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Strengthened Belinskii theorem and its applications

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Abstract. The remarkable theorem of P. Belinskii is a deep underlying result in the variational calculus for quasiconformal maps. It is valid only for maps with small sufficiently regular Beltrami coefficients.

We provide a global version of this theorem connected with the classical results on quasiconformal extensions of conformal maps and new its applications.

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1. Preamble

1.1. The following remarkable theorem of P. Belinskii is a deep underlying result in the variational calculus for quasiconformal maps.

Theorem A. [8] *Let a function $\mu(\zeta)$ be defined on the plane $\mathbb{C}$ and C^1-smooth, up to jumps on a finite number of closed smooth curves. Let*

$$|\mu(\zeta)| < \varepsilon, \ |\partial_\zeta \mu| < \varepsilon, \ |\partial_{\overline{\zeta}} \mu| < \varepsilon,$$

and let either $\mu(1/\zeta)$ or $(\zeta/\overline{\zeta})^2 \mu(1/\overline{\zeta})$ satisfy in a neighborhood of the point $\zeta = 0$ the same assumptions, as the function $\mu(\zeta)$ in the finite points. Then, for sufficiently small $\varepsilon > 0$, the function

$$w(z) = z - \frac{z(z-1)}{\pi} \iint\limits_{|\zeta|<\infty} \frac{\mu(\zeta) d\xi d\eta}{\zeta(\zeta-1)(\zeta-z)} \tag{1}$$

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provides a quasiconformal homeomorphism of the whole plane $\widehat{\mathbb{C}}$ whose Beltrami coefficient is $\widetilde{\mu} = \mu + O(\|\mu\|_\infty^2)$, and this map differs from the map with Beltrami coefficient $\mu(z)$ and the same normalization up to a quantity of order ε^2 uniformly in any bounded domain.

The original proof of this theorem is complicated and relates on the deep results from geometric function theory and from the potential theory. It involves only sufficiently smooth Beltrami coefficients μ with small norm and has been recently strengthened in [23] and applied to complex and potential geometry of the universal Teichmüller space.

This theorem relates to the problem of I.N. Vekua of 1961 on homeomorphy of approximate solutions of the singular two dimensional integral equation intrinsically connected with the Beltrami equation by constructing quasiconformal maps. Consider in the space $L_p(\mathbb{C})$ with $p > 2$ the well-known integral operators

$$T\rho(z) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{\rho(\zeta) d\xi d\eta}{\zeta - z}, \quad \Pi\rho(\zeta) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{\rho(\zeta) d\xi d\eta}{(\zeta - z)^2} = \partial_z T\rho(z)$$

assuming for simplicity that ρ has a compact support in $\mathbb{C}$. Then the second integral exists as a Cauchy principal value, and the derivative $\partial_z T$ generically is understanding as distributional.

Each quasiconformal automorphism w^μ with $\|\mu\|_\infty = k < 1$ of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$ with $\|\mu\|_\infty = k < 1$ is represented in the form $w^\mu(z) = z + T\rho(z)$, where ρ is the solution in L_p (for $2 < p < p_0(k)$) of the integral equation

$$\rho = \mu + \mu\Pi\rho,$$

given by the series

$$\rho = \mu + \mu\Pi\mu + \mu\Pi\mu(\Pi\mu) + \dots. \tag{2}$$

Denote by μ_n be the n-th partial sum of the series (2), and set

$$f_n(z) = z - \frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{\mu_n(\zeta) d\xi d\eta}{\zeta - z}.$$

The question of Vekua was, whether all f_n also are homeomorphisms.

A counterexample of T. Iwaniec shows that the smoothness and smallness assumptions in the Belinskii theorem cannot be dropped completely. A simple modification of his construction allows us to define $\varepsilon \in (0, 1)$ and a Beltrami coefficient μ, so that the second iteration

$$f_2(z) = z + T\mu(z) + T(\mu\Pi\mu)(z)$$

is not injective in $\mathbb{D}$. The details are exposed in survey [21].

1.2. Decomposing any quasiconformal automorphism w^μ of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$ via $w^\mu = w^{\mu_2} \circ w^{\mu_1}$ with the Beltrami coefficients μ_1 and μ_2 supported, respectively, in the unit disk $\mathbb{D} = \{|z| < 1\}$ and in the domain $D_2 = \widehat{\mathbb{C}} \setminus w^{\mu_1}(\mathbb{D})$, one naturally arrives to univalent holomorphic functions having quasiconformal extensions. Such functions play a crucial role in geometric complex analysis and in Teichmüller space theory.

Let us mention also that if $\mu(z) \in C^{m+\sigma}$ with $m \geq 0,\ \sigma > 0$, then $T\rho \in C^{m+1+\sigma}$ and $\Pi\rho \in C^{m+\sigma}$ (see, e.g., [38], Ch. 1).

1.3. First we provide a simplified proof of the Belinskii theorem, illustrating the idea on the case of Beltrami coefficients vanishing near the infinity. For $\mu \in C^\sigma$, we have

$$\partial_{\overline{z}} w = \mu, \quad \partial_z w = 1 - \Pi\mu. \tag{3}$$

Hence, the Jacobian of the map (1) equals

$$J_w(z) = |\partial_z w(z)|^2 - |\partial_{\overline{z}} w(z)|^2 = 1 - O(\varepsilon),$$

which yields, together with the equality $w(z) = z + O(1/z)$ in a neighborhood of the infinite point, that $w(z)$ is locally injective on $\widehat{\mathbb{C}}$ and hence a global homeomorphism of $\widehat{\mathbb{C}}$.

To describe its quasiconformal features, we pass to Beltrami coefficients

$$\mu_t = t\mu/\|\mu\|_\infty,$$

getting the map

$$w_t(z) = z - \frac{t\zeta(z-1)}{\pi} \iint\limits_{|\zeta|<\infty} \frac{\mu(\zeta) d\xi d\eta}{\zeta(\zeta-1)(\zeta-z)},$$

which determines a holomorphic motion $w(z,t) = w_t(z)$ of the Riemann sphere $\widehat{\mathbb{C}}$ (which means that $w(z,t)$ is injective for a fixed t, the function $t \mapsto w(\cdot, t)$ is holomorphic, and $w(z, 0) \equiv z$ on $\widehat{\mathbb{C}}$). By the lambda-lemma for holomorphic motions, each fiber map $w_t(z)$ is a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ whose Beltrami coefficient

$$\mu(z,t) = \partial_{\overline{z}} w(z,t) / \partial_z w(z,t)$$

is a holomorphic function of t as the element from $L_i y(\mathbb{C})$ (see, e.g., [13]). Hence, for admissible (small) t,

$$\mu(z,t) = \mu_1(z)t + \mu_2(z)t^2 + \ldots, \tag{4}$$

and by Schwarz's lemma,

$$\|\mu(\cdot, t) - \mu_1(\cdot)t\|_\infty \le \text{const}\, |t|^2. \tag{5}$$

It remains to observe that in view of (3) $\mu_1 = \mu/\|\mu\|_\infty$, which completes the proof.

2. Main theorem

2.1. As was mentioned above, the aim of this paper is to weaken as possible the rigid assumption that the L_∞-norms of μ and of its first derivatives must be small and provide a global version of this theorem.

Consider the canonical class $\Sigma(0)$ of univalent functions $f(z) = z + b_0 + b_1 z^{-1} + \dots$ in the disk $\mathbb{D}^* = \{z \in \widehat{\mathbb{C}} : |z| > 1\}$ having quasiconformal extension across the unit circle $\mathbf{S}^1$ onto the unit disk $\mathbb{D} = \{|z| < 1\}$. Their Beltrami coefficients run over the unit ball

$$\text{Belt}(\mathbb{D})_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu(z)|\mathbb{D}^* = 0, \ \ \|\mu\|_\infty < 1\}.$$

In accordance with such a normalization, we shall deal with integral

$$w(z) = z - \frac{1}{\pi} \iint\limits_{|\zeta|<1} \mu(\zeta)\Big(\frac{1}{\zeta - z} - \frac{1}{\zeta}\Big) d\xi d\eta. \tag{6}$$

Let $\mu(z, t)$ belong to $\text{Belt}(\mathbb{D})_1$ and for any fixed t be $C^{1+\sigma}$-smooth in z as a function $\mathbb{D} \times \mathbb{D} \to \mathbb{C}$ ($\sigma > 0$), and let μ be holomorphic in t for a fixed z and $\mu(z, 0) \equiv 0$.

Then the well-known property of elements in the functional spaces with sup norms implies that the function

$$t \mapsto \mu(z, t) : \ \mathbb{D} \to \text{Belt}(\mathbb{D})_1$$

is holomorphic in t also in L_∞-norm. This property is based on the following lemma of Earle [11].

Lemma 1. *Let E, T be open subsets of complex Banach spaces X, Y and $B(E)$ be a Banach space of holomorphic functions on E with sup norm. If $\phi(x, t)$ is a bounded map $E \times T \to B(E)$ such that $t \mapsto \phi(x, t)$ is holomorphic for each $x \in E$, then the map ϕ is holomorphic.*

Holomorphy of $\phi(x, t)$ in t for fixed x implies the existence of complex directional derivatives

$$\phi'_t(x, t) = \lim_{\zeta \to 0} \frac{\phi(x, t + \zeta v) - \phi(x, t)}{\zeta} = \frac{1}{2\pi i} \int\limits_{|\xi|=1} \frac{\phi(x, t + \xi v)}{\xi^2} d\xi,$$

while the boundedness of ϕ in sup norm provides the uniform estimate

$$\|\phi(x, t + c\zeta v) - \phi(x, t) - \phi_t'(x, t)cv\|_{B(E)} \le M|c|^2,$$

for sufficiently small $|c|$ and $\|v\|_Y$.

Therefore, we have similar to (4) the expansion

$$\mu(z, t) = \mu_1(z)t + \mu_2(z)t^2 + \dots, \quad (z \in \mathbb{D}^*, \ t \in \mathbb{D}), \tag{7}$$

and the uniform estimates for all $|t| < 1$,

$$\|\mu(\cdot, t)\|_\infty \le |t|, \quad \|\mu(\cdot, t) - \mu_1\|_\infty \le \text{const}\, |t|^2.$$

For $\mu \in \mathrm{Belt}(\mathbb{D})_1$ and $\varphi \in L_1(\mathbb{D})$, we define the pairing

$$\langle \mu, \varphi \rangle_{\mathbb{D}} = -\frac{1}{\pi} \iint\limits_{\mathbb{D}} \mu(\zeta)\varphi(\zeta) d\xi d\eta \quad (\zeta = \xi + i\eta)$$

and call μ infinitesimally trivial, if

$$\langle \mu, \varphi \rangle_{\mathbb{D}} = 0 \quad \text{for all} \quad \varphi \in A_1(\mathbb{D}),$$

where $A_1(\mathbb{D})$ denotes the subspace of $L_1(\mathbb{D})$ formed by integrable holomorphic functions.

We prove the following general theorem giving a global extension of Theorem A.

Theorem 1. *Let $\mu(z, t) \in \mathrm{Belt}(\mathbb{D})_1$ be $C^{1+\sigma}$-smooth in z from a broader disk $\mathbb{D}_d$, $d > 1$, and holomorphic in t and let $\mu(z, 0) \equiv 0$. Suppose that the linear term $\mu_1(z)t$ in expansion* (7) *for $\mu(z, t)$ is not infinitesimally trivial. Then:*

(a) For any t, for which,

$$(|z|^2 - 1)\frac{1}{\pi}\Big|z \iint\limits_{\mathbb{D}} \frac{\mu(\zeta, t) d\xi d\eta}{(\zeta - z)^3}\Big| \le \Big|1 + \frac{1}{\pi} \iint\limits_{\mathbb{D}} \frac{\mu(\zeta, t) d\xi d\eta}{(\zeta - z)^2}\Big|, \tag{8}$$

or equivalently, denoting by $w_t(z)$ the integral (6) *with $\mu = \mu(z, t)$,*

$$(|z|^2 - 1)|z||w_t''(z)| \le |w_t'(z)| \quad \textit{for all} \ \ \zeta \in \overline{\mathbb{D}^*},$$

the integral (6) *for $\mu(z, t)$ provides a homeomorphism of the sphere $\widehat{\mathbb{C}}$, conformal on $\mathbb{D}^*$.*

(b) Under a stronger assumption

$$(|z|^2 - 1)|z||w_t''(z)| \le k|w_t'(z)|, \quad z \in \mathbb{D}^*, \tag{9}$$

with some $k < 1$, the function $w^{\mu(\cdot,t)}|\mathbb{D}^*$ given by (6) admits k-quasiconformal extension across $\mathbf{S}^1$ onto the unit disk with Beltrami coefficient

$$\widetilde{\mu}(z,t) = \Big(\frac{1}{|z|^2} - 1\Big)\frac{1}{\overline{z}}\frac{w_t''(1/\overline{z})}{w_t'(1/\overline{z})}, \quad |z| < 1. \tag{10}$$

This coefficient depends holomorphically on μ and t.

2.2. The following theorem is a special case of Theorem 1, but it has an independent interest, in view of applications.

Theorem 2. *Suppose that $\mu \in Belt(\mathbb{D})_1$ is $C^{1+\sigma}$-smooth (in L_∞ norm) and is not infinitesimally trivial. Then the integral*

$$w^{t\mu}(z) = z - \frac{t}{\pi}\iint\limits_{\mathbb{D}} \mu(\zeta)\Big(\frac{1}{\zeta - z} - \frac{1}{\zeta}\Big)d\xi d\eta \tag{11}$$

provides k-quasiconformal map of $\widehat{\mathbb{C}}$, conformal on $\mathbb{D}^*$, for all $|t| > 0$ satisfying

$$(|z|^2 - 1)\frac{t}{\pi}\Big|z \iint\limits_{\mathbb{D}} \frac{\mu(\zeta)d\xi d\eta}{(\zeta - z)^3}\Big| \le \Big|1 + \frac{t}{\pi}\iint\limits_{\mathbb{D}} \frac{\mu(\zeta,t)d\xi d\eta}{(\zeta - z)^2}\Big|.$$

Note that generically the quasiconformal extensions given by either of these theorems (with the Beltrami coefficients of the form (10)) are not extremal (i.e., do not have the minimal dilatation among the posible extensions of $w^{t\mu}$ to $\mathbb{D}$) for all admissible t.

3. Some integral bounds

We start with the following lemma from [18]:

Lemma 2. *Let D be a domain in $\widehat{\mathbb{C}}$ having at least two boundary points. Then for any univalent, holomorphic in D, function $f(z)$ ($\widehat{\mathbb{C}}$-holomorphic if $D \ni \infty$) in all the finite points $z \in D$ there holds the inequalities*

$$\Big|\frac{f^{(n)}(z)}{f'(z)}\Big| \le \frac{n\, n!}{r_D(z)^{n-1}}, \quad n = 2, 3, \dots, \tag{12}$$

where $r_D(z) = \mathrm{dist}(z, \partial D)$ is the Euclidean from the point $z \in D$ to the boundary ∂D.

This yields, in particular, the following uniform bound for the integral (6):

$$|w''(z)/w'(z)| \le O(1/(|z|-1)) \quad \text{for all} \quad |z| > 1,$$

which allows us to apply the assumptions of Theorems 1 and 2.

We also shall use the well-known identity (see, e.g., [3])

$$\iint\limits_{\mathbb{D}} \frac{d\xi d\eta}{|1-\overline{\zeta}Z|^4} = \frac{\pi}{(1-|Z|^2)^2}$$

which can be rewritten after the change $Z \mapsto z = 1/Z \in \mathbb{D}^*$ in the form

$$\frac{1}{\pi} \iint\limits_{\mathbb{D}} \frac{d\xi d\eta}{|\zeta - z|^4} = \frac{1}{(|z|^2-1)^2}.$$

This identity simply provides (after integration or differentation of both sides in $r = |z|$) the estimate

$$\frac{1}{\pi} \iint\limits_{\mathbb{D}} \frac{d\xi d\eta}{|\zeta - z|^m} = O(1/(|z|^2-1)^{m-2}). \tag{13}$$

4. Proof of Theorem 1

First observe that, in view of assumptions of the theorem, the value $t = 0$ is noncritical for μ.

We shall use Becker's model of the universal Teichmüller space $\mathbf{T}$ (see [7]). In this model, $\mathbf{T}$ is a bounded domain $\mathbf{b}(\mathbf{T})$ in the Banach space $\mathbf{B}_1(\mathbb{D}^*)$ of holomorphic functions ψ on the disk $\mathbb{D}^*$ (containing the origin $\psi = \mathbf{0}$) with the norm

$$\|\psi\| = \sup_{\mathbb{D}^*} (|z|^2 - 1)|z\psi(z)|.$$

This domain is filled by the functions $\mathbf{b}_f = f''/f'$ corresponding to $f \in \Sigma(0)$.

Every $\mu \in \mathrm{Belt}(\mathbb{D})_1$ defines a unique quasiconformal automorphism w^μ of $\widehat{\mathbb{C}}$ as the homeomorphic solution of the Beltrami equation $\partial_{\overline{z}} w = \mu(z)\partial_z w$ on $\mathbb{C}$, whose conformal restriction to $\mathbb{D}^*$ belongs to $\Sigma(0)$.

The space $\mathbf{T}$ is obtained from the ball $\mathrm{Belt}(\mathbb{D})_1$, letting $\mu_1, \ \mu_2 \in \mathrm{Belt}(\mathbb{D})_1$ be equivalent, if the corresponding homeomorphisms w^{μ_1} and w^{μ_1} coincide on the unit circle $\mathbf{S}^1$ (and hence, on $\overline{\mathbb{D}^*}$). The quotient map

$$\phi_{\mathbf{T}}(\mu) = \mathbf{b}_{w^\mu} : \ \mathrm{Belt}(\mathbb{D})_1 \to \mathbf{T} \subset \mathbf{B}_1(\mathbb{D}^*) \tag{14}$$

is holomorphic.

All this implies that the image of the given Beltrami coefficients $\mu(z,t)$ in $\mathbf{T}$ under the map (14) is a non-degenerate holomorphic disk

$$\mathbb{D}(\mu) = \{t\mu : \ |t| < 1\}$$

filled by the points $\mathbf{b}_{w^{\mu(\cdot,t)}}$. Its non-degenerance is a consequence of Theorem A, which provides that for small $|t|$ the points $\mathbf{b}_{w^{\mu(\cdot,t)}}$ are different, and of the uniqueness theorem for holomorphic functions. So, the map $t \mapsto \mathbf{b}_{w^{\mu(\cdot,t)}}$ can have only a countable set of zeros t_n, which determine the cuspidal singularities of the disk $\mathbb{D}(\mu)$.

The following important lemma is a special case of the Ahlfors and Becker results on univalence and quasiconformal extension (see [4, 7]).

Lemma 3. *Let $f(z) = z + b_0 + b_1 z^{-1} + \dots$ be holomorphic on $\mathbb{D}^*$, and let $k \le 1$. If*

$$(|z|^2 - 1)\Big| z\frac{f''(z)}{f'(z)}\Big| \le k \quad \textit{for all} \ \ |z| > 1, \tag{15}$$

then $f(z)$ is univalent on $\mathbb{D}^*$ and for $k < 1$ admits k-quasiconformal extension to $\widehat{\mathbb{C}}$ with complex dilatation

$$\mu(z) = \Big(\frac{1}{|z|^2} - 1\Big)\Big|\frac{1}{\overline{z}}\frac{f''(1/\overline{z})}{f'(1/\overline{z})}\Big|, \quad |z| < 1.$$

We may now prove the assertions of the theorem. In view of assumptions on the linear term $t\mu_1$, we may apply Theorem A, which yields that for a sufficiently small $|t| > 0$ the integral

$$w^{\mu(\cdot,t)}(z) = z - \frac{1}{\pi}\iint\limits_{|\zeta|<1} \mu(\zeta,t)\Big(\frac{1}{\zeta - z} - \frac{1}{\zeta}\Big) d\xi d\eta$$

provides quasiconfomal (hence, injective) automorphism of the sphere $\widehat{\mathbb{C}}$, conformal on the disk $\mathbb{D}^*$, whose Beltramu coefficient $\widetilde{\mu}(z,t)$ in $\mathbb{D}$ satisfies

$$\widetilde{\mu}(z,t) = \mu(z,t) + O(|t|^2),$$

and the remainder is uniform for $|t| < t_0$. Together with nontriviality of the linear term $t\mu_1$, this yields that the image $\mathbb{D}(\mu)$ of this μ under the projection (14) does not degenerate to a point (i.e., the function $\mathbf{b}_{w^{\mu(\cdot,t)}}$ does not be equal identically to zero); it is a holomorphic disk in $\mathbf{T}$ with possible discrete singularities.

For $\mu(z,\cdot) \in C^{1+\sigma}$, we have the growth

$$\iint\limits_{\mathbb{D}} \mu(\zeta,t)(\zeta - z)^{-3} d\xi d\eta = O((1-r)^{\sigma-1}) \quad \text{as} \quad r \to 1.$$

Hence, if $\mu(z,t)$ satisfies the inequality (9), then the corresponding integral (6) satisfies for $z \in \mathbb{D}^*$ and the indicated t the inequality (15) of Lemma 3. Then this lemma implies that the holomorphic function $w_t(z)$ given by this integral has k-quasiconformal extension $w^{\widetilde{\mu}(\cdot,t)}$ onto the disk $\mathbb{D}$.

It remains to show that the coefficient $\widetilde{\mu}(z,t)$ of the extension $w^{\widetilde{\mu}}|\mathbb{D}^*$ to $\widehat{\mathbb{C}}$ depends holomorphically on the original coefficient $\mu(z,t)$ and on the complex parameter $t \in \mathbb{D}$ defining $\mu(z,t)$.

Holomorphy in μ follows trivially from (10), while holomorphy in t is a consequence of holomorphy of both maps $t \mapsto \mu(z,t)$ and the quotient projection (14).

The case of assumption (8) is investigated in a similar way. Now the application of Lemma 3 only provides that the map given by integral (6) is a homeomorphism of $\widehat{\mathbb{C}}$. This completes the proof of Theorem 1.

5. Applications

5.1. Preliminaries. We provide here the applications of Theorem 1 to the Grunsky inequalities and Fredholm eigenvalues. Let us first we recall briefly some needed results in order to formulate the theorems.

The classical Grunsky theorem yields that a holomorphic function $f(z) = z + \text{const} + O(z^{-1})$ in a neighborhood of $z = \infty$ can be extended to a univalent holomorphic function on the $\mathbb{D}^*$ if and only

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| \le 1,$$

where the numbers α_{mn}, called the Grunsky coeffcients of f, are defined from the series

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z,\zeta) \in (\mathbb{D}^*)^2,$$

the sequence $\mathbf{x} = (x_n)$ runs over the unit sphere $S(l^2)$ of the Hilbert space l^2 with norm $\|\mathbf{x}\|^2 = \sum\limits_1^\infty |x_n|^2$, and the principal branch of the logarithmic function is chosen (cf. [15, 31]). The quantity

$$\varkappa(f) = \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| : \ \mathbf{x} = (x_n) \in S(l^2) \Big\} \le 1$$

is called the **Grunsky norm** of f.

For the functions with k-quasiconformal extensions ($k < 1$), we have a stronger bound

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| \le k \quad \text{for any} \ \ \mathbf{x} = (x_n) \in S(l^2), \tag{16}$$

established first in [28] (see also [25]). Then $\varkappa(f) \le k(f)$, where $k(f)$ denotes the **Teichmüller norm** of f which is equal to the infimum of dilatations $k(w^\mu) = \|\mu\|_\infty$ of quasiconformal extensions w^μ of f to $\widehat{\mathbb{C}}$. For most functions f, we have the strong inequality $\varkappa(f) < k(f)$ (moreover, the functions satisfying this inequality form a dense subset of Σ; see, [23, 27]), while the functions with the equal norms play a crucial role in many applications.

The Grunsky coefficients $\alpha_{mn}(f^\mu)$ of the functions $f^\mu \in \Sigma(0)$ generate for each $\mathbf{x} = (x_n) \in l^2$ with $\|\mathbf{x}\| = 1$ the holomorphic maps

$$h_{\mathbf{x}}(S_f) = \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn}(S_f)\ x_m x_n : \ \mathbf{T} \to \mathbb{D} \tag{17}$$

with fixed $\mathbf{x} = (x_n) \in l^2$ with $\|\mathbf{x}\| = 1$ so that $\sup_{\mathbf{x} \in S(l^2)} h_{\mathbf{x}}(S_f) = \varkappa(f)$. Here S_f denote the Schwarzian derivatives

$$S_f(z) = \Big(\frac{f''(z)}{f'(z)}\Big)' - \frac{1}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2 \quad z \in \mathbb{D}^*,$$

which form the canonical model of the space $\mathbf{T}$.

The holomorphy of these maps follows from the holomorphy of coefficients α_{mn} with respect to Beltrami coefficients $\mu \in \mathrm{Belt}(D)_1$ and to Schrazians mentioned above using the estimate

$$\Big| \sum_{m=j}^{M} \sum_{n=l}^{N} \alpha_{mn} x_m x_n \Big|^2 \le \sum_{m=j}^{M} |x_m|^2 \sum_{n=l}^{N} |x_n|^2$$

which holds for any finite M, N and $1 \le j \le M, \ 1 \le l \le N$ (cf. [33], p. 61).

The Grunsky inequalities are intrinsically connected with quasiconformal reflections across quasicircles, Fredholm eigenvalues of the Jordan curves and other quasiinvariants of curves; see, e.g., [23, 25, 30]. These inequalities have been generalized in several directions, even to bordered Riemann surfaces X with a finite number of boundary components.

Consider in the space $A_1(\mathbb{D})$ of integrable holomorphic quadratic differentials on $\mathbb{D}$ its subset

$$A_1^2 = \{\psi \in A_1(\mathbb{D}) : \ \psi = \omega^2, \ \omega \text{ holomorphic}\}$$

which consists of abelian quadratic differentials having in $\mathbb{D}$ only zeros of even order, and put

$$\langle \mu, \psi \rangle_{\mathbb{D}} = \iint_D \mu(z)\psi(z)dxdy, \quad \mu \in L_\infty(\mathbb{D}), \ \psi \in L_1(\mathbb{D}) \ \ (z = x + iy).$$

Given a function $f \in \Sigma_(0)$, we take its extremal quasiconformal extension f^{μ_0} to $\mathbb{D}$ with Beltrami coefficient $\mu_0 \in L_\infty(\mathbb{D})$ (hence, $k(f) = \|\mu_0\|_\infty$) and assign to this function the quantity

$$\alpha_{\mathbb{D}} = \sup\{|\langle \mu_0, \psi \rangle|_{\mathbb{D}} : \ \psi \in A_1^2(\mathbb{D}), \ \|\psi\|_{A_1(\mathbb{D})} = 1\}.$$

Due to [19, 25], the Grunsky norm $\varkappa(f)$ of every function $f \in \Sigma(0)$ is estimated by its Teichmüller norm $k = k(f)$ and the quantity $\alpha(\mathbb{D})$ via

$$\varkappa(f) \le k \frac{k + \alpha_{\mathbb{D}}(f)}{1 + \alpha_{\mathbb{D}}(f)k},$$

and $\varkappa(f) < k$ unless $\alpha_{\mathbb{D}}(f) = \|\mu_0\|_\infty$. The last equality occurs if and only if $\varkappa(f) = k(f)$.

The following important result from [20, 24] implies that the Grunsky norm and its generalization to univalent functions on quasidisks is lower semicontinuous in the weak topology (of locally uniform convergence) on $\Sigma(0)$, and it is locally Lipschitz continuious with respect to Teichmüller metric.

Lemma 4. *(i) If a sequence $\{f_n\} \subset \Sigma(0)$ is convergent locally uniformly on $\mathbb{D}^*$ to f_0, then*

$$\varkappa(f_0) \le \liminf_{n \to \infty} \varkappa(f_n).$$

(ii) The functional $\varkappa(\varphi)$ regarded as a function of points $\varphi = S_f$ from the universal Teichmüller space $\mathbf{T}$ is locally Lipschitz continuous and logarithmically plurisubharmonic on $\mathbf{T}$.

The Teichmüller norm has similar properties. Its continuity and plurisubharmonicity is a consequence, for example, of the following result strengthening of the fundamental Royden–Gardiner theorem.

Lemma 5. [20] *The differential (infinitesimal) Kobayashi metric $\mathcal{K}_{\mathbf{T}}(\varphi, v)$ on the tangent bundle $\mathcal{T}(\mathbf{T})$ of the universal Teichmüller space $\mathbf{T}$ is logarithmically plurisubharmonic in $\varphi \in \mathbf{T}$, equals the canonical Finsler structure $F_{\mathbf{T}}(\varphi, v)$ on $\mathcal{T}(\mathbf{T})$ generating the Teichmüller metric of $\mathbf{T}$ and has constant holomorphic sectional curvature $\kappa_{\mathcal{K}}(\varphi, v) = -4$ on $\mathcal{T}(\mathbf{T})$.*

The proof of these lemmas essentially involves the holomorphy of functions (17) generated by the Grunsky coefficients.

Subharmonicity allows one to apply the maximum principle for estimating the distortion of functionals depending on the Teichmüller and Grunsky norms.

The **generalized Gaussian curvature** κ_λ of an upper semicontinuous Finsler metric $ds = \lambda(t)|dt|$ in a domain $\Omega \subset \mathbb{C}$ is defined by

$$\kappa_\lambda(t) = -\frac{\Delta \log \lambda(t)}{\lambda(t)^2}, \tag{18}$$

where Δ is the **generalized Laplacian**

$$\Delta\lambda(t) = 4 \liminf_{r \to 0} \frac{1}{r^2} \Big\{ \frac{1}{2\pi} \int_0^{2\pi} \lambda(t + re^{i\theta}) d\theta - \lambda(t) \Big\}$$

(provided that $-\infty \le \lambda(t) < \infty$). Similar to C^2 functions, for which Δ coincides with the usual Laplacian, one obtains that λ is subharmonic on Ω if and only if $\Delta\lambda(t) \ge 0$; hence, at the points t_0 of local maximuma of λ with $\lambda(t_0) > -\infty$, we have $\Delta\lambda(t_0) \le 0$.

The sectional **holomorphic curvature** of a Finsler metric on a complex Banach manifold X is defined in a similar way as the supremum of the curvatures (18) over appropriate collections of holomorphic maps from the disk into X for a given tangent direction in the image. The holomorphic curvature of the Kobayashi metric $\mathcal{K}_X(x, v)$ of any complete hyperbolic manifold X satisfies $\kappa_{\mathcal{K}} \ge -4$ at all points (x, v) of the tangent bundle $\mathcal{T}(X)$ of X (while for the Carathéodory metric $\mathcal{C}_X$ we have $\kappa_{\mathcal{C}}(x, v) \le -4$ (cf., e.g., [1, 10, 16]).

It was established in [12] that the metric $\mathcal{K}_{\mathbf{T}}(\varphi, v) = F_{\mathbf{T}}(\varphi, v)$ (the basic Finsler structure on $\mathbf{T}$) is Lipschitz continuous on $\mathbf{T}$ (in its Bers' embedding).

Recall also that a conformal metric $\lambda_0(t)|dt|$ is called **supporting** for $\lambda(t)|dt|$ at a point t_0 if $\lambda_\varkappa(t_0) = \lambda_0(t_0)$ and $\lambda_0(t) < \lambda_\varkappa(t)$ for all $t \in U(t_0) \setminus \{t_0\}$ from a neighborhood $U(t_0)$ of t_0.

5.2. A general theorem. The following theorem shows that any regular with respect to complex parameter t perturbation $\nu(z, t) \in \mathrm{Belt}(\mathbb{D})_1$

with $\nu(z,t) = O(t^2)$ does not disturb the equality of the Teichmüller and Grunsky norms; in other words, the holomorphic disks in $\mathbf{T}$ tangent to an extremal disk inherit this equality.

Theorem 3. *Let $\mu_0 \in \mathrm{Belt}(\mathbb{D})_1$ be extremal in its equivalence class $[\mu_0]$, that means among μ for which $w^\mu(z)|_{\mathbf{S}^1} = w^{\mu_0}(z)|_{\mathbf{S}^1}$, and let*

$$\varkappa(w^{\mu_0}) = k(w^{\mu_0}).$$

Take any $\nu(z,t) \in \mathrm{Belt}(\mathbb{D})_1$ holomorphic in $t \in \mathbb{D}$ and such that

$$\nu(z,t) = \mu_m(z)t^m + \mu_{m+1}(z)t^{m+1} + \dots, \quad m \ge 2,$$

and set

$$\mu_t(z) = t\mu_0(z)/\|\mu_0\|_\infty + \nu(z,t).$$

Then for all $t \in \mathbb{D}$, we have the equality $\varkappa(w^{\mu_t}) = k(w^{\mu_t})$.

In particular, for small $|t|$,

$$\varkappa(w^{\mu_t}) = k(w^{\mu_t}) = |t| + O(|t|^2). \tag{19}$$

Theorem 1 provides additionally that **sufficiently regular** maps w^{μ_t} are presented by integral (11). Some special results of such type are obtained in [26].

Proof. Since the integral (11) determines a quasiconformal (homeomorphic) map for all t running over some simply connected subdomain $D_k \subset \mathbb{D}$ for which the inequality (9) is valid, we can use the Grunsky coefficients of the restrictions $w^{\mu(\cdot,t)}|D^*$ and construct for these maps the corresponding holomorphic functions (17). This yields a collection of holomorphic maps

$$h_{\mathbf{x}}(t) = \sum_{m,n=1}^{\infty} \alpha_{mn}(S_{w_\varphi(\cdot;t)})\, x_m x_n$$

from the indicated domain D_k into the unit disk $\mathbb{D}$ parametrized by points $\mathbf{x} = (x_n) \in S(l^2)$.

Using these maps, we pull back the hyperbolic metric $\lambda_{\mathbb{D}}(t)|dt| = |dt|/(1-|t|^2)$ of the disk $\mathbb{D}$ onto domain D_k , getting on this domain the conformal metrics $\lambda_{h_{\mathbf{x}}}(t)|dt|$ with

$$\lambda_{h_{\mathbf{x}}}(t) = |h'_{\mathbf{x}}(t)|/(1-|h_{\mathbf{x}}(t)|^2)$$

of Gaussian curvature -4 at noncrical points. We take the upper envelope of these metrics

$$\widetilde{\lambda}_{\varkappa}(t) = \sup\{\lambda_{h_{\mathbf{x}}}(t): \ \mathbf{x} \in S(l^2)\}$$

and its upper semicontinuous regularization $\lambda_{\varkappa}(t) = \limsup_{t' \to t} \widetilde{\lambda}_{\varkappa}(t')$, which provides a logarithmically subharmonic metric on the indicated domain D_k.

Lemma 6. *The metric $\lambda_{\varkappa}$ has at any its noncritical point t_0, a supporting subharmonic metric λ_0 of Gaussian curvature at most -4, hence $\kappa_{\lambda_{\varkappa}} \le -4$.*

Proof. Since the space $\mathbf{B}(D^*)$ is dual to $A_1(D^*)$, the sequences $\lambda_{h_{\mathbf{x}}}$ are convergent, by the Alaoglu–Bourbaki theorem, in weak* topology to holomorphic functions $\mathbf{T} \to \mathbb{D}$. This yields that the metric $\lambda_{\varkappa}(t)$ has a supporting metric $\lambda_0(t)$ in a neighborhood of any noncritical point $t_0 \in G$, which means that $\lambda_{\varkappa}(t_0) = \lambda_0(t_0)$ and $\lambda_0(t) < \lambda_{\varkappa}(t)$ for $t \ne t_0$ close to t_0. Hence, for sufficiently small $r > 0$,

$$\frac{1}{2\pi} \int\limits_0^{2\pi} \log \lambda_{\varkappa}(t_0 + re^{i\theta}) d\theta - \lambda_{\varkappa}(t_0) \ge \frac{1}{2\pi} \int\limits_0^{2\pi} \log \lambda_0(t_0 + re^{i\theta}) d\theta - \lambda_0(t_0),$$

which implies $\Delta \log \lambda_{\varkappa}(t_0) \ge \Delta \log \lambda_0(t_0)$, and since $\lambda_{\varkappa}(t_0) = \lambda_0(t_0)$,

$$-\frac{\Delta \log \lambda_{\varkappa}(t_0)}{\lambda_{\varkappa}(t_0)^2} \le -\frac{\Delta \log \lambda_0(t_0)}{\lambda_0(t_0)^2} \le -4,$$

completing the proof of the lemma.

The inequality $\kappa_{\lambda} \le -4$ is equivalent to

$$\Delta \log \lambda \ge 4\lambda^2,$$

or $\Delta u \ge 4e^{2u}$ letting $u = \log \lambda$ (here Δ again means the generalized Laplacian).

Note that the constructed metric $\lambda_{\varkappa}$ is a restriction to the domain D_k (naturally embedded into $\mathbf{T}$) of a plurisubharmonic Finsler metric on this space generated by the Grunsky coefficients (see [24]).This structure is dominated by the infinitesimal Kobayashi-Teichmüller metric $\lambda_{\mathcal{K}}(\varphi, v)$ of the space $\mathbf{T}$.

Our goal now is to establish that on the domain D_k these metrics must coincide, i.e.,

$$\lambda_{\varkappa}(t) = \lambda_{\mathcal{K}}(S_{w_{\varphi}(\cdot;t)}, v), \quad t \in D_k. \tag{20}$$

This is obtained by applying Minda's maximum principle given by

Lemma 7. [32] *If a function* $u : \ \Omega \to [-\infty, +\infty)$ *is upper semicontinuous in a domain* $\Omega \subset \mathbb{C}$ *and its generalized Laplacian satisfies the inequality* $\Delta u(z) \ge K u(z)$ *with some positive constant* K *at any point* $z \in \Omega$*, where* $u(z) > -\infty$*, and if*

$$\limsup_{z \to \zeta} u(z) \le 0 \quad \textit{for all } \zeta \in \partial\Omega,$$

then either $u(z) < 0$ *for all* $z \in \Omega$ *or else* $u(z) = 0$ *for all* $z \in \Omega$*.*

For a sufficiently small neighborhood U_0 of the origin $t = 0$ in $\mathbb{D}_{r_0}$ put

$$M = \{\sup \lambda_{\mathcal{K}}(t) : t \in U_0\};$$

then in this neighborhood, $\lambda_{\mathcal{K}}(t) + \lambda_{\varkappa}(t) \le 2M$. Consider the function

$$u = \log \frac{\lambda_{\varkappa}}{\lambda_{\mathcal{K}}}.$$

Then (cf. [10, 32]) for $t \in U_0$,

$$\Delta u(t) = \Delta \log \lambda_{\varkappa}(t) - \Delta \log \lambda_{\mathcal{K}}(t) = 4(\lambda_{\varkappa}^2 - \lambda_{\mathcal{K}}^2) \ge 8M(\lambda_{\varkappa} - \lambda_{\mathcal{K}}).$$

The elementary estimate

$$M \log(t/s) \ge t - s \quad \text{for} \ \ 0 < s \le t < M$$

(with equality only for $t = s$) implies that

$$M \log \frac{\lambda_{g_0}(t)}{\lambda_d(t)} \ge \lambda_{g_0}(t) - \lambda_d(t),$$

and hence,

$$\Delta u(t) \ge 4M^2 u(t).$$

Now note that the infinitesimal forms of both Grunsky and Teichmüller norms are equal on the Teichmüller disk $\mathbb{D}(\psi) = \{t|\psi|/\psi\}$ and that this disk is tangent at the origin to the image of domain D_k in the space $\mathbf{T}$, because

$$\|w^{\mu_t} - w^{t\mu_0/\|\mu_0\|}\|_\infty = O(|t|^2),$$

and hence the corresponding Schwarzians satisfy

$$\|S_{w^{\mu_t}} - S_{w^{t\mu_0/\|\mu_0\|}}\|_{\mathbf{B}} = O(|t|^2) \to 0 \quad \text{as} \ \ t \to 0.$$

We also need the estimate

$$\lambda_{\varkappa}(t)|_{\mathbb{D}(\psi)} - \lambda_{\varkappa}(t)|_{D_k} = O(t^2), \quad t \to 0 \tag{21}$$

(the similar equality of the metric $\lambda_{\mathcal{K}}$ follows from its continuity on the space $\mathbf{T}$). The estimate (21) follows from the representation

$$w^{\mu_t}(z) = z - \frac{1}{\pi}\iint\limits_{D_L} \frac{\mu_t(\zeta)d\xi d\eta}{\zeta - z} + \mu_t \, \Pi\mu_t + \dots$$

$$= z - \frac{t}{\pi}\iint\limits_{D_L} \frac{\mu_0(\zeta)/\|\mu_0\|_i y d\xi d\eta}{\zeta - z} + O(t^2)$$

and holomorphy of functions (17).

In view of this equality, Lemma 7 implies that both metrics $\lambda_{\varkappa}$ and $\lambda_{\mathcal{K}}$ must be equal on the whole domain D_k, which proves (20).

Now the desired equality (19) for the integrated forms of $\lambda_{\varkappa}$ and $\lambda_{\mathcal{K}}$ is obtained by applying the following reconstruction lemma for Grunsky norm and the similar property of Teichmüller metric (which involves the indicated local Lipschitz continuity).

Lemma 8. [25] *On any extremal Teichmüller disk* $\mathbb{D}(\mu_0) = \{t\mu_0/\|\mu_0\|_\infty\} \subset \mathbf{T}$*, we have the equality*

$$\tanh^{-1}[\varkappa(f^{r\mu_0/\|\mu_0\|_\infty})] = \int\limits_0^r \lambda_{\varkappa}(t)dt.$$

Applying this lemma, one completes the proof of Theorem 3.

5.3. Examples and related results.

1. The simplest example is given by

$$\mu_t(z) = t\frac{|z|^2}{z^2} + t^2\frac{|z|^m}{z^m}, \quad |z| < 1, \quad m \neq 2$$

(extended by zero to $\mathbb{D}^*$). Theorems A and 1 insure quasiconformality of integral (6) with this μ_t for appropriate $|t| > 0$.

2. The following interesting example arises in connection with the problem of starlikness of Teichmüller spaces and represents explicitly the functions which violate this property in the case of the universal Teichmüller space.

Pick unbounded convex rectilinear polygon P_n with finite vertices $A_1, \dots, A_{n-1}$ and $A_n = \infty$. Denote the exterior angles at A_j by $\pi\alpha_j$ so that $\pi < \alpha_j < 2\pi, \ j = 1, \dots, n-1$. The conformal map f_n of the lower half-plane $H^* = \{z : \ \operatorname{Im} z < 0\}$ onto the complementary polygon $P_n^* = \widehat{\mathbb{C}} \setminus \overline{P_n}$ is represented by the Schwarz–Christoffel integral

$$f_n(z) = d_1 \int\limits_0^z (\xi - a_1)^{\alpha_1 - 1}(\xi - a_2)^{\alpha_2 - 1} \dots (\xi - a_{n-1})^{\alpha_{n-1} - 1} d\xi + d_0, \quad (22)$$

with $a_j = f_n^{-1}(A_j) \in \mathbb{R}$ and complex constants d_0, d_1; here $f_n^{-1}(\infty) = \infty$. Its Schwarzian derivative is given by

$$S_{f_n}(z) = \mathbf{b}'_{f_n}(z) - \frac{1}{2} b^2_{f_n}(z) = \sum_1^{n-1} \frac{C_j}{(z - a_j)^2} - \sum_{j,l=1}^{n-1} \frac{C_{jl}}{(z - a_j)(z - a_l)},$$

where $\mathbf{b}_f = f''/f'$ and

$$C_j = -(\alpha_j - 1) - (\alpha_j - 1)^2/2 < 0, \quad C_{jl} = (\alpha_j - 1)(\alpha_l - 1) > 0.$$

It defines a point of the universal Teichmüller space $\mathbf{T}$ modelled as a bounded domain in the space $\mathbf{B}(H^*)$ of hyperbolically bounded holomorphic functions on H^* with norm

$$\|\varphi\|_{\mathbf{B}(H^*)} = \sup_{H^*} |z - \overline{z}|^2 |\varphi(z)|.$$

By the Ahlfors–Weill theorem [5], every $\varphi \in \mathbf{B}(H^*)$ with $\|\varphi\|_{\mathbf{B}(H^*)} < 1/2$ is the Schwarzian derivative of a univalent function f in H^*, and this function has quasiconformal extension onto the upper half-plane $H = \{z : \ \operatorname{Im} z > 0\}$ with Beltrami coefficient of the form

$$\mu_\varphi(z) = -2y^2 \varphi(\overline{z}), \quad \varphi = S_f \ (z = x + iy \in H^*) \quad (23)$$

called harmonic.

Denote by r_0 the positive root of the equation

$$\frac{1}{2}\Big[\sum_1^{n-1} (\alpha_j - 1)^2 + \sum_{j,l=1}^{n-1} (\alpha_j - 1)(\alpha_l - 1)\Big] r^2 - \sum_1^{n-1} (\alpha_j - 1) \ r - 2 = 0,$$

and put $S_{f_n,t} = t b'_{f_n} - b^2_{f_n}/2, \ t > 0$. Then for appropriate α_j, we have

Theorem B. [24] *For any convex polygon P_n, the Schwarzians rS_{f_n,r_0} define for any $0 < r < r_0$ a univalent function $w_r : H^* \to \mathbb{C}$ whose* *harmonic Beltrami coefficient* $\nu_r(z) = -(r/2)y^2 S_{f_n,r_0}(\overline{z})$ *in* H *is extremal in its equivalence class, and*

$$k(w_r) = \varkappa(w_r) = \frac{r}{2}\|S_{f_n,r_0}\|_{\mathbf{B}(H^*)}. \tag{24}$$

This theorem yields that any w_r with $r < r_0$ **does not admit extremal quasiconformal extensions of Teichmüller type** (which is defined by a holomorphic quadratic differential on $\mathbb{D}$).

In view of extremality of harmonic coefficients $\mu_{S_{w_r}}$ the Schwarzians S_{w_r} for some r between r_0 and 1 must lie outside of the space $\mathbf{T}$; so this space is not a starlike domain in $\mathbf{B}(H^*)$ (another model of this space, which was applied in the proof of Theorem 1, also does not be starlike).

Theorem 3 provides that the equality of the Grunsky and Teichmüller norms is preserved under appropriate perturbations of the Beltrami coefficient (23). These perturbations lead to conformal maps $\widetilde{w}_r$ of the disk (half-plane) onto the (generically nonregular) curvelinear polygons $\widetilde{P}_n$ whose sides are quasiconformal arcs. In contrast to Theorem B, we do not have an explicit expression of $k(\widetilde{w}_r) = \varkappa(\widetilde{w}_r)$. For small r, this common value is given by the right-hand side of (24) up to a quantity $O(r^2)$.

The conformal map W_n of the unit disk onto the polygon P_n has the same form (22) with the preimages $a_j \in \mathbf{S}^1$. Composing, if needed W_n with translations and the similarity maps $z \mapsto rz$, one can assume that this polygon contains inside the origin $z = 0$ and $W_n(\zeta) = \zeta + \alpha_2\zeta^2 + \dots \in S$. Since P_n is convex, each stretching function

$$W_{n,r}(z) = \frac{1}{r}W_n(rz) = \zeta + \alpha_2 rz^2 + \dots, \quad 0 < r < 1,$$

also maps the unit disk onto a convex domain, and by [25], $k(W_{n,r}) = \varkappa(W_{n,r})$.

Note also that being analytic on the boundary every map $W_{n,r}$ admits a extremal quasiconformal extension of Teichmüller type. For small $r > 0$, the corresponding integral (6) represents a a quasiconformal map with dilatation $t + O(t^2)$ and equal Grunsky and Teichmüller norms.

5.4. Connection with the Ahfors question. Theorem 3 also relates quantitatively to Ahlfors' question stated, for example, in [4]):

How to characterize the conformal maps of the disk (or half-plane) onto the domains with quasiconformal boundaries ?

Ahlfors conjectured that the characterization should be in analytic properties of the logarithmic derivative $\log f' = f''/f'$, and indeed, many results on quasiconformal extensions of holomorphic maps have been established using this quantity and other invariants (see, e.g., the survey [22]).

The equality (23) implies that in the case of indicated convex polygons P_n the exact quantitative answer is presented in terms of the Schwarzians S_{w_r}.

The indicated question naturally relates to another still not solved problem in geometric complex analysis:

To what extent does the Riemann mapping function f of a Jordan domain $D \subset \widehat{\mathbb{C}}$ determine the geometric and conformal invariants (characteristics) of complementary domain $D^* = \widehat{\mathbb{C}} \setminus \overline{D}$?

There are two quasiinvariant curvelinear functionals naturally associated with the quasicircles: the reflection coefficient and the first nontrivial Fredholm eigenvalue.

Recall that the **quasiconformal relections** (or quasireflections) are the orientation reversing quasiconformal homeomorphisms of the sphere $\widehat{\mathbb{C}}$ which preserve point-wise some (oriented) quasicircle $L \subset \widehat{\mathbb{C}}$ and interchange its interior and exterior domains. In other words, quasireflections are the topological involutions of the sphere $\widehat{\mathbb{C}}$ whose fixed Jordan curve is a quasicircle. Its **reflection coefficient** is determined by

$$q_L = \inf k(f) = \inf \ \|\partial_z f/\partial_{\overline{z}} f\|_\infty,$$

taking the infimum over all quasireflections across L. On the properties of quasireflections and obtained results see, e.g., [3, 23, 30].

In particular, the reflection coefficient q_L of a curve $L = f(|z| = 1)$ determined by a univalent function in the unit disk relates to the Grunsky and Teichmüller norms of this function via

$$\frac{1 + q_L}{1 - q_L} = \left(\frac{1 + \varkappa(f)}{1 - \varkappa(f)}\right)^2. \tag{25}$$

The Fredholm eigenvalues ρ_n of an oriented smooth closed Jordan curve $L \subset \widehat{\mathbb{C}}$ are the eigenvalues of its double-layer potential.

The least positive eigenvalue $\rho_L = \rho_1$ is naturally connected with conformal and quasiconformal maps and naturally arises in various problems. It can be defined for any oriented closed Jordan curve L by

$$\frac{1}{\rho_L} = \sup \ \frac{|\mathcal{D}_G(u) - \mathcal{D}_{G^*}(u)|}{\mathcal{D}_G(u) + \mathcal{D}_{G^*}(u)},$$

where G and G^* are, respectively, the interior and exterior of L; $\mathcal{D}$ denotes the Dirichlet integral, and the supremum is taken over all functions u continuous on $\widehat{\mathbb{C}}$ and harmonic on $G \cup G^*$. In particular, $\rho_L = \infty$ only for the circle.

Using (25) and the Kühnau-Schiffer theorem on reciprocity of ρ_L to the Grunsky norm of the Riemann mapping function of the outer domain of L [29, 36], one explicitly represents the values $q_{\partial P_n}$ and $\rho_{\partial P_n}$ in terms of Schwarzians S_{w_r}.

Another result of such type obtained in [21] states that for any closed oriented unbounded curve L with the convex interior which is $C^{1+\delta}$ smooth at all finite points and has at infinity the asymptotes approaching the interior angle $\pi\alpha < 0$, we have the equalities

$$q_L = 1/\rho_L = 1 - |\alpha|.$$

Here only $z = \infty$ is a substantial point for the boundary values of conformal map $\mathbb{D}^* \to G^*$.

All the above curves are not asymptotically conformal so not enough regular to insure univalence of the corresponding integrals (11).

Note also that since the Schwarzians S_{w^μ} of functions $w^\mu \in \Sigma(0)$ admitting the Teichmüller extremal extensions are dense in the space $\mathbf{T}$, any $w^\mu \in \Sigma(0)$ with small dilatation $\|\mu\|_\infty$ can be approximated locally uniformly on $\mathbb{C}$ by maps represented by integrals of type (6) with compactly supported μ.

6. Connection with the Schwarzian derivatives and invariant differential operators of higher order

We provide here other extensions of Theorems 1 and 3 concerning the quasiconformal extension of conformal maps of quasidisks.

6.1. General smooth quasicircles. Let $L \subset \mathbb{C}$ be a bounded quasicircle from the class C^{1,σ_1} separating the points 0 and ∞. Denote its interior and exterior domains by D_L and D_L^*, (so that $D_L^* \ni \infty$), and consider the functions $\mu \in C^\sigma(\overline{D_L})$ extended by zero to D_L^* (here both σ, σ_1 are positive). Without loss of generality, one can assume that $|\mu(z)| \leq 1$.

The corresponding quasiconformal maps w^μ of the extended complex plane $\widehat{\mathbb{C}}$ are conformal in the domain D_L^* and their Schwarzian derivatives belong to the complex Banach space $\mathbf{B}(\mathbb{D}_L^*)$ with norm

$$\|\varphi\|_{\mathbf{B}(D_L^*)} = \sup_{\mathbf{B}(D_L^*)} \lambda_{D_L^*}(z)^{-2} |\varphi(z)|,$$

where $\lambda_{D_L^*}(z)$ is the hyperbolic metric of domain D_L^* of Gaussian curvature -4. We normalize these maps again by

$$w^\mu(z) - z = O(1/z) \quad \text{as} \quad z \to \infty; \quad w^\mu(0) = 0.$$

Theorem 4. *Let $\mu \in L_\infty(D_L)$ satisfy the indicated smoothness assumptions and $\|\mu\|_\infty = \varepsilon$. Then, for sufficiently small $\varepsilon > 0$:*

(a) The Schwarzian derivative of the map

$$w^\mu(z) = z - \frac{t}{\pi} \iint\limits_{\mathbb{D}_L} \mu(\zeta)\Big(\frac{1}{\zeta - z} - \frac{1}{\zeta}\Big) d\xi d\eta \tag{26}$$

is given by

$$S_{w^\mu}(z) = \frac{1}{\pi} \iint\limits_{\mathbb{D}} \frac{\mu(\zeta) d\xi d\eta}{(\zeta - z)^4} + O(\varepsilon^2), \tag{27}$$

where the remainder is estimated in $\mathbf{B}(D_L^*)$-norm.

(b) The function $w^\mu(z)|D_L^*$ admits quasiconformal extension to D_L with Beltrami coefficient $\widetilde{\mu}$ depending holomorphically on μ and S_{w^μ}.

Proof. Similar to **1.3**, the properties of operators T and Π on $C^{1,\sigma}$ functions imply that for sufficiently small ε, the Jacobian

$$J_{w^\mu}(z) = |\partial_z w^\mu(z)|^2 - |\partial_{\bar{z}} w^\mu(z)|^2 = 1 - O(\varepsilon) > 0$$

for all $z \in \mathbb{C}$; hence, the map w^μ is homeomorphic on $\widehat{\mathbb{C}}$ being conformal on D_L^*. Its Schwarzian derivative is represented for any fixed $z \in D_L^*$ by (27).

Since the quasicircle L is bounded, it is located in a disk $\{|z| < R\}$, $R < \infty$. Applying (12) and the properties of the hyperbolic metric, one obtains for $z \in D_L^*$ the estimate

$$\iint\limits_{D_L} \frac{d\xi d\eta}{|\zeta - z|^4} < \iint\limits_{|\zeta|<R} \frac{d\xi d\eta}{|\zeta - z|^4} = \frac{R^2}{(R^2 - |z|^2)^2} \le \lambda_{D_L^*}(z)^2,$$

which yields that the integral in (26) is a function from the space $\mathbf{B}(D_L^*)$. By Lemma 1, the function S_{w^μ} is holomorphic in μ also in the norm of $\mathbf{B}(D_L^*)$. [1]

[1] This holomorphy also follows from the lambda-lemma for holomorphic motions applied to $w(z,t) = w^{t\widetilde{\mu}}$ with $\widetilde{\mu} = \mu/\|\mu\|_\infty$ and $t \in \mathbb{D}$.

The existence of quasiconformal extensions of univalent functions $w^\mu|D_L^*$, whose Beltrami coefficients depend on μ and S_{w^μ} holomorphically, is a consequence of the Bers extension theorem [9]. We need its special case presented by the following lemma.

Lemma 9. *Let L be a quasicircle on $\widehat{\mathbb{C}}$ with the interior D_L and exterior D_L^*. Then, for some $\varepsilon > 0$, there exists an anti-holomorphic homeomorphism τ (with $\tau(\mathbf{0}) = \mathbf{0}$) of the ball $V_\varepsilon = \{\varphi \in \mathbf{B}(D_L^*) : \|\varphi\|\} < \varepsilon$ into $\mathbf{B}(D_L)$ such that every φ in V_ε is the Schwarzian derivative of some univalent function f which is the restriction to D_L^* of a quasiconformal automorphism $\widehat{f}$ of Riemann sphere $\widehat{\mathbb{C}}$. This $\widehat{f}$ can be chosen in such a way that its Beltrami coefficient is harmonic on D_L, i.e., of the form*

$$\mu_{\widehat{f}}(z) = \lambda_D^{-2}(z)\overline{\psi(z)}, \quad \psi = \tau(\varphi).$$

Applying this lemma completes the proof of the theorem.

6.2. Remarks.
1. Due to [9], the variation formula (27) for the Schwarzian derivatives $S_{w^\mu} \in \mathbf{B}(D_L^*)$ is valid for arbitrary $\mu \in L_\infty(\mathbb{C})$ vanishing on D_L^* and sufficiently small $\|\mu\|_\infty$. The assertion of Theorem 4 that also the Schwarzian of the integral (26) belongs to $\mathbf{B}(D_L^*)$ and is represented by (27) essentially requires the indicated smoothness of μ.
2. Using Lemma 9, one can derive from Theorem 3 that the Teichmüller and Grunsky norms remain equal also after infinitesimally trivial deformations: if $\varphi \in A_1^2$ and the Schwarzians $S = S_{w^{t|\varphi|/\varphi}} + S_{w^\nu}$ with infinitesimally trivial ν, then for all such w, $\varkappa(w) = k(w)$.

6.3. In the case, when L is the unit circle, we have a stronger result.

Theorem 5. *If $\mu \in C^{1,\sigma}(\overline{\mathbb{D}})$ and*

$$\Big| \iint\limits_{\mathbb{D}} \frac{\mu(\zeta)d\xi d\eta}{(\zeta - z)^4} \Big| \le a(|z|^2 - 1)^2, \quad a < 2,$$

then the restriction of the integral (26) to $\mathbb{D}^*$ admits quasiconformal extension to the unit disk with harmonic Beltrami coefficient

$$\widehat{\mu}(z) = -\frac{1}{2\pi}(1 - |z|^2)^2(z/\overline{z}) \iint\limits_{\mathbb{D}} \mu(\zeta)(\zeta - 1/z)^{-4} d\xi d\eta.$$

The *proof* of this theorem is similar to the part (b) of Theorem 1, applying the Ahlfors–Weill theorem on quasiconformal extension of univalent functions $w(z)$ in $\mathbb{D}^*$ with $\|S_w\|_{\mathbf{B}} < 2$.

6.4. It would be interesting to establish the extent to which the above theorems can be generalized to the invariant differential operators of higher order acting on univalent functions.

Using a result of [18], one obtains the following theorem involving the operators of a different type.

Consider the complex Banach space $\widetilde{\mathbf{B}}_n(\mathbb{D})$ of holomorphic functions in the unit disk with the norm

$$\|\varphi\|_{\widetilde{\mathbf{B}}_n(\mathbb{D})} = \sup_{\mathbb{D}}(1 - |z|)^{n-1}|\varphi(z)|,$$

and let $\widetilde{\mathbf{B}}_n(\mathbb{D})_c$ denotes its ball of radius $c > 0$ centered at the origin. Let there be given on the disk $\mathbb{D}$ a differential operator

$$\mathcal{R}_n(f) = \frac{f^{(n)}}{f'} + F_n\Big(\frac{f''}{f'}\Big), \quad n = 2, 3,$$

where F_n is holomorphic in some domain in $\mathbb{C}$ containing the origin and for univalent $f(z)$ in $\mathbb{D}$ the values $F_n(f''/f') \in \widetilde{\mathbf{B}}_n(\mathbb{D})$.

Accordingly, instead of (26), we now consider the integral

$$w^\mu(z) = z - \frac{t}{\pi}\iint\limits_{\mathbb{D}^*} \mu(\zeta)\Big(\frac{1}{\zeta - z} - \frac{1}{\zeta}\Big)d\xi d\eta,$$

for which we have:

Theorem 6. *There exists a number $c = c(P_n) > 0$ such that if the function $w = w^\mu(z)$ satisfies to one of the conditions:*

$$\max_{\mathbb{D}^*}(|z|^2 - 1)|w''(z)/w'(z)| < c, \quad \text{if } n = 2,$$

$$\|S_w\|_{\mathbf{B}(\mathbb{D})} < c, \quad \text{if } n = 3;$$

then this function is univalent on the disk $\mathbb{D}$ and admits quasiconformal extension to $\widehat{\mathbb{C}}$ with the Beltrami coefficient

$$\widetilde{\mu}(z) = (|z|^2 - 1)^2\overline{\tau_n(\phi_n(\mu))}, \quad z \in \mathbb{D}^*,$$

where $\tau_n(\varphi) \in \mathbf{B}(\mathbb{D}^*)$, and $n = 2$ for the first condition, $n = 3$ for the second one, and τ_n is the cross-section of the holomorphic map

$$\phi_n(\mu) = \mathcal{R}_n(w^\mu) : \ \mathrm{Belt}(\mathbb{D}^*)_1 \to \widetilde{\mathbf{B}}_n(\mathbb{D})$$

on the ball $\widetilde{\mathbf{B}}_n(\mathbb{D})_c$.

The proof of this theorem follows the above lines with applying Theorem 3 from [18].

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