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On a boundary properties of functions from a class $H_p(p \geq 1)$

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Abstract. In this paper, the structural properties of a function are characterized by modules of continuity. The classical Hardy–Littlewood theorem describes the connection between the smoothness of the analytic function boundary values at the boundary of its analyticity and the growth rate of the modulus of its derivatives of higher orders. In this paper, we obtain an analogue of the Hardy–Littlewood theorem for functions from class H_p and modules of continuity of higher orders.

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Introduction

The classes $H_p(0 < p < \infty)$, consisting of those functions f which are analytic in the interior D of the unit circle and for which

$$M_r^p(|f|) = \int\limits_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^p d\theta$$

is bounded for $0 < r < l$, were introduced into analysis by G. H. Hardy. The principal facts concerning the behaviour of these functions at the boundary were established by F. Riesz with the aid of the factorization theorem. A.J. Macintyre, W.W. Rogosinski, and H.S. Shapiro have treated linear extremum problems (for $p \leq l$) in great detail. S.S. Walters has discussed the structure of the linear spaceH_p for $0 < p < 1$. R.M. Kovalchuk and Y.I. Volkov studied the properties of the bounded functions depending on L_p-norms of the derivatives for the analytical functions. In this paper we generalize Theorem 1 of the paper [10] for $n = 2, 3 \dots$ and for the modulus of continuity $\omega_n(f,t)$, $\forall n \in N$.

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1. Main result

Denote by $H_p, p \geq 1$ the class of analytic in the circle $\mathrm{D} = \{z : |z| < 1\}$ functions $f(z)$ such that the integral $\int\limits_{-\pi}^{\pi} \left| f\left(re^{i\theta}\right) \right|^p d\theta$ is bounded as $0 < r < 1$. At this case (see the monograph [2, p. 389]) the function $f(z)$ has almost everywhere on $\gamma = \{\, z : |\, z\,| = 1\}$ some boundary values for non-tangent paths, that form boundary function $f(e^{i\theta}) \in L_p\,(-\pi, \pi)$. By definition, put

$$\left\| f\left(e^{i\varphi}\right) \right\|_{L_p} = \left\{ \int\limits_{-\pi}^{\pi} \left| f\left(e^{i\varphi}\right) \right|^p d\varphi \right\}^{\frac{1}{p}},$$

and

$$\omega_n\,(f, t) = \omega_n\,(t) = \sup_{|h| \leq t} \left\| \Delta_h^n\,(f, \theta) \right\|_{L_p};$$

here $\Delta_h^n f(e^{i\theta}) = \sum\limits_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) (-1)^{n-j} f\left(e^{i\,(\theta + jh)}\right)$ is a difference of the order n, $n \in N$, of the function $f(e^{i\theta})$ by the arc coordinate. Suppose that a real function $\omega(t)$ is defined for some segment [0,l]. Then the function $\omega(t)$ belongs to the class Ω_n if the following conditions hold:

1) $\omega(0)$=0 , $\omega(t)$>0 as t$\in (0, l]$;
2) $\omega(t)$ is a non-decreasing function;
3) $\omega(t)$ is a continuous function over $[0, l]$;
4) for any $\lambda > 0$

$$\omega\,(\lambda t) \leq A_1\,(1 + \lambda)^n\,\omega\,(t)\,,$$

where the constant A_1>0 is independent of t and λ.

Define the tensile indicators of a function $\omega\,(t) \in \Omega_n$ as

$$\alpha_\omega = \lim_{k \to 0} \frac{\ln s_\omega\,(k)}{\ln k}, \quad \beta_\omega = \lim_{k \to \infty} \frac{\ln s_\omega\,(k)}{\ln k},$$

where $s_\omega\,(k) = \sup\limits_{t > 0} \frac{\omega(kt)}{\omega(t)}$.

The numbers α_ω and β_ω are called (see the monograph [11]) upper and lower tensile indicators of a function $\omega(t), t > 0$. In the general case the following conditions hold:

$$0 < \alpha_\omega \leq \beta_\omega < \infty.$$

Then, since $\omega(t)$ is continuous, by the monograph [11] it follows that

$$0 \leq \alpha_\omega \leq \beta_\omega \leq 1. \tag{1.1}$$

Suppose $r>0$ and $\omega(t)\in\Omega_n$, $s_\omega(k)<\infty$ for any $k>0$. Then by the papers [1, 11] it follows that:

1) α_ω>0 if and only if

$$\int_0^t \frac{\omega(s)}{s}ds \le A_2\omega(t), \quad \forall t>0; \tag{1.2}$$

2) β_ω<r if and only if

$$\int_t^\infty \frac{\omega(s)}{s^{r+1}}ds \le A_3\frac{\omega(t)}{t^r}, \quad \forall t>0, \tag{1.3}$$

where $A_2>0$, $A_3>0$ are constants.

Theorem 1.1. *Suppose the following conditions hold:*

1) $f(z)\in H_p,\ \ p\ge 1$,

2) the integral modulus of continuity

$$\omega_n(f,t)\le\omega(t), \tag{1.4}$$

where $\omega(t)\in\Omega_n$.

Then fo all $0<r<1$ and $n\in N,\ n\ge 2$

$$\left\| f^{(n)}\left(re^{i\varphi}\right)\right\|_{L_p} \le A_4\frac{\omega(1-r)}{(1-r)^n}, \tag{1.5}$$

where $A_4>0$ is independent from r constant.

Proof. Suppose $f(z)=u(z)+iv(z)$. Since we have condition (1.4) on γ, it follows for $u(z)$ and $v(z)$ that

$$\omega_n(u,t)\le\omega(t), \quad \omega_n(v,t)\le\omega(t). \tag{1.6}$$

Since

$$\left|\Delta_h^n f\right| = \left|\Delta_h^n u + i\Delta_h^n v\right| = \sqrt{\left(\Delta_h^n u\right)^2+\left(\Delta_h^n v\right)^2} \ge \left|\Delta_h^n u\right|,$$

we get

$$\sup_{|h|\le t}\left\|\Delta_h^n f\right\|_{L_p} \ge \sup_{|h|\le t}\left\|\Delta_h^n u\right\|_{L_p}.$$

Therefore $\omega_n(u,t)\le\omega(t)$. Continuing in the same way, we can prove second inequality from (1.6).

Consider $z = re^{i\varphi} \in D$. Then

$$f^{(n)}(z) = \frac{\frac{\partial^n f\left(re^{i\varphi}\right)}{\partial \varphi^n} + B_1(n)\frac{\partial^{n-1} f\left(re^{i\varphi}\right)}{\partial \varphi^{n-1}} + \cdots + B_{n-1}(n)\frac{\partial f\left(re^{i\varphi}\right)}{\partial \varphi}}{z^n}, \tag{1.7}$$

where $B_1(n), ..., B_{n-1}(n)$ are some numerical coefficients. Furthermore for $k = \overline{1,n}$

$$\frac{\partial^k f\left(re^{i\varphi}\right)}{\partial \varphi^k} = \frac{\partial^k u\left(re^{i\varphi}\right)}{\partial \varphi^k} + i\frac{\partial^k v\left(re^{i\varphi}\right)}{\partial \varphi^k}. \tag{1.8}$$

The function $u\left(re^{i\varphi}\right)$ is harmonic on D and both integrals $\int\limits_{-\pi}^{\pi} \left|u\left(re^{i\varphi}\right)\right|^p d\varphi$, $\int\limits_{-\pi}^{\pi} \left|v\left(re^{i\varphi}\right)\right|^p d\varphi$ are bounded for $0 \le r < 1$. Then the value $u\left(re^{i\varphi}\right)$ can be represented by the Poisson integral (see the papers [4,7]) at every point $z = re^{i\varphi} \in D$. Define (see the paper [8,12]) the Δ-like kernel in the Poisson integral as

$$P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

Therefore

$$u\left(re^{i\varphi}\right) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} u\left(e^{i\theta}\right) P\left(r, \theta - \varphi\right) d\theta. \tag{1.9}$$

There are two possible cases:

1) k is odd; 2) k is even.

Consider the first one. Using (1.9) for $k = \overline{1,n}$ and using the methods of the paper [9] we get

$$\frac{\partial^k u\left(re^{i\varphi}\right)}{\partial \varphi^k} = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} u\left(e^{i\theta}\right) \frac{\partial^k P\left(r, \theta - \varphi\right)}{\partial \varphi^k} d\theta$$

$$= \frac{1}{2\pi}\int\limits_{0}^{\pi} \left[u\left(e^{i(\varphi-\theta)}\right) - u\left(e^{i(\varphi+\theta)}\right)\right] \frac{\partial^k P\left(r, \theta\right)}{\partial \theta^k} d\theta. \tag{1.10}$$

Using generalized Minkowski inequality (see the monograph [11]) and (1.10) we get

$$\left\|\frac{\partial^k u\left(re^{i\varphi}\right)}{\partial \varphi^k}\right\|_{L_p} = \left\| \frac{1}{2\pi}\int\limits_{0}^{\pi} \left[u\left(e^{i(\varphi-\theta)}\right) - u\left(e^{i(\varphi+\theta)}\right)\right] \frac{\partial^k P\left(r, \theta\right)}{\partial \theta^k} d\theta \right\|_{L_p} \le$$

$$\leq \frac{1}{2\pi}\int_0^{\pi}\left\|u\left(e^{i(\varphi-\theta)}\right)-u\left(e^{i(\varphi+\theta)}\right)\right\|_{L_p}\left|\frac{\partial^k P(r,\theta)}{\partial\theta^k}\right|d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{\pi}\omega_1(u,\theta)\left|\frac{\partial^k P(r,\theta)}{\partial\theta^k}\right|d\theta. \tag{1.11}$$

An estimate is known from the paper [3]

$$\left|\frac{\partial^k P(r,\theta)}{\partial\theta^k}\right| \leq \frac{C}{(1-2r\cos\theta+r^2)^{\frac{k+1}{2}}}.$$

Since $1-2r\cos\theta+r^2 \geq (1-r)^2+\frac{4r}{\pi^2}\theta^2$, we obtain

$$\left|\frac{\partial^k P(r,\theta)}{\partial\theta^k}\right| \leq \frac{C}{\left((1-r)^2+\frac{4r}{\pi^2}\theta^2\right)^{\frac{k+1}{2}}}. \tag{1.12}$$

From (1.11) we get

$$\left\|\frac{\partial^k u\left(re^{i\varphi}\right)}{\partial\varphi^k}\right\|_{L_p} \leq \frac{1}{2\pi}\left(\int_0^{1-r}+\int_{1-r}^{\pi}\right)\omega_1(u,\theta)\left|\frac{\partial^k P(r,\theta)}{\partial\theta^k}\right|d\theta = I_1+I_2. \tag{1.13}$$

Combining Marchaud inequality (see the monograph [6]), (1.1)–(1.3), (1.6) and (1.12), we obtain

$$I_1=\frac{1}{2\pi}\int_0^{1-r}\omega_1(u,\theta)\left|\frac{\partial^k P(r,\theta)}{\partial\theta^k}\right|d\theta \leq A_5\frac{\omega_1(1-r)(1-r)}{(1-r)^{k+1}}$$

$$\leq A_6\frac{1}{(1-r)^k}(1-r)\int_{1-r}^{\pi}\frac{\omega_n(u,t)}{t^2}dt \leq A_7\frac{\omega(1-r)}{(1-r)^k} \leq A_7\frac{\omega(1-r)}{(1-r)^n}. \tag{1.14}$$

In the same way we get

$$I_2=\frac{1}{2\pi}\int_{1-r}^{\pi}\omega_1(u,\theta)\left|\frac{\partial^k P(r,\theta)}{\partial\theta^k}\right|d\theta \leq A_8\int_{1-r}^{\pi}\frac{\omega_1\left(u,\frac{\theta}{1-r}(1-r)\right)}{\left((1-r)^2+\frac{4r}{\pi^2}\theta^2\right)^{\frac{k+1}{2}}}d\theta \leq$$

$$\leq A_9 \int\limits_{1-r}^{\pi} \frac{\omega_1(1-r)\left(\frac{\theta}{1-r}+1\right)}{\theta^{k+1}} d\theta \leq A_9 \frac{\omega_1(1-r)}{(1-r)} \int\limits_{1-r}^{\pi} \frac{\theta+(1-r)}{\theta^{k+1}} d\theta$$

$$\leq A_{10} \frac{\omega_1(1-r)}{1-r} \int\limits_{1-r}^{\pi} \frac{\theta}{\theta^{k+1}} d\theta \leq A_{11} \frac{\omega_1(1-r)}{(1-r)^k} \leq A_{12} \frac{\omega(1-r)}{(1-r)^n}. \quad (1.15)$$

Combining (1.14) and (1.15) from (1.13) we have

$$\left\| \frac{\partial^k u\left(re^{i\varphi}\right)}{\partial \varphi^k} \right\|_{L_p} \leq A_{13} \frac{\omega(1-r)}{(1-r)^n}. \quad (1.16)$$

Consider the case if k is even. Since $\int\limits_0^{\pi} \frac{\partial^k P(r,\theta)}{\partial \theta^k} d\theta = 0$ for $k = 2m$ and using the methods of Ryazanov (see the paper [13]) it follows that

$$\frac{\partial^k u\left(re^{i\varphi}\right)}{\partial \varphi^k} = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} u\left(e^{i\theta}\right) \frac{\partial^k P(r,\theta-\varphi)}{\partial \varphi^k} d\theta$$

$$= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} u\left(e^{i\theta}\right) \frac{\partial^k P(r,\theta-\varphi)}{\partial \theta^k} d\theta = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} u\left(e^{i(\varphi+\theta)}\right) \frac{\partial^k P(r,\theta)}{\partial \theta^k} d\theta$$

$$= \frac{1}{2\pi} \left(\int\limits_0^{\pi} u\left(e^{i(\varphi+\theta)}\right) \frac{\partial^k P(r,\theta)}{\partial \theta^k} d\theta - \int\limits_0^{-\pi} u\left(e^{i(\varphi+\theta)}\right) \frac{\partial^k P(r,\theta)}{\partial \theta^k} d\theta \right)$$

$$= \frac{1}{2\pi} \int\limits_0^{\pi} \left[u\left(e^{i(\varphi+\theta)}\right) + u\left(e^{i(\varphi-\theta)}\right) \right] \frac{\partial^k P(r,\theta)}{\partial \theta^k} d\theta$$

$$= \frac{1}{2\pi} \int\limits_0^{\pi} \left[u\left(e^{i(\varphi+\theta)}\right) - 2u\left(e^{i\varphi}\right) + u\left(e^{i(\varphi-\theta)}\right) \right] \frac{\partial^k P(r,\theta)}{\partial \theta^k} d\theta.$$

Now if we recall generalized Minkowski inequality, we get an inequality of type (1.11)

$$\left\|\frac{\partial^k u\left(re^{i\varphi}\right)}{\partial\varphi^k}\right\|_{L_p} \le \frac{1}{2\pi}\int_0^{\pi}\omega_2\left(u,\theta\right)\left|\frac{\partial^k P\left(r,\theta\right)}{\partial\theta^k}\right|d\theta$$

$$\le \frac{1}{\pi}\int_0^{\pi}\omega_1\left(u,\theta\right)\left|\frac{\partial^k P\left(r,\theta\right)}{\partial\theta^k}\right|d\theta.$$

Therefore from (1.16) we get

$$\left\|\frac{\partial^k u\left(re^{i\varphi}\right)}{\partial\varphi^k}\right\|_{L_p} \le A_{14}\frac{\omega\left(1-r\right)}{\left(1-r\right)^n}. \tag{1.17}$$

Combining (1.16) and (1.17) for any $k=\overline{1,n}$ we obtain

$$\left\|\frac{\partial^k u\left(re^{i\varphi}\right)}{\partial\varphi^k}\right\|_{L_p} \le A_{15}\frac{\omega\left(1-r\right)}{\left(1-r\right)^n}. \tag{1.18}$$

As above for $v\left(re^{i\varphi}\right)$ we have

$$\left\|\frac{\partial^k v\left(re^{i\varphi}\right)}{\partial\varphi^k}\right\|_{L_p} \le A_{16}\frac{\omega\left(1-r\right)}{\left(1-r\right)^n}. \tag{1.19}$$

Combining (1.8), (1.18) and (1.19), for $k=\overline{1,n}$ we get

$$\left\|\frac{\partial^k f\left(re^{i\varphi}\right)}{\partial\varphi^k}\right\|_{L_p} \le A_{17}\frac{\omega\left(1-r\right)}{\left(1-r\right)^n}. \tag{1.20}$$

Using (1.7) and (1.20) for all $\frac{1}{2}\le r<1$ finally, we obtain

$$\left\|f^{(n)}\left(z\right)\right\|_{L_p} \le A_{18}\frac{\omega\left(1-r\right)}{\left(1-r\right)^n}. \tag{1.21}$$

Since $f^{(n)}\left(z\right)$ is analytical function for $|z|\le\frac{1}{2}$, it follows that (1.21) also holds for $|z|\le\frac{1}{2}$, i.e. for all $0\le r<1$. This completes the proof. □

Remark 1.1. In his paper [10], R.M. Kovalchuk proved Theorem 1 for $n=2$. For analytical functions on domains with a quasi-conformal boundary (and hence for the harmonic functions on such domain) M.Z. Dveirin (see the paper [5]) obtained the estimate (1.5) for a uniform metric in the terms of Tamrazof's (see the monograph [14]) uniform modulus of smoothness.

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