* := \sup_B \frac{1}{m(B)} \int\limits_B |u(z) - u_B|\, dm(z) < \infty\,, \tag{3.8}$$

where the supremum is taken over all discs B in D and

$$u_B = \frac{1}{m(B)} \int\limits_B u(z)\, dm(z)\,.$$

We write $u \in \mathrm{BMO}_{\mathrm{loc}}(D)$ if $u \in \mathrm{BMO}(U)$ for every relatively compact subdomain U of D (we also write BMO or $\mathrm{BMO}_{\mathrm{loc}}$ if it is clear from the context what D is).

The class BMO was introduced by John and Nirenberg (1961) in the paper [20] and soon became an important concept in harmonic analysis, partial differential equations and related areas, see e.g. [16] and [31].

A function φ in BMO is said to have *vanishing mean oscillation*, abbr. $\varphi \in \mathrm{VMO}$, if the supremum in (3.8) taken over all balls B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$. VMO has been introduced by Sarason in [44]. There are a number of papers devoted to the study of partial differential equations with coefficients of the class VMO, see e.g. [9, 18, 23, 27, 28] and [29].

Remark 3.3. Note that the function $\varphi(z) = \log{(1/|z|)}$ belongs to BMO in the unit disk Δ, see, e.g., [31], p. 5, and hence also to FMO. However, $\widetilde{\varphi}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that condition (3.5) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0. Clearly, $\mathrm{BMO}(D) \subset \mathrm{BMO}_{\mathrm{loc}}(D) \subset \mathrm{FMO}(D)$ and as well-known $\mathrm{BMO}_{\mathrm{loc}} \subset L^p_{\mathrm{loc}}$ for all $p \in [1, \infty)$, see, e.g., [20] or [31]. However, FMO is not a subclass of L^p_{loc} for any $p > 1$ but only of L^1_{loc}. Thus, the class FMO is much more wider than $\mathrm{BMO}_{\mathrm{loc}}$.

Since $K^T_\mu(z, z_0) \leqslant K_\mu(z)$ for all z and $z_0 \in \mathbb{C}$, we also obtain the following consequences of Theorem 1.1.

Corollary 3.4. *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a function with a compact support S and $|\mu(z)| < 1$ a.e. Suppose that K_μ has a dominant $Q : \mathbb{C} \to [1, \infty)$ in the class* $\mathrm{BMO}_{\mathrm{loc}}$. *Then the Beltrami equation* (1.1) *has a regular homeomorphic solution f with the hydrodynamic normalization $f(z) = z + o(1)$ as $z \to \infty$.*

Remark 3.4. In particular, the conclusion of Corollary 3.4 holds if $Q \in \mathrm{W}^{1,2}_{\mathrm{loc}}$ because $W^{1,2}_{\mathrm{loc}} \subset \mathrm{VMO}_{\mathrm{loc}}$, see [8], cf. with the Miklyukov-Suvorov result in [26], see Suplement A4 in the monograph [14].

Proof of Theorem 1.2. Set

$$\psi_{z_0}(t) \;=\; \frac{1}{tk^T_\mu(z_0,t)}\,, \qquad t\in(0,\delta(z_0))\,, \tag{3.9}$$

where $k^T_\mu(z_0,t)$ is the average of $K^T_\mu(z,z_0)$ over the circle $S(z_0,t)$. Let

$$I_{z_0}(\varepsilon) \;:=\; \int\limits_{\varepsilon}^{\delta(z_0)} \psi_{z_0}(t)\,dt\,. \tag{3.10}$$

Observe that $I_{z_0}(\varepsilon) > 0$ for every $\varepsilon \in (0,\delta(z_0))$ because, in the contrary case, we would obtain that $k^T_\mu(z_0,t) = \infty$ for almost every $t\in(0,\delta(z_0))$. The latter is impossible because $K^T_\mu(z,z_0) \leqslant K_\mu(z)$ a.e. by (1.4) and $K_\mu(z)$ is locally inregrable by hypotheses of Theorem 1.2 and, thus, $k^T_\mu(z_0,t) < \infty$ a.e. by the Fubini theorem, see e.g. Theorem III.(9.10) in [43].

Moreover, since $K^T_\mu(z,z_0) \geqslant K^{-1}_\mu(z)$ a.e. by (1.4), we have by the Jensen inequality, see e.g. Theorem 2.6.2 in [30], that

$$\frac{1}{k^T_\mu(z_0,t)} \;\leqslant\; k_\mu(z_0,t)\,, \tag{3.11}$$

where $k_\mu(z_0,t)$ is the average of $K_\mu(z)$ over the circle $S(z_0,t)$. Thus,

$$I_{z_0}(t) \leqslant \frac{1}{2\pi\varepsilon^2} \int\limits_{\varepsilon<|z-z_0|<\delta(z_0)} K_\mu(z)\;dm(z)\,, \tag{3.12}$$

i.e., $I_{z_0}(t) < \infty$ for all $\varepsilon\in(0,\delta(z_0))$ by the local integrability of K_μ.

Finally, again by the Fubini theorem,

$$\int\limits_{\varepsilon<|z-z_0|<\delta(z_0)} K^T_\mu(z,z_0)\cdot\psi^2_{z_0}(|z-z_0|)\;dm(z) =$$

$$= 2\pi\int\limits_{\varepsilon}^{\delta(z_0)} rk^T_\mu(z_0,r)\psi^2_{z_0}(r)\,dr = 2\pi I_{z_0}(\varepsilon)o\left(I^2_{z_0}(\varepsilon)\right) \tag{3.13}$$

in view of hypothesis (1.5). Thus, we obtain the conclusion of Theorem 1.2 by Lemma 2.1 with the function family $\psi_{z_0,\varepsilon}(t) := \psi_{z_0}(t)$. □

Corollary 3.5. *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a function with a compact support S and $|\mu(z)| < 1$ a.e. Suppose that $K_\mu \in L^1(S)$ and*

$$k_\mu^T(z_0, \varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \qquad as\ \varepsilon \to 0 \qquad \forall\ z_0 \in K\,. \tag{3.14}$$

Then the Beltrami equation (1.1) has a regular homeomorphic solution f with the hydrodynamic normalization $f(z) = z + o(1)$ as $z \to \infty$.

Remark 3.5. In particular, the conclusion of Corollary 3.5 holds if

$$K_\mu^T(z, z_0) = O\left(\log \frac{1}{|z - z_0|}\right) \qquad \text{as} \quad z \to z_0 \quad \forall\ z_0 \in K\,. \tag{3.15}$$

Moreover, the condition (3.14) can be replaced by the whole series of more weak conditions with the so-called iterated logarithms

$$k_\mu^T(z_0, \varepsilon) = O\left(\left[\log \frac{1}{\varepsilon} \cdot \log\log \frac{1}{\varepsilon} \cdot \ldots \cdot \log \ldots \log \frac{1}{\varepsilon}\right]\right) \qquad \forall\ z_0 \in K\ . \tag{3.16}$$

4. On integral conditions for the existence of solutions of Beltrami equation

The following statement will be also useful later on, see e.g. Theorem 3.2 in [41].

Proposition 4.1. *Let $Q : \mathbb{D} \to [0, \infty]$ be a measurable function such that*

$$\int\limits_{\mathbb{D}} \Phi(Q(z))\, dm(z) < \infty \tag{4.1}$$

where $\Phi : [0, \infty] \to [0, \infty]$ is a non-decreasing convex function such that

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \tag{4.2}$$

for some $\delta > \Phi(+0)$. Then

$$\int\limits_{0}^{1} \frac{dr}{rq(r)} = \infty \tag{4.3}$$

where $q(r)$ is the average of the function $Q(z)$ over the circle $|z| = r$.

Here we use the following notions of the inverse function for monotone functions. Namely, for every non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$ the inverse function $\Phi^{-1} : [0,\infty] \to [0,\infty]$ can be well-defined by setting

$$\Phi^{-1}(\tau) \ := \ \inf_{\Phi(t)\geqslant\tau} t\,. \tag{4.4}$$

Here inf is equal to ∞ if the set of $t \in [0,\infty]$ such that $\Phi(t) \geqslant \tau$ is empty. Note that the function Φ^{-1} is non-decreasing, too. It is evident immediately by the definition that $\Phi^{-1}(\Phi(t)) \leqslant t$ for all $t \in [0,\infty]$ with the equality except intervals of constancy of the function $\Phi(t)$.

Finally, recall connections between some integral conditions, see e.g. Theorem 2.5 in [41].

Remark 4.1. Let $\Phi : [0,\infty] \to [0,\infty]$ be a non-decreasing function and set

$$H(t) \ = \ \log \Phi(t)\ . \tag{4.5}$$

Then the equality

$$\int\limits_{\Delta}^{\infty} H'(t)\,\frac{dt}{t} = \infty, \tag{4.6}$$

implies the equality

$$\int\limits_{\Delta}^{\infty} \frac{dH(t)}{t} = \infty\,, \tag{4.7}$$

and (4.7) is equivalent to

$$\int\limits_{\Delta}^{\infty} H(t)\,\frac{dt}{t^2} = \infty \tag{4.8}$$

for some $\Delta > 0$, and (4.8) is equivalent to each of the equalities

$$\int\limits_{0}^{\delta_*} H\left(\frac{1}{t}\right)\,dt = \infty \tag{4.9}$$

for some $\delta_* > 0$,

$$\int\limits_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} = \infty \tag{4.10}$$

for some $\Delta_* > H(+0)$ and to (4.2) for some $\delta > \Phi(+0)$.

Moreover, (4.6) is equivalent to (4.7) and hence to (4.6)–(4.10) as well as to (4.2) are equivalent to each other if Φ is in addition absolutely continuous. In particular, all the given conditions are equivalent if Φ is convex and non-decreasing.

Note that the integral in (4.7) is understood as the Lebesgue–Stieltjes integral and the integrals in (4.6) and (4.8)–(4.10) as the ordinary Lebesgue integrals. It is necessary to give one more explanation. From the right hand sides in the conditions (4.6)–(4.10) we have in mind $+\infty$. If $\Phi(t) = 0$ for $t \in [0, t_*$, then $H(t) = -\infty$ for $t \in [0, t_*]$ and we complete the definition $H'(t) = 0$ for $t \in [0, t_*]$. Note, the conditions (4.7) and (4.8) exclude that t_* belongs to the interval of integrability because in the contrary case the left hand sides in (4.7) and (4.8) are either equal to $-\infty$ or indeterminate. Hence we may assume in (4.6)–(4.9) that $\delta > t_0$, correspondingly, $\Delta < 1/t_0$ where $t_0 := \sup\limits_{\Phi(t)=0} t$, and set $t_0 = 0$ if $\Phi(0) > 0$.

The most interesting of the above conditions is (4.8) that can be rewritten in the form:

$$\int\limits_{\Delta}^{\infty} \log \Phi(t) \, \frac{dt}{t^2} \;=\; +\infty \qquad \text{for some } \Delta > 0 \;. \tag{4.11}$$

Combining Theorem 1.2, Proposition 4.1 and Remark 4.1, we obtain the following result.

Theorem 4.1. *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a function with a compact support S and $|\mu(z)| < 1$ a.e. Suppose that $K_\mu \in L^1(S)$ and*

$$\int\limits_{U_{z_0}} \Phi_{z_0}\left(K^T_\mu(z, z_0)\right) \, dm(z) < \infty \qquad \forall\ z_0 \in S \tag{4.12}$$

for a neighborhood U_{z_0} of z_0 and a convex non-decreasing function $\Phi_{z_0} : [0,\infty] \to [0,\infty]$ such that, for some $\Delta(z_0) > 0$,

$$\int\limits_{\Delta(z_0)}^{\infty} \log \Phi_{z_0}(t) \, \frac{dt}{t^2} \;=\; +\infty \;. \tag{4.13}$$

Then the Beltrami equation (1.1) has a regular homeomorphic solution f with the hydrodynamic normalization $f(z) = z + o(1)$ as $z \to \infty$.

Corollary 4.1. *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a function with a compact support S and $|\mu(z)| < 1$ a.e. Suppose that $K_\mu \in L^1(S)$ and*

$$\int\limits_{U_{z_0}} e^{\alpha(z_0) K^T_\mu(z,z_0)}\, dm(z) < \infty \qquad \forall\ z_0 \in S \tag{4.14}$$

for some $\alpha(z_0) > 0$ and a neighborhood U_{z_0} of the point z_0.

Then the Beltrami equation (1.1) has a regular homeomorphic solution f with the hydrodynamic normalization $f(z) = z + o(1)$ as $z \to \infty$.

Since by (1.4) $K^T_\mu(z, z_0) \leqslant K_\mu(z)$ for z and $z_0 \in \mathbb{C}$ and $z \in D$, we also obtain the following consequences of Theorem 4.1.

Corollary 4.2. *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a function with a compact support S and $|\mu(z)| < 1$ a.e. Suppose that*

$$\int\limits_{S} \Phi\left(K_\mu(z)\right)\, dm(z) < \infty \tag{4.15}$$

for a convex non-decreasing function $\Phi : [0, \infty] \to [0, \infty]$ with

$$\int\limits_{\delta}^{\infty} \log\, \Phi(t)\, \frac{dt}{t^2} \;=\; +\infty \tag{4.16}$$

for some $\delta > 0$. Then the Beltrami equation (1.1) has a regular homeomorphic solution f with the hydrodynamic normalization $f(z) = z + o(1)$ as $z \to \infty$.

Corollary 4.3. *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a function with a compact support S and $|\mu(z)| < 1$ a.e. Suppose that*

$$\int\limits_{S} e^{\alpha K_\mu(z)}\, dm(z) \;<\; \infty\ . \tag{4.17}$$

Then the Beltrami equation (1.1) has a regular homeomorphic solution f with the hydrodynamic normalization $f(z) = z + o(1)$ as $z \to \infty$.

Remark 4.2. By Theorem 5.1 in [41] the condition (4.16) is not only sufficient but also necessary to have regular solutions for each Beltrami equation (1.1) with the integral constraints (4.15), see also Remark 4.1.

5. Applications to the hydromechanics

We give here significant applications to one of the main equations of mathematical physics in strongly anisotropic and inhomogeneous media. Namely, in this section we give criteria for the existence of solutions u with the classic asymptotics at infinity for the elliptic equations of the form

$$\operatorname{div} A\nabla\, u = 0 \tag{5.1}$$

in the in whole complex plane $\mathbb{C}$ with suitable measurable matrix-valued coefficients $A(z) = \{a_{ij}(z)\}$.

Solutions of the equation (5.1) are called $A-$**harmonic functions**, see e.g. [16], and they satisfied (5.1) in the sense of distributions, i.e., in the sense that $u \in W^{1,1}_{\rm loc}(D)$ and that

$$\int_D \langle A(z)\nabla u(z), \nabla\psi(z)\rangle \; dm(z) = 0 \qquad \forall\ \psi \in C_0^\infty(D)\ , \tag{5.2}$$

where $C_0^\infty(D)$ denotes the collection of all infinitely differentiable functions $\psi : D \to \mathbb{R}$ with compact support in D, $\langle a, b\rangle$ means the scalar product of vectors a and b in $\mathbb{R}^2$, and $dm(z) := dx\,dy$, $z = x + iy$, corresponds to the Lebesgue measure (area) in the plane $\mathbb{C}$.

In this connection, let us describe the relevance of the Beltrami equations (1.1) and the equations (5.1). First of all, recall that the **Hodge operator** J is the counterclockwise rotation by the angle $\pi/2$ in $\mathbb{R}^2$:

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \; : \; \mathbb{R}^2 \to \mathbb{R}^2\ , \qquad J^2 \;=\; -\, I\ , \tag{5.3}$$

where I denotes the unit 2×2 matrix. Thus, the matrix J plays the role of an imaginary unit in the space of two-dimensional square matrices with real-valued elements.

By Theorem 16.1.6 in [3], if f is a $W^{1,1}_{\rm loc}$ solution of the Beltrami equation (1.1), then the functions $u := {\rm Re} f$ and $v := {\rm Im} f$ satisfy the equation:

$$\nabla\, v(z) \;=\; J\, A(z)\nabla\, u(z)\ , \tag{5.4}$$

where the matrix-valued function $A(z)$ is calculated through $\mu(z)$ in the following way:

$$A \;=\; \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \;:=\; \begin{bmatrix} \frac{|1-\mu|^2}{1-|\mu|^2} & \frac{-2{\rm Im}\,\mu}{1-|\mu|^2} \\ \frac{-2{\rm Im}\,\mu}{1-|\mu|^2} & \frac{|1+\mu|^2}{1-|\mu|^2} \end{bmatrix} . \tag{5.5}$$

The function v is called the $A-$**harmonic conjugate of** u or sometimes a **stream function of the potential** u. Note that by (5.3) the equation

(5.4) is equivalent to the equation

$$A(z)\nabla u(z) \;=\; -J\nabla v(z)\;. \tag{5.6}$$

As known, the curl of any gradient field is zero in the sense of distributions and the Hodge operator J transforms curl-free fields into divergence-free fields, and vice versa, see e.g. 16.1.3 in [3]. Hence (5.6) implies (5.1).

We see from (5.5) that the matrix A is symmetric and it is clear by elementary calculations that $\det A = 1$. Moreover, since $|\mu(z)| < 1$ a.e., from ellipticity of this matrix A follows that $\det(I + A) > 0$ a.e., which in terms of its elements means that $(1 + a_{11})(1 + a_{22}) > a_{12}a_{21}$ a.e. Further $\mathbb{S}^{2\times 2}$ denotes the collection of all such matrices. Thus, by Theorem 16.1.6 in [3], the Beltrami equation is the complex form of one of the main equations of mathematical physics, the potential equation (5.1) with the matrix-valued coefficient A in the class $\mathbb{S}^{2\times 2}$.

Note that the matrix identities in (5.5) can be converted a.e. to express the coefficient $\mu(z)$ of the Beltrami equation (1.1) through the elements of the matrices $A(z)$:

$$\mu \;=\; \mu_A \;:=\; -\frac{a_{11} - a_{22} + i(a_{12} + a_{21})}{2 + a_{11} + a_{22}}\;. \tag{5.7}$$

Thus, we obtain the latter expression as a criterion for the solvability of the potential equation (5.1). Namely, by the above arguments in this section as well as Lemma 2.1 we come to the following general criteria.

Lemma 5.1. *Let $A : \mathbb{C} \to \mathbb{S}^{2\times 2}$ be a measurable function with $K_{\mu_A} \in L^1(S)$, where S is the compact support of the function $A(z) - I$, and*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_{\mu_A}(z, z_0)\cdot\psi^2_{z_0,\varepsilon}(|z-z_0|)\, dm(z) = o(I^2_{z_0}(\varepsilon)) \qquad \forall\, z_0 \in S \tag{5.8}$$

as $\varepsilon \to 0$ for some $\varepsilon_0 = \varepsilon(z_0) > 0$ and a family of measurable functions $\psi_{z_0,\varepsilon} : (0, \varepsilon_0) \to (0,\infty)$ with

$$I_{z_0}(\varepsilon) \colon = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\, dt < \infty \qquad \forall\, \varepsilon \in (0, \varepsilon_0)\,. \tag{5.9}$$

Then the potential equation (5.1) *has its continuous $A-$harmonic solution u with the asymptotics at infinity*

$$u(z) \;=\; x \;+\; o(1)\;,\; z \;=\; x \;+\; i\cdot y\;, \qquad as \quad z \to \infty\;. \tag{5.10}$$

Moreover, there is a continuous $A-$harmonic conjugate v of u with the asymptotics at infinity

$$v(z) \;=\; y \;+\; o(1)\;,\; z \;=\; x \;+\; i\cdot y\;, \qquad as \quad z \to \infty\;. \tag{5.11}$$

In other words, relations of (5.10) and (5.11) in Lemma 5.1 show that independently on a wide range of singularities in S of the equation (5.1) the asymptotic behavior at infinity of these solutions is perfectly similar to the absence of singularities at all and, in particular, equipotential lines are parallel to the imaginary axis and the stream lines are parallel to the real axis at infinity.

Remark 5.1. Note that if the family of the functions $\psi_{z_0,\varepsilon}(t) \equiv \psi_{z_0}(t)$ is independent on the parameter ε, then the condition (5.8) implies that $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. This follows immediately from arguments by contradiction, applying for this (1.4) and the condition $K_{\mu_A} \in L^1(D)$. Note also that (5.8) holds, in particular, if, for some $\varepsilon_0 = \varepsilon(z_0)$,

$$\int\limits_{|z-z_0|<\varepsilon_0} K^T_{\mu_A}(z, z_0) \cdot \psi^2_{z_0}(|z - z_0|)\, dm(z) < \infty \qquad \forall\, z_0 \in S \tag{5.12}$$

and $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In other words, for the existence of $A-$harmonic solutions to the potential equation (5.1) with normalization (5.10), it is sufficient that the integral in (5.12) converges for some nonnegative function $\psi_{z_0}(t)$ that is locally integrable over $(0, \varepsilon_0]$ but has a nonintegrable singularity at 0.

The functions $\log^{\lambda}(e/|z - z_0|)$, $\lambda \in (0, 1)$, $z \in \mathbb{D}$, $z_0 \in \overline{\mathbb{D}}$, and $\psi(t) = 1/(t\ \log(e/t))$, $t \in (0, 1)$, show that the condition (5.12) is compatible with the condition $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Furthermore, the condition (5.8) in Lemma 5.1 shows that it is sufficient for the existence of $A-$harmonic solutions with the normalization at infinity (5.10) to the potential equation (5.1) even that the integral in (5.12) to be divergent in a controlled way.

Arguing similarly to Section 4, we derive from Lemma 5.1 the next series of results.

For instance, choosing $\psi(t) = 1/\left(t \log\left(1/t\right)\right)$ in Lemma 5.1, we obtain by Lemma 3.1 the following.

Theorem 5.1. *Let $A : \mathbb{C} \to \mathbb{S}^{2\times 2}$ be a measurable function with $K_{\mu_A} \in L^1(S)$, where S is the compact support of the function $A(z) - I$. Suppose that $K^T_{\mu_A}(z, z_0) \leqslant Q_{z_0}(z)$ a.e. in U_{z_0} for every point $z_0 \in S$, a neighborhood U_{z_0} of z_0 and a function $Q_{z_0} : U_{z_0} \to [0, \infty]$ in the class* $\mathrm{FMO}(z_0)$.

Then the potential equation (5.1) *has a continuous $A-$harmonic solution u and a continuous conjugate $A-$harmonic function v of u with the normalizations at infinity* (5.10) *and* (5.11), *correspondingly.*

In particular, by Proposition 3.1 the conclusion of Theorem 5.1 holds if every point $z_0 \in S$ is the Lebesgue point of a suitable dominant Q_{z_0} for the tangent dilatataion $K^T_{\mu_A}(z, z_0)$ in a neighborhood of z_0.

By Corollary 3.1 we obtain also the following nice consequence of Theorem 5.1.

Corollary 5.1. *Let $A : \mathbb{C} \to \mathbb{S}^{2\times 2}$ be a measurable function with $K_{\mu_A} \in L^1(S)$, where S is the compact support of the function $A(z) - I$, and*

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{B(z_0,\varepsilon)} K^T_{\mu_A}(z, z_0)\, dm(z) < \infty \qquad \forall\ z_0 \in S\,. \tag{5.13}$$

Then the potential equation (5.1) *has a continuous $A-$harmonic solution u and a continuous conjugate $A-$harmonic function v of u with the normalizations at infinity* (5.10) *and* (5.11)*, correspondingly.*

Since $K^T_{\mu_A}(z, z_0) \leqslant K_{\mu_A}(z)$ for all z and $z_0 \in \mathbb{C}$, we also obtain the following consequences of Theorem 5.1.

Corollary 5.2. *Let $A : \mathbb{C} \to \mathbb{S}^{2\times 2}$ be a measurable function with the compact support S of the function $A(z) - I$. Suppose that K_{μ_A} have a dominant $Q : \mathbb{C} \to [1, \infty)$ in the class* $\mathrm{BMO}_{\mathrm{loc}}(\mathbb{C})$*. Then the potential equation (5.1) has a continuous $A-$harmonic solution u and a continuous conjugate $A-$harmonic function v of u with the normalizations at infinity* (5.10) *and* (5.11)*, correspondingly.*

Remark 5.2. In particular, the conclusion of Corollary 5.2 holds if $Q \in \mathrm{W}^{1,2}_{\mathrm{loc}}$ because $W^{1,2}_{\mathrm{loc}} \subset \mathrm{VMO}_{\mathrm{loc}}$, see [8].

Corollary 5.3. *Let $A : \mathbb{C} \to \mathbb{S}^{2\times 2}$ be a measurable function with the compact support S of the function $A(z) - I$. Suppose that $K_{\mu_A}(z) \leqslant Q(z)$ a.e. in a neighborhood U of S with $Q \in \mathrm{FMO}(U)$. Then the potential equation* (5.1) *has a continuous $A-$harmonic solution u and a continuous conjugate $A-$harmonic function v of u with the normalizations at infinity* (5.10) *and* (5.11)*, correspondingly.*

Similarly, choosing in Lemma 5.1 the function $\psi(t) = 1/t$, we come to the next statement.

Theorem 5.2. *Let $A : \mathbb{C} \to \mathbb{S}^{2\times 2}$ be a measurable function with $K_{\mu_A} \in L^1(S)$, where S is the compact support of the function $A(z) - I$. Suppose that*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_{\mu_A}(z, z_0)\, \frac{dm(z)}{|z - z_0|^2} = o\left(\left[\log \frac{1}{\varepsilon}\right]^2\right) \qquad \forall\ z_0 \in S \tag{5.14}$$

as $\varepsilon \to 0$ *for some* $\varepsilon_0 = \varepsilon(z_0) > 0$. *Then the potential equation* (5.1) *has a continuous* $A-$*harmonic solution* u *and a continuous conjugate* $A-$*harmonic function* v *of* u *with the normalizations at infinity* (5.10) *and* (5.11), *correspondingly.*

Remark 5.3. Choosing in Lemma 5.1 the function $\psi(t) = 1/(t\log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (5.14) by

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{K^T_{\mu_A}(z,z_0)\, dm(z)}{\left(|z-z_0|\log\frac{1}{|z-z_0|}\right)^2} = o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right) \qquad \forall\ z_0 \in S\ . \quad (5.15)$$

In general, we are able to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t\log 1/t\cdot\log\log 1/t\cdot \ldots\cdot\log\ldots\log 1/t)$.

Choosing in Lemma 5.1 the functional parameter $\psi_{z_0,\varepsilon}(t) \equiv \psi_{z_0}(t) :=$ $1/[tk^T_{\mu_A}(z_0,t)]$, where $k^T_{\mu_A}(z_0,r)$ is the integral mean of $K^T_{\mu_A}(z,z_0)$ over the circle $S(z_0,r) := \{z\in\mathbb{C} : |z-z_0| = r\}$, we obtain one more important conclusion.

Theorem 5.3. *Let* $A : \mathbb{C}\to\mathbb{S}^{2\times 2}$ *be a measurable function with* $K_{\mu_A}\in L^1(S)$, *where* S *is the compact support of the function* $A(z)-I$. *Suppose that*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{rk^T_{\mu_A}(z_0,r)} = \infty \qquad \forall\ z_0\in S \qquad (5.16)$$

for some $\varepsilon_0 = \varepsilon(z_0) > 0$. *Then the potential equation* (5.1) *has a continuous* $A-$*harmonic solution* u *and a continuous conjugate* $A-$*harmonic function* v *of* u *with the normalizations at infinity* (5.10) *and* (5.11), *correspondingly.*

Corollary 5.4. *Let* $A : \mathbb{C}\to\mathbb{S}^{2\times 2}$ *be a measurable function with* $K_{\mu_A}\in L^1(S)$, *where* S *is the compact support of the function* $A(z)-I$ *and*

$$k^T_{\mu_A}(z_0,\varepsilon) = O\left(\log\frac{1}{\varepsilon}\right) \qquad \text{as } \varepsilon\to 0 \qquad \forall\ z_0\in S\ . \qquad (5.17)$$

Then the potential equation (5.1) *has a continuous* $A-$*harmonic solution* u *and a continuous conjugate* $A-$*harmonic function* v *of* u *with the normalizations at infinity* (5.10) *and* (5.11), *correspondingly.*

Remark 5.4. In particular, the conclusion of Corollary 5.4 holds if

$$K^T_{\mu_A}(z,z_0) = O\left(\log\frac{1}{|z-z_0|}\right) \qquad \text{as} \quad z\to z_0 \quad \forall\ z_0\in S\,. \qquad (5.18)$$

Moreover, the condition (5.17) can be replaced by the whole series of more weak conditions

$$k_{\mu_A}^T(z_0,\varepsilon) = O\left(\left[\log\frac{1}{\varepsilon}\cdot\log\log\frac{1}{\varepsilon}\cdot\ldots\cdot\log\ldots\log\frac{1}{\varepsilon}\right]\right) \qquad \forall\ z_0\in S\ . \tag{5.19}$$

Combining Theorems 5.3, Proposition 4.1 and Remark 4.1, we obtain the following result.

Theorem 5.4. *Let $A:\mathbb{C}\to\mathbb{S}^{2\times 2}$ be a measurable function with $K_{\mu_A}\in L^1(S)$, where S is the compact support of the function $A(z)-I$. Suppose that*

$$\int\limits_{U_{z_0}} \Phi_{z_0}\left(K_{\mu_A}^T(z,z_0)\right)\,dm(z)<\infty \qquad \forall\ z_0\in S \tag{5.20}$$

for a neighborhood U_{z_0} of z_0 and a convex non-decreasing function $\Phi_{z_0}:[0,\infty]\to[0,\infty]$ with

$$\int\limits_{\Delta(z_0)}^{\infty} \log\,\Phi_{z_0}(t)\,\frac{dt}{t^2} \;=\; +\infty \tag{5.21}$$

for some $\Delta(z_0)>0$. Then the potential equation (5.1) *has a continuous $A-$harmonic solution u and a continuous conjugate $A-$harmonic function v of u with the normalizations at infinity* (5.10) *and* (5.11)*, correspondingly.*

Corollary 5.5. *Let $A:\mathbb{C}\to\mathbb{S}^{2\times 2}$ be a measurable function with $K_{\mu_A}\in L^1(S)$, where S is the compact support of the function $A(z)-I$ and*

$$\int\limits_{U_{z_0}} e^{\alpha(z_0)K_{\mu_A}^T(z,z_0)}\,dm(z)<\infty \qquad \forall\ z_0\in S \tag{5.22}$$

for some $\alpha(z_0)>0$ and a neighborhood U_{z_0} of the point z_0. Then the potential equation (5.1) *has a continuous $A-$harmonic solution u and a continuous conjugate $A-$harmonic function v of u with the normalizations at infinity* (5.10) *and* (5.11)*, correspondingly.*

Since $K_{\mu_A}^T(z,z_0)\leqslant K_{\mu_A}(z)$ for z and $z_0\in\mathbb{C}$ and $z\in D$, we also obtain the following consequences of Theorem 5.4.

Corollary 5.6. *Let $A:\mathbb{C}\to\mathbb{S}^{2\times 2}$ be a measurable function with $K_{\mu_A}\in L^1(S)$, where S is the compact support of the function $A(z)-I$. Suppose that*

$$\int\limits_{S} \Phi\left(K_{\mu_A}(z)\right)\,dm(z)<\infty \tag{5.23}$$

for a convex non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$ *with*

$$\int\limits_{\delta}^{\infty} \log\, \Phi(t)\, \frac{dt}{t^2} \;=\; +\infty \tag{5.24}$$

for some $\delta > 0$. *Then the potential equation* (5.1) *has a continuous* $A-$*harmonic solution* u *and a continuous conjugate* $A-$*harmonic function* v *of* u *with the normalizations at infinity* (5.10) *and* (5.11), *correspondingly.*

Corollary 5.7. *Let* $A : \mathbb{C} \to \mathbb{S}^{2\times 2}$ *be a measurable function with* $K_{\mu_A} \in L^1(S)$, *where* S *is the compact support of the function* $A(z) - I$ *and, for some* $\alpha > 0$,

$$\int\limits_{S} e^{\alpha K_{\mu_A}(z)}\, dm(z) \;<\; \infty\ . \tag{5.25}$$

Then the potential equation (5.1) *has a continuous* $A-$*harmonic solution* u *and a continuous conjugate* $A-$*harmonic function* v *of* u *with the normalizations at infinity* (5.10) *and* (5.11), *correspondingly.*

Thus, we have a number of effective criteria for solvability of the main equation (5.1) of the hydromechanics (fluid mechanics) in strongly anisotropic and inhomogeneous media with the classic normalizations at infinity (5.10) and (5.11) for its potential function u and its stream function v, correspondingly.

Remark 5.5. By Theorem 5.1 in [41], see also Theorem 16.1.6 in [3], the condition (5.24) is not only sufficient but also necessary to have $A-$harmonic solutions u of the potential equation (5.1) with the integral constraints (5.23), see also Remark 4.1.

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