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Linearly ordered coarse spaces

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Abstract. A coarse space X, endowed with a linear order compatible with the coarse structure of X, is called linearly ordered. We prove that every linearly ordered coarse space X is locally convex and the asymptotic dimension of X is either 0 or 1. If X is metrizable then the family of all right bounded subsets of X has a selector.

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1. Introduction and preliminaries

Let $\mathcal{K}$ be a class of coarse spaces. Given $X \in \mathcal{K}$, how can one detect whether there exists a linear order on X, compatible with the coarse structure of X? We used selectors to answer this question if $\mathcal{K}$ is one of the following classes: discrete coarse spaces [6], [7]; finitary coarse spaces of groups [8]; finitary coarse spaces of graphs [9].

In this paper, we continue the investigations of the structure of a linearly ordered coarse space initiated in [7].

In Section 2, we prove that every linearly ordered coarse space is locally convex, but the coarse structure of X needs not to be interval. Given a linear order $\leq$ on a set X, we characterize the minimal and maximal coarse structures on X, compatible with the interval bornology of $(X, \leq)$.

In Section 3, we prove that the asymptotic dimension of a linearly ordered coarse space is either 0 or 1.

In Section 4, we construct a selector of the family of right bounded subsets of a metrizable lineary ordered coarse space.

We conclude the paper with Section 5 of comments and open questions.

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We recall some basic definitions. Given a set X, a family $\mathcal{E}$ of subsets of $X \times X$ is called a *coarse structure* on X if

- each $E \in \mathcal{E}$ contains the diagonal $\triangle_X := \{(x,x) : x \in X\}$ of X;

- if E, $E' \in \mathcal{E}$ then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x,y) : \exists z \ \ ((x,z) \in E, \ (z,y) \in E')\}$, $E^{-1} = \{(y,x) : (x,y) \in E\}$;

- if $E \in \mathcal{E}$ and $\triangle_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$.

Elements $E \in \mathcal{E}$ of the coarse structure are called *entourages* on X.

For $x \in X$ and $E \in \mathcal{E}$, the set $E[x] := \{y \in X : (x,y) \in \mathcal{E}\}$ is called the *ball of radius E centered at x.* Since $E = \bigcup_{x \in X}(\{x\} \times E[x])$, the entourage E is uniquely determined by the family of balls $\{E[x] : x \in X\}$. A subfamily $\mathcal{E}' \subseteq \mathcal{E}$ is called a *base* of the coarse structure $\mathcal{E}$ if each set $E \in \mathcal{E}$ is contained in some $E' \in \mathcal{E}'$.

The pair $(X, \mathcal{E})$ is called a *coarse space* [13] or a *ballean* [10, 11].

A coarse spaces $(X, \mathcal{E})$ is called *connected* if, for any $x, y \in X$, there exists $E \in \mathcal{E}$ such that $y \in E[x]$. A subset $Y \subseteq X$ is called *bounded* if $Y \subseteq E[x]$ for some $E \in \mathcal{E}$, and $x \in X$. If $(X, \mathcal{E})$ is connected then the family $\mathcal{B}_X$ of all bounded subsets of X is a bornology on X. We recall that a family $\mathcal{B}$ of subsets of a set X is a *bornology* if $\mathcal{B}$ contains the family $[X]^{<\omega}$ of all finite subsets of X and $\mathcal{B}$ is closed under finite unions and taking subsets. A bornology $\mathcal{B}$ on a set X is called *unbounded* if $X \notin \mathcal{B}$. A subfamily $\mathcal{B}'$ of $\mathcal{B}$ is called a base for $\mathcal{B}$ if, for each $B \in \mathcal{B}$, there exists $B' \in \mathcal{B}'$ such that $B \subseteq B'$.

Each subset $Y \subseteq X$ defines a *subspace* $(Y, \mathcal{E}|_Y)$ of $(X, \mathcal{E})$, where $\mathcal{E}|_Y = \{E \cap (Y \times Y) : E \in \mathcal{E}\}$. A subspace $(Y, \mathcal{E}|_Y)$ is called *large* if there exists $E \in \mathcal{E}$ such that $X = E[Y]$, where $E[Y] = \bigcup_{y \in Y} E[y]$.

Let $(X, \mathcal{E})$, $(X', \mathcal{E}')$ be coarse spaces. A mapping $f : X \to X'$ is called *macro-uniform* if for every $E \in \mathcal{E}$ there exists $E' \in \mathcal{E}'$ such that $f(E(x)) \subseteq E'(f(x))$ for each $x \in X$. If f is a bijection such that f and f^{-1} are macro-uniform, then f is called an *asymorphism.* If $(X, \mathcal{E})$ and $(X', \mathcal{E}')$ contain large asymorphic subspaces, then they are called *coarsely equivalent.*

For a coarse space $(X, \mathcal{E})$, we denote by *exp* X the family of all non-empty subsets of X and by *exp* $\mathcal{E}$ the coarse structure on *exp* X with the base $\{exp\ E : E \in \mathcal{E}\}$, where

$$(A, B) \in exp\ E \Leftrightarrow A \subseteq E[B], \quad B \subseteq E[A],$$

and say that $(exp\ X, exp\ \mathcal{E})$ is the *hyperballean* of $(X, \mathcal{E})$.

Let $\mathcal{F}$ be a non-empty subspace of $exp\ X$. We say that a macro-uniform mapping $f : \mathcal{F} \longrightarrow X$ is an *$\mathcal{F}$-selector* of $(X, \mathcal{E})$ if $f(A) \in A$ for each $A \in \mathcal{F}$. In the case $\mathcal{F} \in [X]^2$, $\mathcal{F} = \mathcal{B}_X$ and $\mathcal{F} = exp\ X$, an $\mathcal{F}$-selector is called a 2-*selector*, a *bornologous selector* and a *global selector* respectively.

We recall that a connected coarse space $(X, \mathcal{E})$ is *discrete* if, for each $E \in \mathcal{E}$, there exists a bounded subset B of $(X, \mathcal{E})$ such that $E[x] = \{x\}$ for each $x \in X \setminus B$. Every bornology $\mathcal{B}$ on a set X defines the discrete coarse space $X_{\mathcal{B}} = (X, \mathcal{E}_{\mathcal{B}})$, where $\mathcal{E}_{\mathcal{B}}$ is a coarse structure with the base $\{E_B : B \in \mathcal{B}\}$, $E_B[x] = B$ if $x \in B$ and $E_B[x] = \{x\}$ if $x \in X \setminus B$. On the other hand, every discrete coarse space $(X, \mathcal{E})$ coincides with $X_{\mathcal{B}}$, where $\mathcal{B}$ is the bornology of bounded subsets of $(X, \mathcal{E})$.

2. Local convexity and interval bases

Let $(X, \mathcal{E})$ be a coarse space. Following [7], we say that a linear order $\leq$ or X is *compatible* with the coarse structure $\mathcal{E}$ if one of the following equivalent conditions holds

- for every $E \in \mathcal{E}$, there exists $F \in \mathcal{E}$ such that if $x < y$ and $y \in X \setminus F[x]$ then $x' < y$ for each $x' \in E[x]$;
- for every $E \in \mathcal{E}$, there exists $H \in \mathcal{E}$ such that if $y < x$ and $y \in X \setminus H[x]$ then $y < x'$ for each $x' \in E[x]$;
- for every $E \in \mathcal{E}$, there exists $K \in \mathcal{E}$ such that if $x < y$ and $y \in X \setminus K[x]$ then $x' < y'$ for all $x' \in E[x]$, $y' \in E[y]$.

A coarse space $(X, \mathcal{E})$, endowed with a linear order $\leq$ compatible with $\mathcal{E}$ is called *linearly ordered*. In this case, by [7, Proposition 2], the mapping $f : [X]^2 \longrightarrow X$, defined by $f(A) = min\ A$ is a 2-selector of $(X, \mathcal{E})$ and if $(X, \mathcal{E})$ is connected then each interval $[a, b]$, where $[a, b] = \{x \in X : a \leq x \leq b\}$ is bounded. In what follows, all linearly ordered coarse spaces are suppose to be connected.

We recall that a subset Y of a linearly ordered set $(X, \leq)$ is called *convex* if $[a, b] \subseteq Y$ for all $a, b \in Y$, and observe that Y is convex if and only if there exists $x \in Y$ such that $[x, y] \subseteq Y$ for each $y \in X$.

Theorem 1. *For a coarse space $(X, \mathcal{E})$ and a linear order $\leq$ on X, the following statements are equivalent*

(i) $(X, \mathcal{E}, \leq)$ *is linearly ordered;*

(ii) $\mathcal{E}$ *has a base $\mathcal{E}'$ such that $E'[x]$ is convex for all $x \in X$, $E' \in \mathcal{E}'$.*

Proof. $(i) \Rightarrow (ii)$. For each $E \in \mathcal{E}$, we denote $E' = \bigcup \{[x, y] : x, y \in E\}$. Since E' is convex, it suffices to show that $E' \in \mathcal{E}$. Since $\leq$ is compatible with $\mathcal{E}$, there exists $F \in \mathcal{E}$ such that $E \subseteq F$ and if $x < z$ $(z < x)$ and $z \in X \setminus F[x]$ then $y < z$ $(z < y)$ for each $y \in E[x]$. It follows that $E'[x] \subseteq F[x]$ and $E' \in \mathcal{E}$.

$(ii) \Rightarrow (i)$. Given $E \in \mathcal{E}$, we choose $E' \in \mathcal{E}$ such that $E \in E'$ and $E'[x]$ is convex for each $x \in X$. If $z \in X \setminus E'[x]$ then either $z < x'$ for each $x' \in E[x]$ or $x' < z$ for each $x' \in E[x]$. Hence, $\leq$ is compatible with $\mathcal{E}$. □

We say that a base E' of $\mathcal{E}$, satisfying (ii) is *locally convex.*

Let $\mathcal{B}$ be a bornology on a set X. Following [1], we say that a coarse structure $\mathcal{E}$ on X is *compatible* with $\mathcal{B}$ if $\mathcal{B}$ is the bornology of bounded subsets of the coarse space $(X, \mathcal{E})$.

For a linear order $\leq$ on a set X, $\mathcal{B}_{\leq}$ denotes the interval bornology on X with the base $\{[a, b] : a, b \in X\}$. In the following two examples, we describe the smallest locally convex coarse structure $\downarrow \mathcal{E}_{\leq}$ and the strongest locally convex coarse structure $\uparrow \mathcal{E}_{\leq}$ compatible with $\mathcal{B}_{\leq}$.

Example 1. Let $\leq$ be a linear order on a set X. Then $\downarrow \mathcal{B}_{\leq}$ is the discrete coarse structure on X defined by the bornology $\mathcal{B}_{\leq}$.

Example 2. Let $\leq$ be a linear order on a set X, $\mathcal{C}_{\leq}$ denotes the family of all bounded convex subsets of $(X, \leq)$. We consider the family Φ of all mappings $\varphi : X \to \mathcal{C}$ such that, for all $a, b \in X$, we have

$$\bigcup \{\varphi(x) : x \in [a, b]\} \in \mathcal{B}_{\leq}, \quad \{x \in X : \varphi(x) \bigcap [a, b] \neq \varnothing\} \in \mathcal{B}_{\leq}.$$

Then the family $\{E_{\varphi} : \varphi \in \Phi\}$, where $E_{\varphi} = \{(x, y) : y \in \varphi(x)\}$, is a base for $\uparrow \mathcal{E}_{\leq}$.

Following [7], we say that a coarse structure $\mathcal{E}$ on $(X, \leq)$ is *interval* if there is a base $\mathcal{E}'$ of $\mathcal{E}$ such that, for all E', $x \in X$, $E'[x]$ is an interval in $(X, \leq)$. Clearly, $\mathcal{E}$ is locally convex and, by Theorem 1, $\leq$ is compatible with $\mathcal{E}$. On the other hand, let $(X, \mathcal{E}, \leq)$ be a linearly ordered coarse space. Is $\mathcal{E}$ an interval coarse structure? We give the negative answer to this question.

The following example also shows that a subspace of a coarse space with interval base may not have an interval base.

Example 3. We denote by X the subset $\bigcup\{(2^n - 1,\ 2^n + 1) :\ n > 1\}$ of $\mathbb{R}$, put $E_0 = \{(x, y) \in X \times X :\ |x - y| < 2\}$.

$$E_n = \{(x, y) \in X \times X :\ x, y \in (3,\ 2^{n+1})\},\ n > 1,$$

endow X with a coarse structure $\mathcal{E}$ with the base $\{E_n \cup E_0 : n > 1\}$. Then $\mathcal{E}$ is locally convex. To see that $\mathcal{E}$ does not have an interval base, we observe that if $H \subseteq X \times X$, $H[x]$ is an interval for each $x \in X$ and $E_0 \subseteq H$ then $2^{n+1} \in H[2^n]$ for each $n > 1$. Hence, $H \notin \mathcal{E}$.

3. Asymptotic dimension

Let $(X, \mathcal{E})$ be a coarse space, $E \in \mathcal{E}$. A family Im of subsets of X is called E-bounded (E-disjoint) if, for each $A \in$ Im, there exists $x \in X$ such that $A \subseteq E[x]$ ($E[A] \cap B = \emptyset$ for all distinct $A, B \in$ Im).

By the definition [13, Chapter 10], $asdim(X, \mathcal{E}) \leq n$ if, for each $E \in \mathcal{E}$, there exist $F \in \mathcal{E}$ and F-bounded covering $\mathcal{M}$ of X which can be partitioned $\mathcal{M} = \mathcal{M}_0 \cup \dots \cup \mathcal{M}_n$ so that each family $\mathcal{M}_i$ is E-disjoint. If there is the minimal natural number n with this property then $asdim(X, \mathcal{E}) = n$, otherwise $asdim(X, \mathcal{E}) = \infty$.

Theorem 2. *Let $(X, \mathcal{E}, \leq)$ be a linearly ordered coarse space. Then $asdim(X, \mathcal{E}) \in \{0, 1\}$.*

Proof. Let $E \in \mathcal{E}$, $E = E^{-1}$ and $E[x]$ is convex for each $x \in X$, see Theorem 1. We fix $x \in X$, observe that $E^n[x]$ is convex for each $n \in \mathbb{N}$ and put $E^{\omega}[x] = \bigcup\{E^n[x] : x \in \mathbb{N}\}$. We show that there exist E^2-bounded covering $\mathcal{M}(x)$ of $E^{\omega}[x]$ and a partition $\mathcal{M}(x) = \mathcal{M}_0(x) \bigcup \mathcal{M}_1(x)$ such that $\mathcal{M}_0(x)$, $\mathcal{M}_1(x)$ are E-disjoint.

For each $n \in \mathbb{N}$, we denote $R_n = (E^{n+1}[x] \setminus E^n[x]) \bigcap \{y \in X : x \leq y\}$, $L_n = (E^{n+1}[x] \setminus E^n[x]) \bigcap \{y \in X : y < x\}$ and observe that $E[R_i] \bigcap L_j = \emptyset$ for all $i, j \in \mathbb{N}$ and $E[R_i] \bigcap R_j = \emptyset$, $E[L_i] \bigcap L_j = \emptyset$ for all $i, j \in \mathbb{N}$ such that $|i - j| > 1$. Clearly, R_n and L_n are convex for each $n \in \mathbb{N}$. If $R_n = \emptyset$ ($L_n = \emptyset$) then $R_i = \emptyset$ ($L_i = \emptyset$) for every $i > n$.

We put

$$\mathcal{M}_0(x) = \{E[x],\ R_{2n}, L_{2n} : n \in \mathbb{N}\},$$

$$\mathcal{M}_1(x) = \{R_{2n-1}, L_{2n-1} : n \in \mathbb{N}\}$$

and note that $\mathcal{M}_0(x)$, $\mathcal{M}_1(x)$ are E-disjoint.

We show that each R_n is E^2-bounded, the case L_n is analogous. For $n = 1$, we have $R_1 \subseteq E^2[x]$. Let $n > 1$ and $R_n \neq \emptyset$. We take $y \in R_{n-1}$ such that $E[y] \cap R_n \neq \emptyset$ and show that $R_n \subseteq E^2[y]$. Given $z \in R_n$, we choose $t \in R_{n-1}$ such that $z \in E[t]$. Since R_{n-1} is convex and $E[y] \cap R_n \neq \emptyset$, we have $t \in E[y]$ so $z \in E^2[y]$.

To conclude the proof, we choose a subset Z of X such that $\bigcup\{E^{\omega}[z] : z \in Z\} = X$ and $E^{\omega}[z] \bigcap E^{\omega}[z'] = \emptyset$ for all distinct $z, z' \in Z$. For each $z \in Z$ and $E^{\omega}[z]$, we use above construction to choose $\mathcal{M}_0(z)$ and $\mathcal{M}_1(z)$.

We put $\mathcal{M}_0 = \bigcup\{\mathcal{M}_0(z) : z \in Z\}$, $\mathcal{M}_1 = \bigcup\{\mathcal{M}_1(z) : z \in Z\}$. Then $\mathcal{M}_0,\ \mathcal{M}_1$ are E^2-bounded, $\mathcal{M}_0,\ \mathcal{M}_1$ are E-disjoint. □

For every discrete coarse structure $\downarrow \mathcal{E}_{\leq}$ on a linearly ordered set $(X, \leq)$, we have $asdim(X,\ \mathcal{E}_{\leq},\ \leq) = 0$.

Let $\leq$ be the natural well ordering on ω. Then $\uparrow \mathcal{E}_{\leq}$ coincides with the universal locally finite coarse structure and, by [5, Theorem 1], $asdim(\omega,\ \uparrow \mathcal{E}_{\leq},\ \leq) = 1$.

4. Selectors

Let $(X,\ \mathcal{E},\ \leq)$ be linearly ordered coarse space, $A \subseteq X$, $E \in \mathcal{E}$. We say that $a \in A$ is a *right (left) E-end* of A if $x < a$ $(a < x)$ for each $x \in A \setminus E[x]$. If a is the maximal (minimal) element of A then a is a right (left) E-end for each $E \in \mathcal{E}$.

Example 4. Let $(X,\ \mathcal{E},\ \leq)$ be linearly ordered coarse space, metrizable by a metric d on X, for metrizability of coarse spaces see [11, Chapter 2]. We take an arbitrary $\varepsilon > 0$ and show that every right bounded subset A of X has a right ε-end. To this end, we take $a_0 \in A$. If a_0 is not a right ε-end then we choose $a_1 \in A$ such that $a_0 < a_1,\ \ d(a_0, a_1) > \varepsilon$. Repeating this procedure, after finite number of steps, we get a right ε-end a_n of A.

Example 5. Let $(X,\ \mathcal{E},\ \leq)$ be a discrete coarse space, defined by the interval bornology $\mathcal{B}_{\leq}$ on $(X, \leq)$. Let $A \subseteq X,\ \ B \in \mathcal{B}_{\leq}$ and let there exists $a \in A$ such that $b < a$ for each $b \in \mathcal{B}$. Then A has a right E_B-end if and only if A has the maximal element.

Theorem 3. *Let $(X,\ \mathcal{E},\ \leq)$ be a linearly ordered coarse space, $E \in \mathcal{E}$,* Im *be a family of subsets of X. If every subsets $A \in$* Im *has a right E-end then* Im *has a selector.*

Proof. For each $A \in \mathrm{Im}$, we take some right E-end $f(A)$ of A and show that the mapping $f : \mathrm{Im} \to X$ is macro-uniform.

We take an arbitrary $H \in \mathcal{E},\ H = H^{-1}$ such that $H[x]$ is convex for each $x \in X$. Let $Y, Z \in \mathrm{Im}$, $(Y, Z) \in expH$ and $f(Y) \leq f(Z)$. We take $y \in Y$ such that $y \in H[f(Z)]$. If $y < f(Y)$ then, by the convexity of $H[f(Z)]$ and $f(Y) \leq f(Z)$, we have $f(Y) \in H[f(Z)]$. If $y \geqslant f(Y)$ then $y \in E[f(Y)]$. Hence, $H[f(Z)] \bigcap E[f(Y)] \neq \emptyset$ and $f(Z) \in HE[f(Y)]$. □

Applying Theorem 3 to Example 4, we conclude that the family of all right bounded subsets of a lineary ordered metric space has a selector.

5. Comments and open questions

1. Coarse spaces can be considered as asymptotic counterparts of uniform topological spaces, see [11, Chapter 1]. Selectors and orderings of topological spaces, studied in a plenty of papers, take an important place in *Topology*, see surveys [2], [3], [4], [12].

2. Example 3 answers negatively Question 1 from [7], Question 4 was answered negatively in [8], Questions 2, 3, 5 from [7] remain open.

3. In light of Theorem 3, we ask the following question.

Question 1. *Does the family of all right bounded subsets of a linearly ordered coarse space have a selector?*

4. It is well-knows that every linearly ordered topological space is normal. For normality of coarse spaces, see [11, Chapter 4].

Question 2. *Is every linearly ordered coarse space normal?*

5. We conclude with the following question.

Question 3. *Is every linearly ordered coarse space asymorphic to a subspace of a linearly ordered coarse space with an interval base?*

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