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Taylor series of biharmonic Poisson integral for upper half plane

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Abstract. The fourth order partial differential equation for the biharmonic Poisson integral is presented in the case of the upper half plane ($y > 0$). To solve this equation, the two boundary conditions must be taken into account. The boundary value problem is solved by a way to transform the presented boundary value problem for the biharmonic Poisson integral into the two boundary value problems for the two-dimensional functions $\mathcal{A}(q, y)$ and $\mathcal{B}(q, y)$. After that, the biharmonic Poisson integral for the upper half plane is obtained. It was found that the derived Taylor series of biharmonic Poisson integral for the upper half plane contains the remainder in the integral form.

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1. Introduction

The properties of the biharmonic Poisson integral for the unit disk were first studied in the paper [1]. The further investigations were performed in the papers [2–10]. It is significant to note that the properties of the biharmonic Poisson integral can also be studied in the case of the upper half plane [11, 12].

In our theoretical investigation, we are going to derive the Taylor series of the biharmonic Poisson integral for the upper half plane. Let a two-dimensional function $U(x, y)$ be a general solution to the following boundary value problem:

$$\nabla^2\left(\nabla^2 U\right) = 0, \quad \lim_{y\to 0} U(x,y) = f(x), \quad \lim_{y\to 0} \frac{\partial U(x,y)}{\partial y} = g(x). \tag{1.1}$$

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In the formula (1.1), the functions $f(x)$ and $g(x)$ are the bounded uniformly continuous functions for $x \in \mathbb{R}$. The formula (1.1) contains the Laplacian $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ in the two-dimensional case. It is also important to note that $y > 0$, because we are going to solve the boundary value problem (1.1) for the upper half plane.

The presented boundary value problem (1.1) can be solved by a way to introduce the new variables $z = x + iy$ and $\bar{z} = x - iy$. This enables one to represent the Laplacian $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ via the second order partial derivative $\dfrac{\partial^2}{\partial \bar{z} \partial z}$ [13–15].

In this paper, the problem will be solved by a way to apply the integral representation

$$U(x,y) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \left\{ \int\limits_{-\infty}^{+\infty} \mathcal{A}(q,y)\, e^{iq(x-x')} dq \right\} f\left(x'\right) dx' + \\ + \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \left\{ \int\limits_{-\infty}^{+\infty} \mathcal{B}(q,y)\, e^{iq(x-x')} dq \right\} g\left(x'\right) dx' \qquad (1.2)$$

that can enable one to transform the boundary value problem (1.1) for a two-dimensional function $U(x,y)$ into the boundary value problems for the functions $\mathcal{A}(q,y)$ and $\mathcal{B}(q,y)$.

The boundary value problem for a function $\mathcal{A}(q,y)$ is the following:

$$\begin{aligned} \frac{\partial^4 \mathcal{A}(q,y)}{\partial y^4} - 2q^2 \frac{\partial^2 \mathcal{A}(q,y)}{\partial y^2} + q^4 \mathcal{A}(q,y) &= 0, \\ \lim_{y \to 0} \mathcal{A}(q,y) &= 1, \\ \lim_{y \to 0} \frac{\partial \mathcal{A}(q,y)}{\partial y} &= 0. \end{aligned} \qquad (1.3)$$

The boundary value problem for a function $\mathcal{B}(q,y)$ is the following:

$$\begin{aligned} \frac{\partial^4 \mathcal{B}(q,y)}{\partial y^4} - 2q^2 \frac{\partial^2 \mathcal{B}(q,y)}{\partial y^2} + q^4 \mathcal{B}(q,y) &= 0, \\ \lim_{y \to 0} \mathcal{B}(q,y) &= 0, \\ \lim_{y \to 0} \frac{\partial \mathcal{B}(q,y)}{\partial y} &= 1. \end{aligned} \qquad (1.4)$$

2. Closed form solution

The solutions to the boundary value problems (1.3) and (1.4) are presented by the following expressions:

$$\mathcal{A}(q,y) = (1 + y|q|)\, e^{-y|q|}, \quad \mathcal{B}(q,y) = ye^{-y|q|}. \tag{2.1}$$

After that, the obtained solutions (2.1) to the boundary value problems (1.3) and (1.4) must be used with the aim to evaluate the integrals of the two functions $\mathcal{A}(q,y)\, e^{iq(x-x')}$ and $\mathcal{B}(q,y)\, e^{iq(x-x')}$ over the interval $-\infty < q < +\infty$. Substituting the obtained results into the integral representation (1.2) of a two-dimensional function $U(x,y)$, we can derive a closed form solution

$$U(x,y) = \frac{2y^3}{\pi} \int\limits_{-\infty}^{+\infty} \frac{f(x')\, dx'}{\left\{(x-x')^2 + y^2\right\}^2} + \frac{y^2}{\pi} \int\limits_{-\infty}^{+\infty} \frac{g(x')\, dx'}{(x-x')^2 + y^2} \tag{2.2}$$

to the boundary value problem (1.1). The obtained result is a sum of the two integrals that contain the delta-shaped kernels [16–19]. Taking into account the substitution $x' = t + x$, the integral (2.2) can be transformed into the integral

$$U(x,y) = \frac{2y^3}{\pi} \int\limits_{-\infty}^{+\infty} \frac{f(t+x)\, dt}{(t^2+y^2)^2} + \frac{y^2}{\pi} \int\limits_{-\infty}^{+\infty} \frac{g(t+x)\, dt}{t^2+y^2}. \tag{2.3}$$

The integral (2.3) is called the biharmonic Poisson integral for the upper half plane. It is also important to note that the integral (2.3) can be transformed into the integral

$$U(x,y) = \frac{2y^3}{\pi} \int\limits_{0}^{+\infty} \frac{f(x+t) + f(x-t)}{(t^2+y^2)^2} dt + \\ + \frac{y^2}{\pi} \int\limits_{0}^{+\infty} \frac{g(x+t) + g(x-t)}{t^2+y^2} dt \tag{2.4}$$

that contains the integrals on the semiaxis $t > 0$.

3. Taylor's theorem

Let us now prove that the biharmonic Poisson integral (2.4) can be considered to be a Taylor series with the remainder in the integral form.

Theorem 3.1. *Let a function $f(x)$ be a triple differentiable function at the point $x \in \mathbb{R}$. Let a function $g(x)$ be a differentiable function at the point $x \in \mathbb{R}$. If the derivatives of the functions $f(x)$ and $g(x)$ are the bounded uniformly continuous functions at the point $x \in \mathbb{R}$, then the biharmonic Poisson integral (2.4) for the upper half plane can be transformed into the Taylor series expansion*

$$\begin{aligned} U(x,y) &= f(x) + g(x)\,y + \frac{d^2 f(x)}{dx^2}\frac{y^2}{2} + \\ &+ \int\limits_x^{+\infty} \left\{ \mathcal{K}(\tau - x, y)\frac{d^3 f(\tau)}{d\tau^3} + \mathcal{L}(\tau - x, y)\frac{dg(\tau)}{d\tau} \right\} d\tau - \\ &- \int\limits_{-\infty}^{x} \left\{ \mathcal{K}(x - \tau, y)\frac{d^3 f(\tau)}{d\tau^3} + \mathcal{L}(x - \tau, y)\frac{dg(\tau)}{d\tau} \right\} d\tau \end{aligned} \tag{3.1}$$

with the integral form of the remainder that contains the notations for the integral kernels

$$\begin{aligned} \mathcal{K}(\lambda, y) &= \frac{y^3}{\pi}\int\limits_{\lambda}^{+\infty} \frac{(t-\lambda)^2}{(t^2+y^2)^2} dt = \\ &= \frac{1}{2\pi}\left\{ \left(\lambda^2 + y^2\right)\left(\frac{\pi}{2} - \arctan\frac{\lambda}{y}\right) - \lambda y \right\} \end{aligned} \tag{3.2}$$

and

$$\mathcal{L}(\lambda, y) = \frac{y^2}{\pi}\int\limits_{\lambda}^{+\infty}\frac{dt}{t^2+y^2} = \frac{y}{\pi}\left(\frac{\pi}{2} - \arctan\frac{\lambda}{y}\right). \tag{3.3}$$

Proof. The theorem can be proved by a way to apply the following Taylor series expansions:

$$\begin{aligned} f(x+t) &= f(x) + f'(x)\,t + f''(x)\frac{t^2}{2} + \int\limits_x^{x+t} (x+t-\tau)^2 f'''(\tau)\frac{d\tau}{2}, \\ g(x+t) &= g(x) + \int\limits_x^{x+t} \frac{dg(\tau)}{d\tau} d\tau, \\ f(x-t) &= f(x) - f'(x)\,t + f''(x)\frac{t^2}{2} - \int\limits_{x-t}^{x} (x-t-\tau)^2 f'''(\tau)\frac{d\tau}{2}, \\ g(x-t) &= g(x) - \int\limits_{x-t}^{x} \frac{dg(\tau)}{d\tau} d\tau. \end{aligned} \tag{3.4}$$

The Taylor series expansions (3.4) must be substituted into the biharmonic Poisson integral (2.4) for the upper half plane. As a result, we need to evaluate the integrals

$$\int\limits_{0}^{+\infty} \frac{f\left(x+t\right)+f\left(x-t\right)}{\left(t^2+y^2\right)^2} dt = \frac{\pi f\left(x\right)}{2y^3} + \frac{\pi f''\left(x\right)}{4y} + \\ + \frac{1}{2} \int\limits_{0}^{+\infty} \frac{dt}{\left(t^2+y^2\right)^2} \int\limits_{x}^{x+t} \left(x+t-\tau\right)^2 f'''\left(\tau\right) d\tau - \\ - \frac{1}{2} \int\limits_{0}^{+\infty} \frac{dt}{\left(t^2+y^2\right)^2} \int\limits_{x-t}^{x} \left(x-t-\tau\right)^2 f'''\left(\tau\right) d\tau \tag{3.5}$$

and

$$\int\limits_{0}^{+\infty} \frac{g\left(x+t\right)+g\left(x-t\right)}{t^2+y^2} dt = \frac{\pi g\left(x\right)}{y} + \int\limits_{0}^{+\infty} \frac{dt}{t^2+y^2} \int\limits_{x}^{x+t} \frac{dg\left(\tau\right)}{d\tau} d\tau - \\ - \int\limits_{0}^{+\infty} \frac{dt}{t^2+y^2} \int\limits_{x-t}^{x} \frac{dg\left(\tau\right)}{d\tau} d\tau. \tag{3.6}$$

The double integrals of the formulas (3.5) and (3.6) must be transformed by a way to change the order of integration. So, we can obtain the results

$$\int\limits_{0}^{+\infty} \frac{f\left(x+t\right)+f\left(x-t\right)}{\left(t^2+y^2\right)^2} dt = \frac{\pi f\left(x\right)}{2y^3} + \frac{\pi f''\left(x\right)}{4y} + \\ + \frac{1}{2} \int\limits_{x}^{+\infty} \frac{d^3 f\left(\tau\right)}{d\tau^3} d\tau \int\limits_{\tau-x}^{+\infty} \frac{\left(t+x-\tau\right)^2}{\left(t^2+y^2\right)^2} dt - \\ - \frac{1}{2} \int\limits_{-\infty}^{x} \frac{d^3 f\left(\tau\right)}{d\tau^3} d\tau \int\limits_{x-\tau}^{+\infty} \frac{\left(t+\tau-x\right)^2}{\left(t^2+y^2\right)^2} dt \tag{3.7}$$

and

$$\int\limits_{0}^{+\infty} \frac{g\left(x+t\right)+g\left(x-t\right)}{t^2+y^2} dt = \frac{\pi g\left(x\right)}{y} + \int\limits_{x}^{+\infty} \frac{dg\left(\tau\right)}{d\tau} d\tau \int\limits_{\tau-x}^{+\infty} \frac{dt}{t^2+y^2} - \\ - \int\limits_{-\infty}^{x} \frac{dg\left(\tau\right)}{d\tau} d\tau \int\limits_{x-\tau}^{+\infty} \frac{dt}{t^2+y^2}. \tag{3.8}$$

After that, we need to substitute the obtained results for the integrals (3.7) and (3.8) into the biharmonic Poisson integral (2.4) for the upper half plane. Taking into account the notations (3.2) and (3.3), we obtain the Taylor series expansion (3.1). The theorem is proved. $\square$

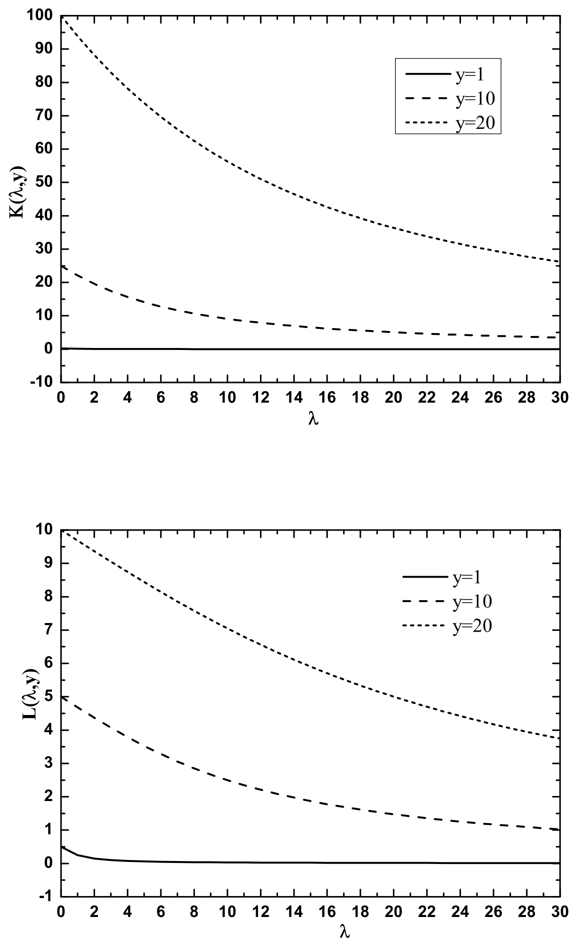

Figure 1: The dependencies of the integral kernels on λ for various values of the parameter y.

The contributions of the integral terms to the values of the biharmonic Poisson integral $U(x,y)$ are significantly determined by the integral kernels $\mathcal{K}(\lambda,y)$ and $\mathcal{L}(\lambda,y)$. The dependencies of these kernels on the variables λ and y are presented on the figures (1) and (2). The analysis of

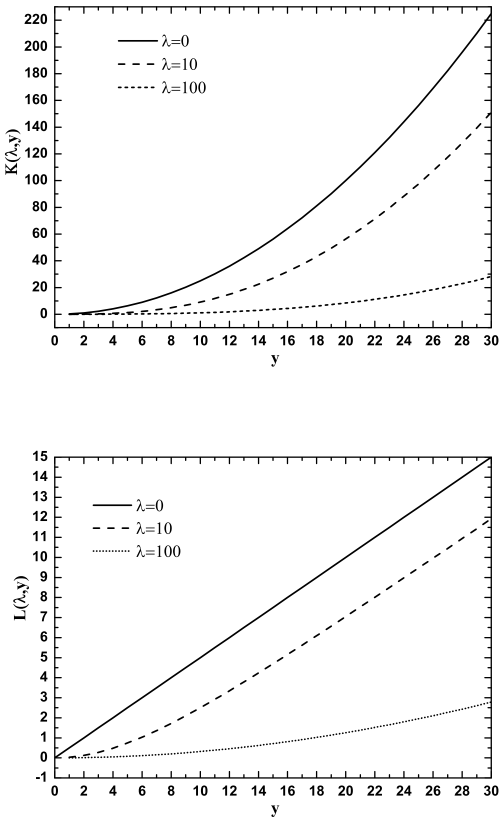

Figure 2: The dependencies of the integral kernels on y for various values of the parameter λ.

the figure (2) shows that the dependencies of the integral kernels on the variable y for the constant values of the parameter λ are the increasing functions. This means that the dependencies of the integral terms in the formula (3.1) are also increasing functions. Moreover, the increasing rate of the integral kernels $\mathcal{K}\left(\lambda,y\right)$ and $\mathcal{L}\left(\lambda,y\right)$ depends on λ: the integral kernels increase more slowly for the larger values of the parameter λ. It is also important to note that the increase of the integral kernel $\mathcal{K}$ is one order of magnitude faster than the increase of the integral kernel $\mathcal{L}$. It is significant to note that the asymptotic results in the case $\lambda = 0$ can also be presented: $\mathcal{K}\left(0,y\right) = \dfrac{y^2}{4}$ and $\mathcal{L}\left(0,y\right) = \dfrac{y}{2}$.

It is significant to note that an interesting conclusion can be made considering the figure (1) that presents the dependencies of the integral kernels on the variable $\lambda = |x - \tau|$. In fact, the variable λ characterizes the proximity of the variable x to the variable τ. We can see that the values of the integral kernels decrease rapidly for the larger values of the parameter λ. This means that the main contribution to the integral terms in the formula (3.1) can be obtained for the values of the integration variable τ that are close to x. Moreover, this is more tangible for small values of the parameter y. The value range of the integration variable τ with a prominent contribution to the integral terms is more extensive for the larger values of the parameter y.

Conclusion

In our theoretical investigation, the boundary value problem (1.1) is solved for the upper half plane $(y > 0)$. It was found that a general solution of the boundary value problem (1.1) can be represented by the double integral (1.2). As a result, the presented boundary value problem (1.1) is replaced by the boundary value problems (1.3) and (1.4) for the two-dimensional functions $\mathcal{A}(q, y)$ and $\mathcal{B}(q, y)$. The solutions of the above-mentioned boundary value problems are presented by the formula (2.1). The two-dimensional functions (2.1) are substituted into the double integral (1.2). As a result, the closed form solution (2.3) to the boundary value problem (1.1) is obtained.

To derive the Taylor series of biharmonic Poisson integral for the upper half plane, the closed form solution (2.3) is replaced by the closed form solution (2.4) that contains the integrals on the semiaxis $t > 0$. It was found that the Taylor series expansions (3.4) can enable one to transform the closed form solution (2.4) into the Taylor series (3.1) that contains the remainder in the integral form.

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