Український математичний вісник
Том 19 **(2022)**, № 3, 305 – 314
https://doi.org/10.32782/1810-3200-2022-19-3-1

On one inverse problem for a hyperbolic equation

Adalat Ya. Akhundov, Arasta Sh. Habibova

(Presented by F. Abdullayev)

Abstract. The paper considers the inverse problem of determining the unknown coefficient on the right-hand side of the hyperbolic equation. An additional condition for finding the unknown coefficient, which depends on the variable time, is given in the integral form. Theorems on the uniqueness, stability, and existence of the solution have been proved.

2010 MSC. 35R30, 35L70, 65M06.

Key words and phrases. Inverse problem, hyperbolic equation, uniqueness, "conditional" stability, existence.

We consider the following inverse problem on determining a pair of function $\{f(t), u(x,t)\}$

$$u_{tt} - u_{xx} = f(t)g(x), (x,t) \in D = (0,1) \times (0,T], \tag{1}$$

$$u(x,0) = \varphi(x), u_t(x,0) = \psi(x), x \in [0,1], \tag{2}$$

$$u(0,t) = u(1,t) = 0, t \in [0,T], \tag{3}$$

$$\int_0^1 u(x,t)dx = h(t), t \in [0,T], \tag{4}$$

where $g(x), \varphi(x), \psi(x), h(t)$ are the given functions,

$$u_t = \frac{\partial u}{\partial t},\ u_{tt} = \frac{\partial^2 u}{\partial t^2},\ u_x = \frac{\partial u}{\partial x},\ u_{xx} = \frac{\partial^2 u}{\partial x^2}.$$

Direct problems for hyperbolic equations are studied in the works [1, 2, 3] and others.

The coefficient inverse problem for a hyperbolic equation were studied in the papers (see [4, 5, 6]).

Received 21.06.2022

ISSN 1810 – 3200.

Problem (1)–(4) belongs to the class Hadamard ill-posed problems. Therefore, thes problem should be treated proceeding from the general concepts of the theory of ill-posed problems. We make the following assumptions for the data of problem (1)–(4):

1^0. $g(x) \in C[0,1], \int\limits_0^1 g(x)dx = g_0 \neq 0;$

2^0. $\varphi(x), \psi(x) \in C^2[0,1], \int\limits_0^1 \varphi(x)dx = h(0), \int\limits_0^1 \psi(x)dx = h'(0);$

3^0. $h(t) \in C^2[0,T].$

Definition 1. *The pair of functions* $\{f(t), u(x,t)\}$ *is called the solution of problem* (1)–(4) *if :*

1) $f(t) \in C[0,T]$;

2) $u(x,t) \in C^{2,2}(D) \cap C^{1,1}(\overline{D})$;

3) the conditions (1)–(4) *hold for these functions.*

First we bring problem (1)–(4) to an equivalent problem

Lemma 1. *Let conditions,* 1^0–3^0 *be satisfied. Then problems* (1)–(4) *and* (1), (2), (3) *and*

$$f(t) = \left[h''(t) - u_x(1,t) + u_x(0,t)\right]/g_0, t \in [0,T], \tag{5}$$

on finding the $\{f(t), u(x,t)\}$ *are equivalent in sense of definition 1.*

Where $h''(t) = \frac{d^2h(t)}{dt^2}$.

Proof. Let the pair of functions $\{f(t), u(x,t)\}$ be the solution of problem (1)–(4) in the sense of definition 1. If we integrate equation (1) in the interval $(0,1)$ with respect to the variable x, we get:

$$\int_0^1 u_{tt}dx - \int_0^1 u_{xx}dx = f(t)\int_0^1 g(x)dx. \tag{6}$$

Taking into account the conditions of Lemma 1, we obtain.

$$h''(t) - u_x(1,t) + u_x(0,t) = f(t)g_0.$$

Hence the reliability of formula (5) is obvious.

Now suppose that the pair of functions $\{f(t), u(x,t)\}$ is the classical solution of problem (1), (2), (3), (5). If we take into account formula (5) in (6), then for $y(t) = \int\limits_0^1 u(x,t)dx - h(t)$ we can write

$$y'' = 0$$

$$y(0) = 0, y'(0) = 0.$$

It is clear that the only solution to the problem for ther unkown $y(t)$ is $y(t) \equiv 0$. From here we get $\int\limits_0^1 u(x,t)dx = h(t), t \in [0,T]$.

The lemma 1 is proved. □

The uniqueness theorem and estimate of stability for the solutions of inverse problems occupy a central place in investigation of their well-poseduess. Define the following set :

$$K = \{(f,u)|f(t) \in C[0,T], u(x,t) \in C^{2,1}(D) \cap C^{1,1}(\overline{D}), |f(t)| \\ \leq c_1, |u_x(x,t)| \leq c_2, \\ (x,t) \in \overline{D}, u_{1x}(0,t) = u_{2x}(0,t), u_{1x}(1,t) = u_{2x}(1,t), t \in [0,T], \\ \forall (f_1,u_1),(f_2,u_2) \in K, c_1,c_2 = const > 0\}$$

Let us assume that the two input sets, $\{g_1(x), \varphi_1(x), \psi_1(x), h_1(t)\}$ and $\{g_2(x),\ \varphi_2(x), \psi_2(x), h_2(t)\}$ are given for problem (1),(2),(3),(5). For brevity of the further exposition, problem I_1, with the second input set we will call problem I_2. Let $\{f_1(t), u_1(x,t)\}$ and $\{f_2(t), u_2(x,t)\}$ be solutions of problems I_1 and I_2 respectively.

Theorem 1. *Let the following conditions hold*

1) *the functions* $g_i(x), \varphi_i(x), \psi_i(x), h_i(t), i = 1,2$ *satisfy conditions* 1^0–3^0 *respectively;*

2) *Solutions of problems* I_1 *and* I_2 *exist in the sense of definition 1 and they belong to the set* K.

Then there exists a $T^*(0 < T^* \leq T)$, *such that for* $(x,t) \in \overline{D}_* = [0,1] \times [0,T^*]$ *the solution of problem* (1), (2), (3), (5) *is unique, and the stability estimate*

$$\int\limits_0^1 \left[(u_{1t}(x,t) - u_{2t}(x,t))^2 + (u_{1x}(x,t) - u_{2x}(x,t))^2\right] dx$$

$$+c_3 \|f_1(t) - f_2(t)\|_0^2 \leq c_4 \left\{ \int\limits_0^1 \left[(g_1(x) - g_2(x))^2 + (\varphi_1'(x) - \varphi_2'(x))^2 \right.\right.$$

$$\left.\left. + (\psi_1(x) - \psi_2(x))^2\right] dx + \|h_1(t) - h_2(t)\|_2^2 \right\} \tag{7}$$

is valid, where $c_3, c_4 > 0$ *depends on the data of problem* I_1 *and* I_2 *in the set* K, $\|q(t)\|_T^k = \sum\limits_{i=0}^{k} \max\limits_{[0,T]} \left|\frac{d^i q(t)}{dt^i}\right|$.

Proof. First, we prove inequality (7) under the condition $g_1 = g_2, \varphi_1 = \varphi_2, \psi_1 = \psi_2, h_1 = h_2$.

Denote

$$z(x,t) = u_1(x,t) - u_2(x,t), \lambda(t) = f_1(t) - f_2(t), \delta_1(x) = g_1(x) - g_2(x),$$

$$\delta_2(x) = \varphi_1(x) - \varphi_2(x), \delta_3(x) = \psi_1(x) - \psi_2(x), \delta_4(t) = h_1(t) - h_2(t).$$

Subtracting from the relations of problem I_1 the corresponding relations of problem I_2 we obtain the problem of determining a pair of functions $\{\lambda(t), z(x,t)\}$

$$z_{tt} - z_{xx} = \lambda(t) g_1(x) + f_2(t)\delta_1(x), (x,t) \in D, \tag{8}$$

$$z(x,0) = \delta_2(x), z_t(x,0) = \delta_3(x), x \in [0,1], \tag{9}$$

$$z(0,t) = z(1,t) = 0,\ t \in [0,T], \tag{10}$$

$$\lambda(t) = \delta_4^{''}(t) \setminus g_{01} + H(t), t \in [0,T] \tag{11}$$

where

$$g_{0i} = \int_0^1 g_i(x)dx, i = 1,2,\ H(t) = \left[h_2^{''}(t) - u_{2x}(1,t) + u_{2x}(0,t)\right] \times$$

$$\times \left(g_{02} - g_{01}\right) \setminus \left(g_{01} \cdot g_{02}\right).$$

Multiply equations (8) by $2z_t(x,t)$ and integrate over the domain D:

$$2\int_0^t\int_0^1 \left[z_{tt} - z_{xx}\right] z_t dx dt = 2\int_0^t\int_0^1 \left[\lambda(t) g_1(x) + f_2(t)\delta_1(x)\right] z_t dx dt. \tag{12}$$

If consider

$$2\int_0^t\int_0^1 z_{tt} z_t dx dt = \int_0^1 \left[z_t^2(x,t) - \delta_3^2(x)\right] dx,$$

$$2\int_0^t\int_0^1 z_{xx} z_t dx dt = -\int_0^1 \left[z_x^2(x,t) - (\delta_2^{'}(x))^2\right] dx,$$

$$2\int_0^t\int_0^1 \left[\lambda(\tau) g_1(x) + f_2(\tau)\delta_1(x)\right] z_\tau dx d\tau \le \int_0^t\int_0^1 \lambda^2(\tau) g_1^2 dx d\tau$$

$$+\int_0^t\int_0^1 f_2^2(\tau)\delta_1^2(x)dxd\tau+\int_0^t\int_0^1\left[z_\tau^2(x,\tau)+z_x^2(x,\tau)\right]dxd\tau$$

then from (12) we get:

$$\int_0^1\left[z_t^2(x,t)+z_x^2(x,t)\right]dx\le\int_0^1\left[\delta_3^2(x)+(\delta_2'(x))^2\right]dx$$

$$+c_5\int_0^1\delta_1^2(x)dx+c_6t\,\|\lambda\|_0^2+\int_0^t\int_0^1\left[z_\tau^2(x,\tau)+z_x^2(x,\tau)\right]dxd\tau,$$

where $c_5, c_6>0$ depend on the data of problems I_1 and I_2 on the set K.

Denoting

$$w(t)=\int_0^t\int_0^1\left[z_\tau^2(x,\tau)+z_x^2(x,\tau)\right]dxd\tau,$$

we get

$$\begin{aligned}&\tfrac{dw}{dt}\le c_7+c_8t+w(t)\\&w(0)=0.\end{aligned}\tag{13}$$

The initial data of problem (13) satisfy the conditions of lemma 1 [3, p. 164]. Therefore, we can write

$$\int_0^1\left[z_t^2(x,t)+z_x^2(x,t)\right]dx\le c_9\int_0^1\left[\delta_1^2(x)+\delta_2^{\prime 2}(x)+\delta_3^2(x)\right]dx+c_{10}t\,\|\lambda\|_0^2.\tag{14}$$

Let us estimate the function $\lambda(t)$. From (11) we have

$$|\lambda(t)|\le\left|\delta_4''(t)\right|/\left|g_{01}\right|+|H(t)|,$$

$$\lambda(t)^2\le c_{11}\|\delta_4\|_2^2+c_{12}\int_0^1\delta_1^2(x)dx,\ \ t\in[0,T].$$

The last inequality is satisfied for each $t\in[0,T]$, so it must be satisfied for the maximum value of the left side:

$$\|\lambda\|_0^2\le c_{11}\|\delta_4\|_2^2+c_{12}\int_0^1\delta_1^2(x)dx.\tag{15}$$

From (14) and (15) we get:

$$\int_0^1 \left[z_t^2(x,t)+z_x^2(x,t)\right]dx+\|\lambda\|_0^2 \le c_{13}\int_0^1 \left[\delta_1^2(x)+\delta_2^{'2}(x)\right.$$
$$\left.+\delta_3^2(x)\right]dx+c_{14}t\,\|\lambda\|_0^2+c_{15}\,\|\delta_4\|_2^2\,.$$

Let $T* \in (0,T]$ be a number such that $c_{14}T^* < 1$. Then we get that $\overline{D}_* = [0,1]\times[0,T^*]$ the estimater of "conditional" stability is true (7).

The uniquenuess of the solution problem (1), (2), (3), (5) is obtained from uniquality (7) for $g_1(x) = g_2(x)$, $\varphi_1(x) = \varphi_2(x)$, $\psi_1(x) = \psi_2(x)$, $h_1(t) = h_2(t)$.

The theorem 1 has been proved. □

For A.N. Tikhonov correct problems, the existence of a solution is a priori assumed and justified by the physical meaning of the problem under consideration.

Despite the fact that the proof of the existence of a solution to ill-posed problems requires some additional conditions for the input data, but from the point of view of constructing algorithims for the exact or appoximate fiuding of a solution to the problem, it is ceratinly of practical interest.

Theorem 2. *Let be*

1) $g(x)\in C^1[0,1], g(0)=g(1)=0, \int_0^1 g(x)dx = g_0 \neq 0;$

2) $\varphi(x)\in C^2[0,1], \varphi(0)=\varphi(1)=0, \varphi''(0)=\varphi''(1)=0, \int_0^1 \varphi(x)dx = h(0);$

3) $\psi(x)\in C^2[0,1], \psi(0)=\psi(1)=0, \int_0^1 \psi(x)dx = h'(0);$

4) $h(t)\in C^2[0,T]$

The problem (1), (2), (3), (5) *in* $\overline{D}=[0,1]\times[0,T]$ *has a solution in the sense of the definition 1.*

Proof. For a given $f(t)\in C[0,T]$, the solution of problem (1), (2), (3) on the determination of $u(x,t)$ will be sought in the form

$$u(x,t)=\vartheta(x,t)+w(x,t)$$

Here $\vartheta(x,t)$ is the solution of the following problem

$$\vartheta_{tt}-\vartheta_{xx}=0,(x,t)\in D, \tag{16}$$

$$\vartheta(x,0)=\varphi(x), \vartheta_t(x,0)=\psi(x), x\in[0,1], \tag{17}$$

$$\vartheta(0,t)=\vartheta(1,t)=0, t\in[0,T], \tag{18}$$

and $w(x,t)$ is the solution of the following

$$w_{tt} - w_{xx} = f(t)g(x), (x,t) \in D, \tag{19}$$

$$w(x,0) = w_t(x,0) = 0, x \in [0,1], \tag{20}$$

$$w(0,t) = w(1,t) = 0, t \in [0,T]. \tag{21}$$

Let us first consider a homogeneous equation (16). We will look for all solutions of this equation that can be represented in the form $\vartheta(x,t) = y(t)q(x)$ and satisfy the boundary condition (18).

We can say that each of the functions

$$\vartheta_n(x,t) = (A_n \cos\lambda_n t + B_n \sin\lambda_n t)\sin\lambda_n x,\ \lambda_n = n\pi,\ n = 1,2,...$$

(for any constants A_n and B_n) is a solution to the equation (16) that satisfies the boundary condition (18).

Taking into account the initial conditions leads us to the following expression for the coefficients A_n and B_n:

$$A_n = \varphi_n, B_n = \frac{1}{\lambda_n}\psi_n. \tag{22}$$

Here φ_n and ψ_n denote the Fourier coefficients of the functions $\varphi(x)$ and $\psi(x)$ on the system $\{\sin\lambda_n x\}$.

Thus, formally we came to the following representation of the solution of the mixed problem (16)–(18)

$$\vartheta(x,t) = \sum_{n=1}^{\infty} y_n(t)q_n(x) = \sum_{n=1}^{\infty}\left[\varphi_n \cos\lambda_n t + \frac{1}{\lambda_n}\psi_n \sin\lambda_n t\right]\sin\lambda_n x. \tag{23}$$

Formally expanding the desired solution $w(x,t)$ of problem (19)–(21) and the right side of the equation (19) $f(t)g(x)$ in a series of eigen functions $\{\sin\lambda_n x\} : w(x,t) = \sum\limits_{n=1}^{\infty}\theta_n(t)\sin\lambda_n x$, and $f(t)g(x) = \sum\limits_{n=1}^{\infty} f(t)g_n \sin\lambda_n x$ and taking into account these functions in (19)-(21) we obtain:

$$\begin{gathered} \theta_n''(t) + \lambda_n^2\theta_n(t) = f_n(t), \\ \theta_n(0) = \theta_n'(0) = 0, \end{gathered} \tag{24}$$

here $f_n(t) = f(t)g_n, n = 1,2,....$

The solution to problem (24) in the function

$$\theta_n(t) = \frac{1}{\lambda_n}\int\limits_0^t f_n(\tau)\sin\lambda_n(t-\tau)d\tau.$$

Given that $w_n(x,t)=\theta_n(t)\sin\lambda_n x$ we get:

$$u(x,t)=\sum_{n=1}^{\infty}\left\{\left(\varphi_n\cos\lambda_n t+\frac{\psi_n}{\lambda_n}\sin\lambda_n t\right)+\frac{1}{\lambda_n}\int_0^t f_n(\tau)\sin\lambda_n(t-\tau)d\tau\right\}\sin\lambda_n x. \tag{25}$$

In order for the function (25) to be a solution problem (1)–(3) for each $f(t)\in C[0,T]$ series (25) and the formally composed following series must converge uniformly

$$u_t(x,t)=\sum_{n=1}^{\infty}\{(-\lambda_n\varphi_n\sin\lambda_n t+\psi_n\cos\lambda_n t)$$

$$+\int_0^t f_n(\tau)\cos\lambda_n(t-\tau)d\tau\Bigg\}\sin\lambda_n x, \tag{26}$$

$$u_{tt}(x,t)=\sum_{n=1}^{\infty}\{(-\lambda_n^2\varphi_n\cos\lambda_n t-\lambda_n\psi_n\sin\lambda_n t)$$

$$-\lambda_n\int_0^t f_n(\tau)\sin\lambda_n(t-\tau)d\tau\Bigg\}\sin\lambda_n x, \tag{27}$$

$$u_x(x,t)=\sum_{n=1}^{\infty}\lambda_n\left\{\left(\varphi_n\cos\lambda_n t+\frac{\psi_n}{\lambda_n}\sin\lambda_n t\right)\right.$$

$$\left.+\frac{1}{\lambda_n}\int_0^t f_n(\tau)\sin\lambda_n(t-\tau)d\tau\right\}\cos\lambda_n x, \tag{28}$$

$$u_{xx}(x,t)=-\sum_{n=1}^{\infty}\lambda_n^2\left\{\left(\varphi_n\cos\lambda_n t+\frac{\psi_n}{\lambda_n}\sin\lambda_n t\right)\right.$$

$$\left.+\frac{1}{\lambda_n}\int_0^t f_n(\tau)\sin\lambda_n(t-\tau)d\tau\right\}\sin\lambda_n x. \tag{29}$$

The following series are majorants of series (25)-(28) respectively.

$$\sum_{n=1}^{\infty}|u_n(x,t)|\le c_{16}\sum_{n=1}^{\infty}\left[|\varphi_n|+\frac{|\psi_n|}{\lambda_n}\right],$$

$$\sum_{n=1}^{\infty}|u_{nt}(x,t)|\le c_{17}\sum_{n=1}^{\infty}\left[\lambda_n|\varphi_n|+|\psi_n|+|g_n|\right],$$

$$\sum_{n=1}^{\infty} |u_{ntt}(x,t)| \le c_{18} \sum_{n=1}^{\infty} \left[\lambda_n^2 |\varphi_n| + \lambda_n |\psi_n| + \lambda_n |g_n|\right],$$

$$\sum_{n=1}^{\infty} |u_{nx}(x,t)| \le c_{19} \sum_{n=1}^{\infty} \left[\lambda_n |\varphi_n| + |\psi_n| + |g_n|\right],$$

$$\sum_{n=1}^{\infty} |u_{nxx}(x,t)| \le c_{20} \sum_{n=1}^{\infty} \left[\lambda_n^2 |\varphi_n| + \lambda_n |\psi_n| + \lambda_n |g_n|\right].$$

Under the conditions of Theorem 2 the majourant series converge [1].

Thus, for each given $f(t) \in C[0,T]$, the function (25) is a solution to problem (1)–(3) in the sence of definition 1.

Now we will show the existence of the function $f(t) \in C[0,T]$.

Let denote $Q = C[0,T]$. Write equation (5) in operator form:

$$M[f(t)] = f(t), M : Q \to Q$$

$$M[f(t)] = \left\{ h''(t) + \sum_{m=1}^{\infty} [(\lambda_{2m-1}\varphi_{2m-1} \cos \lambda_{2m-1} t \right.$$

$$\left. + \psi_{2m-1} \sin \lambda_{2m-1} t) + \int_0^t f_{2m-1}(\tau) \sin \lambda_{2m-1}(t-\tau) d\tau \right\} / g_0.$$

Denote

$$Q' = \{ f | f(t) \in C[0,T], |f(t) \le f_0, t \in [0,T] \}$$

$f_0 > 0$ some constant.

It is clear that $M[Q'] \subset Q$. Show that the set $M[Q']$ in uniformly bounded and eqicontinous:

$$M[f(t)] = \left\{ |h''(t)| + \sum_{m=1}^{\infty} (\lambda_{2m-1} |\varphi_{2m-1}| + |\psi_{2m-1}| + |f(t)| T) \right\} / |g_0|.$$

Under the conditions of Theorem 2 and belouging to $f(t) \in Q'$ from the last inequality we obtain the uniform boundedness of the set $M[Q']$.

We show the equicontinuity of the set $M[Q']$. Estimate the difference $M[f(t_1)] - M[f(t_2)]$ for any $t_1, t_2 \in [0,T]$.

$$|M[f(t_1)] - M[f(t_2)]| \le \left\{ |h''(t_1) - h''(t_2)| \right.$$

$$+ \sum_{m=1}^{\infty} [\lambda_{2m-1} |\varphi_{2m-1}| |\cos \lambda_{2m-1} t_1 - \cos \lambda_{2m-1} t_2| + |\psi_{2m-1}|$$

$$\times |\sin \lambda_{2m-1} t_1 - \sin \lambda_{2m-1} t_2|$$

$$+\int_0^{t_2} |f_{2m-1}(\tau)| \left|\sin\lambda_{2m-1}(t_1-\tau) - \sin\lambda_{2m-1}(t_2-\tau)\right| d\tau$$

$$\left. \left. + \int_{t_2}^{t_1} |f_{2m-1}(\tau)| \left|\sin\lambda_{2m-1}(t_1-\tau)\right| d\tau \right] \right\} / |g_0|.$$

Taking into account the conditions of Theorem 2, for the last inequality we have

$$|M[f(t_1)] - M[f(t_2)]| \le c_{21} |t_1 - t_2| .$$

Thus, by the Arsels theorem, the set $M[Q']$ is compact in Q [7]. In this case, according to the Schounder theorem, the operator $M[f(t)]$ has at least one fixed point, in other words, the operator equation $M[f(t)] = f(t)$ has solution $f(t) \in Q = C[0,T]$.

Theorem 2 is proved. □

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Contact information

Adalat Ya. Akhundov

Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
E-Mail: adalatakhund@gmail.com

Arasta Sh. Habibova

Lankaran State University, Lankaran, Azerbaijan
E-Mail: arasta.h@mail.ru