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On the Carathéodory metric of universal Teichmüller space

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Abstract. In contrast to finite dimensional Teichmüller spaces, all non-expanding invariant metrics on the universal Teichmüller space coincide. This important fact found various applications. We give its new, simplified proof based on some deep features of the Grunsky operator, which intrinsically relate to the universal Teichmüller space.

This approach also yields a quantitative answer to Ahlfors' question.

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1. Introductory remarks

As is well known, the Carathéodory, Kobayashi and Teichmüller metrics on any Teichmüller space $\widetilde{\mathbf{T}}$ are related by

$$c_{\widetilde{\mathbf{T}}}(\cdot,\cdot) \le d_{\widetilde{\mathbf{T}}}(\cdot,\cdot) \le \tau_{\widetilde{\mathbf{T}}}(\cdot,\cdot),$$

and similarly for the infinitesimal forms of these metrics.

Recall, that the first two metrics arise from the complex Banach structure on the space $\widetilde{\mathbf{T}}$. Namely, the Kobayashi metric $d_{\widetilde{\mathbf{T}}}$ on $\widetilde{\mathbf{T}}$ is the largest pseudometric d on $\widetilde{\mathbf{T}}$ which does not get increased by holomorphic maps h from the unit disk $\mathbb{D}$ into $\widetilde{\mathbf{T}}$ so that for any two points $\psi_1,\ \psi_2 \in \widetilde{\mathbf{T}}$,

$$d_{\wr\mathbf{T}}(\psi_1, \psi_2) \le \inf\{d_{\mathbb{D}}(0,t):\ h(0) = \psi_1,\ h(t) = \psi_2\},$$

where $d_{\mathbb{D}}$ is the hyperbolic metric on $\mathbb{D}$ of Gaussian curvature -4; while the Carathéodory distance between the points ψ_1 and ψ_2 in $\widetilde{\mathbf{T}}$ is

$$c_{\widetilde{\mathbf{T}}}(\psi_1, \psi_2) = \sup d_{\mathbb{D}}(h(\psi_1), h(\psi_2)),$$

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where the supremum is taken over all holomorphic maps $h : \widetilde{\mathbf{T}} \to \mathbb{D}$. The Teichmüller metric on $\widetilde{\mathbf{T}}$ is canonically defined using the corresponding extremal quasiconformal maps

The fundamental Royden–Gardiner theorem states that $d_{\widetilde{\mathbf{T}}} = \tau_{\widetilde{\mathbf{T}}}$; for details see, e.g., [2, 4].

The aim of this paper is to provide a new proof of the following basic theorem.

Theorem 1. *The Carathéodory metric of the universal Teichmüller space* $\mathbf{T}$ *coincides with its Kobayashi metric; hence all non-expanding invariant metrics on* $\mathbf{T}$ *are equal its Teichmüller metric, and*

$$\begin{aligned} c_{\mathbf{T}}(\varphi, \psi) &= d_{\mathbf{T}}(\varphi, \psi) = \tau_{\mathbf{T}}(\varphi, \psi) \\ &= \inf\{d_{\mathbb{D}}(h^{-1}(\varphi), h^{-1}(\psi)) : \ h \in \mathrm{Hol}(\mathbb{D}, \mathbf{T})\}. \end{aligned}$$

A more general theorem including the infinitesimal version of Theorem 1 (i.e., the equality of the infinitesimal forms of all these metrics) and a similar result for Teichmüller space $\mathbf{T}_1$ of the punctured disk $\mathbb{D} \setminus \{0\}$ has been proved by the author in [13].

The arguments applied in the proof in [13] involve the features of the differential subharmonic metrics and the Grunsky inequalities technique. This proof is based on the deep result of Kühnau [16] that for any quasiconformally extendible univalent function $f(z) = z + a_2 z^2 + \dots$ in the unit disk, with $a_3 - a_2^2 \neq 0$ (hence, on a dense set in the space $\mathbf{T}$), there exists a number $r_0(f) \in (0, 1)$ such that for all $|t| \le r_0(f)$ the corresponding homotopy functions $f_t(z) = f(tz)/t, \ |t| \le r_0(f)$, have equal Teichmüller and Grunsky norms. For any such t, the extremal quasiconformal extension of the function f_t is defined by a nonvanishing holomorphic quadratic differential.

Theorem 1 and its generalization in [13], besides of their own interest, found many important applications, in particular, in the variational theory of univalent functions with quasiconformal extension, which bridge geometric complex analysis with the Teichmüller space theory.

In view of its importance, we provide here an alternate, much simpler proof of Theorem 1, using other features of the Grunsky operator, which intrinsically relate to the universal Teichmüller space.

2. Bachground

2.1. Class of functions

Consider the univalent functions $f(z) = z + a_2 z^2 + \dots$ in the unit disk $\mathbb{D} = \{|z| < 1\}$ admitting quasiconformal extensions to the whole Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ preserving the infinite point. Such functions form the class $S(\infty)$. The Beltrami coefficients of extensions are supported in the complementary disk

$$\mathbb{D}^* = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} = \{z \in \widehat{\mathbb{C}} : \ |z| > 1\}$$

and run over the unit ball

$$\mathrm{Belt}(\mathbb{D}^*)_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu(z)|D = 0, \ \ \|\mu\|_\infty < 1\}.$$

By w^μ will be denoted the homeomorphic solutions to the Beltrami equation $\overline{\partial} w = \mu \partial w$ on $\mathbb{C}$ with $\mu \in \mathrm{Belt}(\mathbb{D}^*)_1$ (quasiconformal automorphisms of $\widehat{\mathbb{C}}$) normalized by $w^\mu(0) = 0, \ (w^\mu)'(0) = 1, \ w^\mu(\infty) = \infty$, i.e., with restrictions $w^\mu|\mathbb{D} \in S(\infty)$.

We point out that for any $\mu \in \mathrm{Belt}(\mathbb{D}^*)_1$ **such solution exists and is unique**. Indeed, the generalized Riemann mapping theorem for the indicated Beltrami equation implies its homeomorphic solution $w(z)$ on $\widehat{\mathbb{C}}$ satisfying $w(0) = 0, \ w(1) = 1, \ w(\infty) = \infty$, and such solution is unique. It remains to pass to functions $w^\mu(z) = w(z)/w'(0)$.

This implies that modeling the universal Teichmüller space in the standard way as a bounded domains filled by the Schwarzian derivatives

$$S_w(z) = \Big(\frac{w''(z)}{w'(z)}\Big)' - \frac{1}{2}\Big(\frac{w''(z)}{w'(z)}\Big)^2$$

of appropriately normalized univalent functions in $\mathbb{D}$, one can use only the functions from $S(\infty)$.

Recall that these Schwarzian derivatives belong to the complex Banach space $\mathbf{B} = \mathbf{B}(\mathbb{D})$ of hyperbolically bounded holomorphic functions in the disk $\mathbb{D}$ with norm

$$\|\varphi\|_{\mathbf{B}} = \sup_{\mathbb{D}} \ (1 - |z|^2)^2 |\varphi(z)|.$$

2.2. The Grunsky coefficients

The underlying features are created by the Grunsky inequalities. We recall some needed results on the Grunsky coefficients involved in order to prove Theorem 1.

Given $f \in S(\infty)$, its Grunsky coefficients are determined from the expansion

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=1}^{\infty} \alpha_{mn} z^m \zeta^n \quad (z, \zeta) \in \mathbb{D}^2, \tag{1}$$

where the principal branch of the logarithmic function is chosen. These coefficients satisfy the inequality

$$\Big|\sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn}(f)x_m x_n\Big| \le 1$$

for any sequence $\mathbf{x} = (x_n)$ from the unit sphere $S(l^2)$ of the Hilbert space l^2 with norm $\|\mathbf{x}\| = \big(\sum\limits_1^\infty |x_n|^2\big)^{1/2}$ (cf. [6]).

The minimum $k(f)$ of dilatations $k(f^\mu) = \|\mu\|_\infty$ among all quasiconformal extensions $f^\mu(z)$ of f onto the whole plane $\widehat{\mathbb{C}}$ (forming the equivalence class of f) is called the **Teichmüller norm** of this function. Hence,

$$k(f) = \tanh d_{\mathbf{T}}(\mathbf{0}, S_f). \tag{2}$$

This quantity dominates the **Grunsky norm**

$$\varkappa(f) = \sup\Big\{\Big|\sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n\Big| :\ \mathbf{x} = (x_n) \in S(l^2)\Big\}$$

by $\varkappa(f) \le k(f)$ (see, e.g., [14, 16]). These norms coincide only when any extremal Beltrami coefficient μ_0 for f (i.e., with $\|\mu_0\|_\infty = k(f)$) satisfies

$$\|\mu_0\|_\infty = \sup\Big\{\Big|\iint_{D^*} \mu(z)\psi(z)dxdy\Big|\Big\} : \psi \in A_1^2(\mathbb{D}^*), \|\psi\|_{A_1} = 1\}$$

$$(z = x + iy). \quad (3)$$

Here $A_1(\mathbb{D}^*)$ denotes the subspace in $L_1(\mathbb{D}^*)$ formed by integrable holomorphic functions (quadratic differentials) on $\mathbb{D}^*$ (hence, $\psi(z) = c_4 z^{-4} + c_5 z^{-5} + \dots$), so $\psi(z) = O(z^{-4})$ as $z \to \infty$, and $A_1^2(\mathbb{D}^*)$ is its subset consisting of ψ with zeros of even order in $\mathbb{D}$, i.e., of the squares of holomorphic functions.

Moreover, if $\varkappa(f) = k(f)$ and the equivalence class of f is a **Strebel point**, which means that it contains the Teichmüller extremal extension $f^{k|\psi_0|/\psi_0}$ with $\psi_0 \in A_1(\mathbb{D})$, then necessarily $\psi_0 = \omega^2 \in A_1^2$ (cf. [10, 14, 17, 21]).

An important fact is that *the Strebel points are dense in any Teichmüller space* (see [4]).

Every Grunsky coefficient $\alpha_{mn}(f)$ in (1) is represented as a polynomial of a finite number of the initial Taylor coefficients $a_2, \dots, a_s$ and hence depends holomorphically on Beltrami coefficients $\mu_f \in \mathrm{Belt}(\mathbb{D}^*)_1$ and on the Schwarzians $S_f \in \mathbf{T}$. This generates holomorphic maps

$$h_{\mathbf{x}}(S_f) = \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn}(S_f)\ x_m x_n :\ \mathbf{T} \to \mathbb{D} \tag{4}$$

with fixed $\mathbf{x} = (x_n) \in l^2$ with $\|\mathbf{x}\| = 1$ so that

$$\sup_{\mathbf{x} \in S(l^2)} h_{\mathbf{x}}(S_f) = \varkappa(f). \tag{5}$$

The holomorphy of these functions follows from the holomorphy of coefficients α_{mn} with respect to Beltrami coefficients $\mu \in \mathrm{Belt}(D^*)_1$ mentioned above and the estimate

$$\Big| \sum_{m=j}^{M} \sum_{n=l}^{N} \beta_{mn} x_m x_n \Big|^2 \le \sum_{m=j}^{M} |x_m|^2 \sum_{n=l}^{N} |x_n|^2,$$

which holds for any finite M, N and $1 \le j \le M,\ 1 \le l \le N$ (see [19, p. 61]).

Both norms $\varkappa(f)$ and $k(f)$ are continuous plurisubharmonic functions of S_f in the norm of $\mathbf{B}$, hence, on the space $\mathbf{T}$ (see, e.g. [14]).

Note that the Grunsky (matrix) operator

$$\mathcal{G}(f) = (\sqrt{mn}\ \alpha_{mn}(f))_{m,n=1}^{\infty}$$

acts as a linear operator $l^2 \to l^2$ contracting the norms of elements $\mathbf{x} \in l^2$; the norm of this operator equals $\varkappa(f)$ (cf. [5]).

2.3. The root transform

One can apply to $f \in S(\infty)$ the rotational conjugation

$$\mathcal{R}_p :\ f(z) \mapsto f_p(z) := f(z^p)^{1/p} = z + \frac{a_2}{p} z^{p-1} + \dots$$

with integer $p \ge 2$ which transforms $f \in S(\infty)$ into p-symmetric univalent functions accordingly to the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbb{C}}_p & \xrightarrow{\mathcal{R}_p f} & \widetilde{\mathbb{C}}_p \\ \downarrow \pi_p & & \downarrow \pi_p \\ \widehat{\mathbb{C}} & \xrightarrow{f} & \widehat{\mathbb{C}} \end{array}$$

where $\widetilde{\mathbb{C}}_p$ denotes the p-sheeted sphere $\widehat{\mathbb{C}}$ branched over 0 and ∞, and the projection $\pi_p(z) = z^p$.

This transform acts on $\mu \in \mathrm{Belt}(\mathbb{D}^*)_1$ and $\psi \in L_1(\mathbb{D}^*)$ by

$$\mathcal{R}_p^* \mu = \mu(z^p) \overline{z}^{p-1} / z^{p-1}, \quad \mathcal{R}_p^* \psi = \psi(z^{-p}) p^2 z^{2p-2};$$

then

$$k(\mathcal{R}_p f) = k(f), \quad \varkappa(\mathcal{R}_p f) \ge \varkappa(f).$$

The Grunsky coefficients of every function $\mathcal{R}_p f$ also are polynomials of $a_2, \dots, a_l$, which implies, similar to (4), the holomorphy of maps

$$h_{\mathbf{x},p}(\mu) = \sum_{m,n=1}^{\infty} \sqrt{mn}\, \alpha_{mn}(\mathcal{R}_p f^\mu)\, x_m x_n : \ \mathrm{Belt}(D^*)_1 \to \mathbb{D} \qquad (6)$$

for any fixed p and any $\mathbf{x} = (x_n) \in S(l^2)$. Similar to (5), $\sup_{\mathbf{x}\in S(l^2)} h_{\mathbf{x},p}(\mu) = \varkappa(\mathcal{R}_p f^\mu)$. Every function $h_{\mathbf{x},p}(\mu)$ descends to a holomorphic functions on the space $\mathbf{T}$, which implies that the Grunsky norms $\varkappa(\mathcal{R}_p f^\mu)$ are continuous and plurisubharmonic on $\mathbf{T}$ [14].

By the Kühnau–Schiffer theorem, the Grunsky norm $\varkappa(f)$ of any $f \in S(\infty)$ is reciprocal to the least posituve Fredholm eigenvalue ϱ_L of the quasicircle $L = f(|z| = 1)$ given by

$$\frac{1}{\varrho_L} = \sup \frac{|\mathcal{D}_G(u) - \mathcal{D}_{G^*}(u)|}{\mathcal{D}_G(u) + \mathcal{D}_{G^*}(u)},$$

where G and G^* are, respectively, the interior and exterior of L; $\mathcal{D}$ denotes the Dirichlet integral, and the supremum is taken over all functions u continuous on $\widehat{\mathbb{C}}$ and harmonic on $G \cup G^*$ (see [16, 20]). This yields, in particular, that

$$\varkappa(\mathcal{R}_p f) \ge \varkappa(f) \quad \text{for any} \ \ p > 1,$$

while $k(\mathcal{R}_p f) = k(f)$.

2.4. Truncation

Fix $0 < r < 1$ and consider for $\mu \in \mathrm{Belt}(\mathbb{D}^*)_1$ the maps

$$f_r^{\widetilde{\mu}}(z) = r^{-1} f^\mu(rz), \quad z \in \mathbb{C}$$

with Beltrami coefficients $\widetilde{\mu}(z) = \mu(rz)$. Truncating the Beltrami coefficients by

$$\mu_r(z) = \begin{cases} \mu(rz), & |z| > 1, \\ 0, & |z| < 1, \end{cases}$$

one obtains a linear (hence holomorphic) map

$$\iota_r : \ \mu \mapsto \mu_r : \ \mathrm{Belt}(\mathbb{D}^*)_1 \to \mathrm{Belt}(\mathbb{D}^*_{1/r})_1.$$

We compare this map with the holomorphic homotopy $f_t(z) = \frac{1}{t}f(tz)$ with $|t| \le 1$, which determines for $|t| < 1$ a holomorphic map of the space $\mathbf{T}$ into itself by

$$S_f(z) = S_{f_t}(z) = t^2 S_f(tz).$$

Passing if needed from f to f_ρ with ρ arbitrarily close to 1, we may assume that S_f is continuous on the closed dik $\overline{\mathbb{D}}$.

Since the Beltrami coefficients of maps f_r^μ and f_r differ only on the annulus $\{1 < |z| < 1/r\}$, one can apply for $r = |t|$ close to 1 the known distortion estimates for quasiconformal maps with integrally small dilatations given in [8, p. 179].

If $f(z)$ is asymptotically conformal on the unit circle, then $S_f(z) = o((1-|z|)^2)$ as $|z| \nearrow 1$, and the indicated estimates imply the uniform bound

$$\|S_{f_r^\mu} - S_{f_r}\|_{\mathbf{B}} = \alpha_1(1-r), \tag{7}$$

where $\alpha_1(1-r) \to 0$ as $r \to 1$. Hence, for such maps,

$$\varkappa(f_r) - \varkappa(f^{\mu_r}) = \alpha_2(1-r) \to 0, \quad r \to 1, \tag{8}$$

regarding these Grunsky norms as the functions from the Schwarzians, i.e., on $\mathbf{T}$ (under the same assumption of conformality of $f(z)$).

Note also that $\varkappa(f_t)$ is a radial subharmonic function on the unit disk $\{|t| < 1\}$; hence, $\varkappa(f_t) = \varkappa(f_{|t|})$ continuous and monotone increasing on $[0, 1]$, and $\lim_{r\to1} \varkappa(f_r) = \varkappa(f)$ (see also [16]). On the other hand, the uniqueness of the extremal extensions of f_t implies that $\lim_{r\to1} k(f_r) = k(f)$.

The higher norms $\varkappa(\mathcal{R}_p f^{\mu_r})$ inherit the indicated properties. Any of these norms is approximated by the corresponding holomorphic functions $h_{\mathbf{x},p}(S_{R_p f_r})$ on $\mathbf{T}$.

2.5. Some additional facts

For any $f(z) \in S(\infty)$, its inverted function

$$F(z) = 1/f(1/z) = z + b_1 z^2 + b_2 z^2 + \dots$$

is univalent and holomorphic on the punctured disk $\mathbb{D}^* \setminus \{0\}$, and has a simple pole at $z = \infty$. The Grunsky coefficients of F are defined similar to (1), and its Grunsky norm $\varkappa(F) = \varkappa(f)$.

The Grunsky coefficients of functions univalent in the disk $\mathbb{D}_\rho = \{|z| < \rho\},\ 0 < \rho < \infty$, are defined from the expansion

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=1}^{\infty} \alpha_{mn} \rho^{m+n} z^m \zeta^n$$

and satisfy

$$\Big|\sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn}(f)\rho^{m+n}x_m x_n\Big| \le 1, \quad \mathbf{x} = (x_n) \in S(l^2).$$

Accordingly, we have the holomorphic maps

$$h_{\mathbf{x},p}(\mu) = \sum_{m,n=1}^{\infty} \sqrt{mn}\alpha_{mn}(\mathcal{R}_p f^{\mu})\rho^{m+n}\, x_m x_n : \mathrm{Belt}(\mathbb{D}_\rho^*)_1 \to \mathbb{D}(\mathbb{D}_\rho^* = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}_\rho}).$$

This theory is extended to univalent functions in arbitrary quasidisks (see [14, 18]).

3. Proof of Theorem 1

1⁰. We shall use the following two lemmas, which present the special cases of more general results from [12, 14] and involve the functions with extremal extensions of Teichmüller type.

Given a function $f \in S(\infty)$, consider its extremal quasiconformal extension f^{μ_0} to $\mathbb{D}^*$ with Beltrami coefficient $\mu_0 \in L_\infty(\mathbb{D}^*)$ and assign to this function the quantity

$$\alpha_{\mathbb{D}^*}(f) = \sup\left\{\Big|\iint\limits_{D^*} \mu_0(z)\psi(z)dxdy\Big| :\ \psi \in A_1^2(D^*),\ \|\psi\|_{A_1(\mathbb{D}^*)} = 1\right\}. \tag{9}$$

Lemma 2. *(a) The Grunsky norm $\varkappa(f)$ of every function $f \in S(\infty)$ is estimated by its Teichmüller norm $k = k(f)$ via*

$$\varkappa(f) \le k\frac{k + \alpha_{\mathbb{D}^*}(f)}{1 + \alpha_{\mathbb{D}^*}(f)k}, \tag{10}$$

and $\varkappa(f) < k$ unless $\alpha_{\mathbb{D}^*}(f) = \|\mu_0\|_\infty$.

The last equality occurs if and only if $\varkappa(f) = k(f)$, and if in addition the equivalence class of f (the collection of maps equal to f on ∂D) is a Strebel point, then μ_0 is necessarily of the form

$$\mu_0(z) = \|\mu_0\|_\infty|\psi_0(z)|/\psi_0(z) \quad \textit{with} \ \ \psi_0 \in A_1^2(\mathbb{D}^*).$$

(b) If f admits a Teichmüller quasiconformal extension f^{μ_0} onto the disk $\mathbb{D}^*$, then for small $\|\mu_0\|_\infty$

$$\varkappa(f) = \sup\Big|\iint\limits_{D^*} \mu_0(z)\psi(z)dxdy\Big| + O(\|\mu_0\|_\infty^2).$$

with the same supremum as in (9).

Lemma 3. *For any function $f \in S(\infty)$ having the Teichmüller extremal extension f^{μ_0} to $\mathbb{D}^*$, its Teichmüller norm $k(f) = \|\mu_0\|_\infty$ is given by*

$$k(f) = \widehat{\varkappa}(f) := \sup_{\mu \in [f]} \limsup_{r \to 1} \sup_{p} \sup_{\in A_1^2(\mathbb{D}^*), \|\psi\|_{A_1} = 1} \Big| \iint_{\mathbb{D}^*} \mathcal{R}_p^* \mu_{0r}(z) \psi(z) dxdy \Big|. \tag{11}$$

This quantity $\widehat{\varkappa}(f)$ can be regarded as the **outer limit Grunsky norm** of f, and (11) is an essential strengthening of the relation (3).

In fact, as is shown in [12], **any function** $f \in S(\infty)$ admits quasiconformal extension with dilatations $k \geq \widehat{\varkappa}(f)$, and this lower admissible bound $\widehat{\varkappa}(f)$ for dilatations of extensions is sharp in the sense that it cannot be replaced by a smaller quantity for each $f \in S(\infty)$.

Lemma 3 plays a crucial role in the proof of Theorem 1, thus we provide here its **proof**.

By assumption, the Beltrami coefficient of extremal extension of f to the disk $\mathbb{D}^*$ has the form

$$\mu_0(z) = \|\mu_0\|_\infty |\psi_0(z)|/\psi_0(z)$$

with $\widehat{\mathbb{C}}$-holomorphic quadratic differential

$$\psi_0(z) = c_3 z^{-3} + c_4 z^{-4} + \dots, \quad |z| > 1, \tag{12}$$

having at most simple pole at the infinite point.

If $c_3 \neq 0,\ c_4 \neq 0$, then, noting that $\widehat{\varkappa}(\mathcal{R}_2 f^{\mu_0}) = \widehat{\varkappa}(f^{\mu_0})$, one can start with the squared map $\mathcal{R}_2 f^{\mu_0}$ whose defining quadratic differential is of the form

$$\mathcal{R}_2^* \psi_0(z^2) = 4(c_3 z^{-4} + c_4 z^{-6} + \dots)$$

and has at $z = \infty$ zero of even order. To avoid a complication of notations, assume that this holds for ψ_0 (hence in (12) $c_3 = 0$).

We only need to consider the case when ψ_0 has at least two zeros of odd order. After applying to f^{μ_0} the root transform, we get the Teichmüller map $\mathcal{R}_p^* f = f^{k|\mathcal{R}_p^* \psi_0|/\mathcal{R}_p^* \psi_0}$ determined by quadratic differential

$$\mathcal{R}_p^* \psi_0 = \psi_0(z^p) p^2 z^{2p-2}.$$

Fix r_j arbitrarily close to 1 and pick p_j so large that all zeros of odd order of $\mathcal{R}_p^* \psi_0$ are placed in the annulus $\{1 < |z| < 1/r_j\}$.

Then, taking the truncated Beltrami coefficients $(\mathcal{R}_{p_j}^* \mu_0)_{1/r_j}$ for

$$\mathcal{R}_{p_j}^* \mu_0 = k|\mathcal{R}_{p_j}^* \psi_0|/\mathcal{R}_{p_j}^* \psi_0,$$

vanishing in the disk $\mathbb{D}_{1/r_j}$, and applying to these coefficients Lemma 2, one obtains that on the disk $D^*_{1/r}$ the corresponding extremal maps $f^{(\mathcal{R}^*_{p_j}\mu_0)_{1/r_j}}$ are determined by holomorphic quadratic differentials with zeros of even order. [1] Therefore,

$$\varkappa(f^{(\mathcal{R}^*_{p_j}\mu_0)_{1/r_j}}) = \sup_{(x_n)\in S(l^2)} \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn}(f^{(\mathcal{R}^*_{p_j}\mu_0)_{1/r_j}})\ r_j^{m+n} x_m x_n \Big|.$$

Using this equality, one can find the appropriate sequences $\{r_n\} \to 1$, $\{p_n\} \to \infty$ and $\{\psi_n\} \in A_1^2$ with $\|\psi_n\|_{A_1(\mathbb{D}^*)} = 1$ such that in the limit as $n \to \infty$ the above relations result in the desired equality (10), completing the proof of Lemma 3.

2⁰. We may now prove Theorem 1.

First assume that $f \in S(\infty)$ is univalent in a broader disk $\mathbb{D}_d$ with $d > 1$. Then it admits the Teichmüller extremal extension across any circle $\{|z| = d'\}$, $d' < d$, so the Schwarzian S_f is a Strebel point in the space $\mathbf{T}$.

For such f, the proof of Lemma 3 and the estimates established in section **2.4** for the associated homotopies $f_r(z) = r^{-1}f(rz)$ provide the sequences $\{r_n\} \to 1$, $\{p_n\} \to \infty$ and $\{\psi_n\} \in A_1^2$ defining the extremal extensions $f^{(\mathcal{R}^*_{p_n}\mu_0)_{1/r_n}}$ generated by f. Combining with (8), on obtains for any n the corresponding holomorphic function $h_n(S_f) : \mathbf{T} \to \mathbb{D}$ of type (4) which satisfies

$$|h_n(S_f)| \geq \widehat{\varkappa}(f) - \varepsilon_n \tag{13}$$

with $\varepsilon_n > 0$ monotone decreasing to zero as $n \to \infty$.

Letting $n \to \infty$, one obtains from (2), (3), (11) and (13) the equalities

$$d_{\mathbf{T}}(\mathbf{0}, S_f) = c_{\mathbf{T}}(\mathbf{0}, S_f) = \tanh^{-1} \widehat{\varkappa}(f). \tag{14}$$

Let now f be an arbitrary function from $S(\infty)$ determining a Strebel point in $\mathbf{T}$. Then its homotopy functions $f_r(z)$ satisfy (13) and (14). Since for any $p \geq 1$, we have

$$\lim_{r\to 1} \varkappa_p(f_r) = \varkappa_p(f)$$

and all $\varkappa_p(f_r) \leq k = k(f)$, every function $h_{\mathbf{x},p}(S_{R_p f_r})$ of type (4) and (6) is dominated (for r close to 1) by appropriate function $h_{\mathbf{x},p}(S_{\mathcal{R}_p f})$. The

[1] Note that the extremal extension of $f^{(\mathcal{R}^*_{p_j}\mu_0)_{1/r_j}}$ across the unit circle $\{|z| = 1\}$ also is of Teichmüller type and has dilatation at most $r_j k$ (see, e.g. [11]). We do not use these extensions and deal, instead, with extensions across the circles $\{|z| = 1/r_j\}$.

inequalities (13) provide for these majorizing functions

$$h_{\mathbf{x},p_n}(S_{R_{p_n f}}) =: \widetilde{h}_n(S_f)$$

a collection of the relations

$$|\widetilde{h}_n(S_f)| \ge \widehat{\varkappa}(f_r) - \varepsilon_n, \quad n = 1, 2, \dots$$

(though generically the Schwarzians S_{f_r} do not converge to S_f as $r \to 1$). Going to the limits $n \to \infty$ and $r \to 1$, one derives from the last inequalities that the equalities (14) are valid also for the limit function f.

This provides the assertion of the theorem for the distances between a Strebel point $\varphi = S_f$ and the origin of $\mathbf{T}$. Since the Strebel points are dense in the space $\mathbf{T}$ and both metrics $d_{\mathbf{T}}$ and $c_{\mathbf{T}}$ are continuous, the equalities (14) hold for any point of the space $\mathbf{T}$.

Now consider two arbitrary points $\varphi_1 = S_{f_1}$ and $\varphi_2 = S_{f_2}$ in $\mathbf{T}$. Since the universal Teichmüller space is a complex homogeneous domain in $\mathbf{B}$, this general case is reduced to the previous step by moving one of these points to the origin $\varphi = \mathbf{0}$, applying a right translation of the space $\mathbf{T}$. Such translations preserve the invariant distances; hence, from (14),

$$d_{\mathbf{T}}(\varphi_1, \varphi_2) = c_{\mathbf{T}}(\varphi_1, \varphi_2).$$

This completes the proof of Theorem 1. □

4. Additional remarks

1. The situation is different in the case of finite dimensional Teichmüller spaces $\mathbf{T}(g, n)$ of dimension greater than 1.

The well-known theorem of Kra [7] yields that all invariant metrics on $\mathbf{T}(g, n)$ coincide on the Abelian Teichmüller disks determined by holomorphic quadratic differentials with zeros of even order. Its more general extension to the universal Teichmüller space (applied above) was given in [10].

Recently, Gardiner [3] established that any space $\mathbf{T}(g, n)$ of dimension greater than 1 contains the holomorphic disks on which the Carathéodory and Kobayashi metrics are not equal. [2]

2. The equality (14) has also another important application: **this equality provides a quantitative answer to Ahlfors' question** (stated, for example, in [1]):

[2]This fact was claimed in [9]; the arguments outlined there contain a gap.

How to characterize the conformal maps of the disk (or half-plane) onto the domains with quasiconformal boundaries?

It follows from (14) and from the well-known properties of quasicircles (presented, for example, in the survey [11]) that if a function $f \in S(\infty)$ admits k-quasiconformal extensions across the unit circle to $\widehat{\mathbb{C}}$, then $k \geq \widehat{\varkappa}(f)$, and this lower admissible bound is sharp. Hence, the reflection coefficient q_L of the curve $L = f(|z| = 1)$ relates to the Grunsky and Teichmüller norms of this function via

$$\frac{1+q_L}{1-q_L} = \left(\frac{1+\widehat{\varkappa}(f)}{1-\widehat{\varkappa}(f)}\right)^2.$$

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