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BMO and Dirichlet problem for degenerate Beltrami equation

Vladimir Gutlyanskii, Vladimir Ryazanov,
Evgeny Sevost'yanov, Eduard Yakubov

Dedicated to the memory of Professor Uri Srebro (1936–2016)

Abstract. Following Bojarski and Vekua, we study the Dirichlet problem $\lim\limits_{z\to\zeta} \operatorname{Re} f(z) = \varphi(\zeta)$ as $z \to \zeta$, $z \in D, \zeta \in \partial D$, with continuous boundary data $\varphi(\zeta)$ in bounded domains D of the complex plane $\mathbb{C}$, where f satisfies the degenerate Beltrami equation $f_{\bar z} = \mu(z) f_z$, $|\mu(z)| < 1$, a.e. in D. Assuming that D is arbitrary simply connected, we have established in terms of μ the BMO and FMO criteria as well as a number of other integral criteria on the existence and representation of regular discrete open solutions to the stated above problem. We have also proven similar theorems on the existence of multi-valued solutions to the problem with single-valued real parts in an arbitrary bounded domain D with no boundary component degenerated to a single point. Finally, we have given similar solvability and representation results concerning the Dirichlet problem in such domains for the degenerate A - harmonic equation associated with the Beltrami equation.

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1. Introduction

Let D be a domain in the complex plane $\mathbb{C}$ and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. in D. A **Beltrami equation** is an equation of the form

$$f_{\bar z} = \mu(z)\, f_z \tag{1.1}$$

with the formal complex derivatives $f_{\bar z} = \overline{\partial} f = (f_x + i f_y)/2$, $f_z = \partial f = (f_x - i f_y)/2$, $z = x + iy$, where f_x and f_y are usual partial derivatives of

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f in x and y, correspondingly. The function μ is said to be the **complex coefficient** and

$$K_\mu(z) \;:=\; \frac{1+|\mu(z)|}{1-|\mu(z)|} \tag{1.2}$$

the **dilatation quotient** of the equation (1.1). The Beltrami equation is called **degenerate** if $\operatorname{ess\,sup} K_\mu(z) = \infty$.

It is known that if K_μ is bounded, then the Beltrami equation has homeomorphic solutions, see e.g. monographs [1, 5] and [26]. Recently, a series of effective criteria for existence of homeomorphic solutions have been also established for degenerate Beltrami equations, see e.g. historic comments with relevant references in monographs [3, 15] and [27], in BMO-article [37] and in surveys [16] and [46].

These criteria were formulated both in terms of K_μ and the more refined quantity that takes into account not only the modulus of the complex coefficient μ but also its argument

$$K^T_\mu(z, z_0) \;:=\; \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1-|\mu(z)|^2} \tag{1.3}$$

that is called the **tangent dilatation quotient** of the Beltrami equation with respect to a point $z_0 \in \mathbb{C}$, see e.g. [2, 6, 7, 13, 25] and [37–42]. Note that

$$K^{-1}_\mu(z) \leqslant K^T_\mu(z, z_0) \leqslant K_\mu(z) \qquad \forall\, z \in D\,,\ \ z_0 \in \mathbb{C}\ . \tag{1.4}$$

The geometrical sense of K^T_μ can be found e.g. in monographs [15] and [27].

Following [4] and [49], the **Dirichlet problem** for the Beltrami equation (1.1) in a domain $D \subset \mathbb{C}$ is the problem on the existence of a continuous function $f : D \to \mathbb{C}$ with generalized derivatives by Sobolev of the first order, satisfying (1.1) a.e., such that

$$\lim_{z\to\zeta} \operatorname{Re} f(z) = \varphi(\zeta) \qquad \forall\, \zeta \in \partial D \tag{1.5}$$

for each prescribed continuous function $\varphi : \partial D \to \mathbb{R}$.

If D is the unit disk, some criteria for the solvability of the Dirichlet problem for the degenerate Beltrami equation can be found in monograph [15], see also survey [16]. The case of domains bounded by a finite collection of Jordan curves has been studied in [23] and [36]. With the help of the concept of the prime end by Caratheodory, we have extended the above criteria to more general domains in [8] and [17]. However, from the point of view of applications, such an approach is quite complicated. That is why, in this paper we continue to study the Dirichlet problem in the classic setting (1.5) for the degenerate Beltrami equation in arbitrary bounded domains $D \subset \mathbb{C}$.

The paper is organized as follows. In Section 2 we give Lemma 1 on the existence of regular homeomorphic solutions f with hydrodynamic normalization $f(z) = z + o(1)$ as $z \to \infty$ to the degenerate Beltrami equations in $\mathbb{C}$ whose complex coefficient μ has compact support. Section 3 contains criteria for existence and representation of regular discrete open solutions for the Dirichlet problem with continuous data to deg enerate Beltrami equations in arbitrary simply connected bounded domains D in $\mathbb{C}$. In Section 4 we obtain similar criteria for the existence of multi-valued solutions f with single–valued real parts in the spirit of the theory of multi-valued analytic functions in arbitrary bounded domains $D \subset \mathbb{C}$ with no boundary component degenerated to a single point. Note that the real part u of such a solution f is the $A-$harmonic function, i.e., a single-valued continuous solution of degenerate elliptic equation $\operatorname{div}(A\nabla u) = 0$ with a matrix-valued coefficient A associated with μ. Section 5 contains a number of solvability criteria to the Dirichlet problem for the $A-$harmonic equation.

2. More definitions and preliminary remarks

Let D be a domain in the complex plane $\mathbb{C}$. A function $f : D \to \mathbb{C}$ in the Sobolev class $W^{1,1}_{\text{loc}}$ is called a **regular solution** of the Beltrami equation (1.1) if f satisfies (1.1) a.e. and its Jacobian $J_f(z) > 0$ a.e.

Lemma 1. *Let a function $\mu : \mathbb{C} \to \mathbb{C}$ be with compact support S, $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1(S)$. Suppose that, for every $z_0 \in S$, there is a family of measurable functions $\psi_{z_0,\varepsilon} : (0, \varepsilon_0) \to (0, \infty)$, $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 = \varepsilon(z_0) > 0$, such that*

$$I_{z_0}(\varepsilon) \, : = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\, dt \, < \, \infty \qquad \forall\, \varepsilon \in (0, \varepsilon_0) \tag{2.1}$$

and

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z, z_0) \cdot \psi^2_{z_0,\varepsilon}(|z - z_0|)\, dm(z) = o(I^2_{z_0}(\varepsilon)) \quad \text{as } \varepsilon \to 0 \ \forall z_0 \in S. \tag{2.2}$$

Then the Beltrami equation (1.1) *has a regular homeomorphic solution f_μ with the hydrodynamic normalization $f_\mu(z) = z + o(1)$ as $z \to \infty$.*

Here and further $dm(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$. This lemma was first proved as Lemma 2.1 in our paper [14]. In view of its importance, we give here its alternative proof.

Proof. By Lemma 3 and Remark 2 in [41] the Beltrami equation (1.1) has a regular homeomorphic solution f in $\mathbb{C}$ under the hypotheses on μ given above. Note that f is holomorphic and univalent (one-to-one), i.e. conformal, and with no zeros outside of a closed disk

$$\{z \in \mathbb{C} : |z| \le R\}, \qquad R > 0\ ,$$

because the support S of μ is compact.

Let us consider the function $F(\zeta) := f(1/\zeta)$, $\zeta \in \mathbb{C}_0 := \overline{\mathbb{C}} \setminus \{0\}$, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, that is conformal in a punctured disk $\mathbb{D}_r \setminus \{0\}$, where $\mathbb{D}_r = \{\zeta \in \mathbb{C} : |\zeta| < r\}$, $r = 1/R$, and 0 is its isolated singular point. In view of the Casorati-Weierstrass theorem, see e.g. Proposition II.6.3 in [11], 0 cannot be essential singular point because the mapping F is homeomorphic.

Moreover, 0 cannot be a removable singular point of F. Indeed, let us assume that F has a finite limit $\lim\limits_{\zeta \to 0} F(\zeta) = c$. Then the extended mapping $\tilde{F}$ is a homeomorphism of $\overline{\mathbb{C}}$ into $\mathbb{C}$. However, by stereographic projection $\overline{\mathbb{C}}$ is homeomorphic to the sphere $\mathbb{S}^2$ and, consequently, by the Brouwer theorem on the invariance of domain the set $C := \tilde{F}(\overline{\mathbb{C}})$ is open in $\overline{\mathbb{C}}$, see e.g. Theorem 4.8.16 in [45]. In addition, the set C is compact as a continuous image of the compact space $\overline{\mathbb{C}}$. Hence the set $\overline{\mathbb{C}} \setminus C \neq \emptyset$ is also open in $\overline{\mathbb{C}}$. The latter contradicts the connectivity of $\overline{\mathbb{C}}$, see e.g. Proposition I.1.1 in [11].

Thus, 0 is a (unique) pole of the function F in the disk $\mathbb{D}_r$. Hence the function $\Phi(\zeta) := 1/F(\zeta)$ has a removable singularity at 0 and $\Phi(0) = 0$. By the Riemann extension theorem, see e.g. Proposition II.3.7 in [11], the extended function $\tilde{\Phi}$ is conformal in $\mathbb{D}_r$. By the Rouche theorem $\tilde{\Phi}'(0) \neq 0$, see e.g. Theorem 63 in [48], and, consequently, the function $\tilde{\Phi}$ has the expansion of the form $c_1\zeta + c_2\zeta^2 + \ldots$ in the disk $\mathbb{D}_r$ with $c_1 \neq 0$. Consequently, along the set $\{z \in \mathbb{C} : |z| > R\}$

$$f(z) \;=\; \frac{1}{\Phi(\frac{1}{z})} \;=\frac{1}{c_1 z^{-1} + c_2 z^{-2} + \ldots} \;=\; \frac{z}{c_1}\left(1 + \frac{c_2}{c_1} z^{-1} + \ldots\right)^{-1}$$

$$= \; c_1^{-1} z \;-\; c_1^{-2} c_2 \;+\; o(1)\ ,$$

i.e. the function $f_\mu(z) := c_1 f(z) + c_2/c_1$ gives the desired regular homeomorphic solution of the Beltrami equation with the hydrodynamic normalization $f_\mu(z) = z + o(1)$ as $z \to \infty$. □

In particular, by relations (1.4) we obtain from Lemma 1 the following consequence.

Corollary 1. *Let a function* $\mu : \mathbb{C} \to \mathbb{C}$ *be with compact support* S, $|\mu(z)| < 1$ *a.e. and* $K_\mu \in L^1(S)$ *and let* $\psi : (0, \varepsilon_0) \to (0, \infty)$ *for some* $\varepsilon_0 > 0$ *be a measurable function such that*

$$\int_0^{\varepsilon_0} \psi(t)\, dt = \infty\,, \qquad \int_\varepsilon^{\varepsilon_0} \psi(t)\, dt < \infty \qquad \forall\, \varepsilon \in (0, \varepsilon_0)\,. \tag{2.3}$$

Suppose that

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu(z)\cdot\psi^2(|z-z_0|)\, dm(z) \le O\left(\int_\varepsilon^{\varepsilon_0} \psi(t)dt\right) \text{ as } \varepsilon \to 0\ \forall\, z_0 \in S. \tag{2.4}$$

Then the Beltrami equation (1.1) *has a regular homeomorphic solution* f *with the hydrodynamic normalization* $f(z) = z + o(1)$ *as* $z \to \infty$.

Recall that a real-valued function u in a domain D in $\mathbb{C}$ is said to be of **bounded mean oscillation** in D, abbr. $u \in \mathrm{BMO}(D)$, if $u \in L^1_{\mathrm{loc}}(D)$ and

$$\|u\|_* := \sup_B \frac{1}{|B|} \int_B |u(z) - u_B|\, dm(z) < \infty\,, \tag{2.5}$$

where the supremum is taken over all discs B in D and

$$u_B = \frac{1}{|B|} \int_B u(z)\, dm(z)\,.$$

We write $u \in \mathrm{BMO}_{\mathrm{loc}}(D)$ if $u \in \mathrm{BMO}(U)$ for every relatively compact subdomain U of D (we also write BMO or $\mathrm{BMO}_{\mathrm{loc}}$ if it is clear from the context what D is).

The class BMO was introduced by John and Nirenberg (1961) in the paper [21] and soon became an important concept in harmonic analysis, partial differential equations and related areas, see e.g. [18] and [34].

A function φ in BMO is said to have **vanishing mean oscillation**, abbr. $\varphi \in$ VMO, if the supremum in (2.5) taken over all balls B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$. VMO has been introduced by Sarason in [44]. There are a number of papers devoted to the study of partial differential equations with coefficients of the class VMO, see e.g. [10, 20, 28, 30, 31] and [32].

Remark 1. Note that $W^{1,2}(D) \subset VMO(D)$, see e.g. [9].

Following [19], we say that a function $\varphi : D \to \mathbb{R}$ has **finite mean oscillation** at a point $z_0 \in D$, abbr. $\varphi \in \mathrm{FMO}(z_0)$, if

$$\overline{\lim_{\varepsilon\to 0}} \;\; -\!\!\!\!\!\!\int_{B(z_0,\varepsilon)} |\varphi(z) - \widetilde{\varphi}_\varepsilon(z_0)|\, dm(z) < \infty\,, \tag{2.6}$$

where

$$\widetilde{\varphi}_{\varepsilon}(z_0) = \fint_{B(z_0,\varepsilon)} \varphi(z)\, dm(z) \tag{2.7}$$

is the mean value of the function $\varphi(z)$ over the disk $B(z_0, \varepsilon) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$. Note that the condition (2.6) includes the assumption that φ is integrable in some neighborhood of the point z_0. We say also that a function $\varphi : D \to \mathbb{R}$ is of **finite mean oscillation in** D, abbr. $\varphi \in \mathrm{FMO}(D)$ or simply $\varphi \in \mathrm{FMO}$, if $\varphi \in \mathrm{FMO}(z_0)$ for all points $z_0 \in D$. We write $\varphi \in \mathrm{FMO}(\overline{D})$ if φ is given in a domain G in $\mathbb{C}$ such that $\overline{D} \subset G$ and $\varphi \in \mathrm{FMO}(G)$.

The following statement is obvious by the triangle inequality.

Proposition 1. *If, for a collection of numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,*

$$\overline{\lim_{\varepsilon \to 0}} \fint_{B(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon|\, dm(z) < \infty\,, \tag{2.8}$$

then φ is of finite mean oscillation at z_0.

In particular, choosing here $\varphi_\varepsilon \equiv 0$, $\varepsilon \in (0, \varepsilon_0]$ in Proposition 1, we obtain the following.

Corollary 2. *If, for a point $z_0 \in D$,*

$$\overline{\lim_{\varepsilon \to 0}} \fint_{B(z_0,\varepsilon)} |\varphi(z)|\, dm(z) < \infty\,, \tag{2.9}$$

then φ has finite mean oscillation at z_0.

Recall that a point $z_0 \in D$ is called a **Lebesgue point** of a function $\varphi : D \to \mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon \to 0} \fint_{B(z_0,\varepsilon)} |\varphi(z) - \varphi(z_0)|\, dm(z) = 0\,. \tag{2.10}$$

It is known that, almost every point in D is a Lebesgue point for every function $\varphi \in L^1(D)$. Thus, we have by Proposition 1 the next corollary.

Corollary 3. *Every locally integrable function $\varphi : D \to \mathbb{R}$ has a finite mean oscillation at almost every point in D.*

Remark 2. Note that the function $\varphi(z) = \log(1/|z|)$ belongs to BMO in the unit disk Δ, see, e.g., [34], p. 5, and hence also to FMO. However, $\widetilde{\varphi}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that condition (2.9) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0. Clearly, $\mathrm{BMO}(D) \subset \mathrm{BMO}_{\mathrm{loc}}(D) \subset \mathrm{FMO}(D)$ and as well-known $\mathrm{BMO}_{\mathrm{loc}} \subset L^p_{\mathrm{loc}}$ for all $p \in [1, \infty)$, see, e.g., [21] or [34]. However, FMO is not a subclass of $L^p_{\rm loc}$ for any $p > 1$ but only of $L^1_{\rm loc}$. Thus, the class FMO is much more wider than $\rm BMO_{loc}$.

Versions of the next lemma has been first proved for the class BMO in [37]. For the FMO case, see the papers [1,19,39,40] and the monographs [15] and [27].

Lemma 2. *Let D be a domain in $\mathbb{C}$ and let $\varphi : D \to \mathbb{R}$ be a non-negative function of the class* $\mathrm{FMO}(z_0)$ *for some $z_0 \in D$. Then*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{\varphi(z)\, dm(z)}{\left(|z-z_0|\log\frac{1}{|z-z_0|}\right)^2} = O\left(\log\log\frac{1}{\varepsilon}\right) \qquad as \quad \varepsilon \to 0 \quad (2.11)$$

for some $\varepsilon_0 \in (0, \delta_0)$ where $\delta_0 = \min(e^{-e}, d_0)$, $d_0 = \sup\limits_{z\in D}|z - z_0|$.

The following statement will be also useful later on, see e.g. Theorem 3.2 in [42].

Proposition 2. *Let $Q : \mathbb{D} \to [0,\infty]$ be a measurable function such that*

$$\int\limits_{\mathbb{D}} \Phi(Q(z))\, dm(z) < \infty \tag{2.12}$$

where $\Phi : [0,\infty] \to [0,\infty]$ is a non-decreasing convex function such that

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau\Phi^{-1}(\tau)} = \infty \tag{2.13}$$

for some $\delta > \Phi(+0)$. Then

$$\int\limits_{0}^{1} \frac{dr}{rq(r)} = \infty \tag{2.14}$$

where $q(r)$ is the average of the function $Q(z)$ over the circle $|z| = r$.

Here we use the following notions of the inverse function for monotone functions. Namely, for every non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$ the inverse function $\Phi^{-1} : [0,\infty] \to [0,\infty]$ can be well-defined by setting

$$\Phi^{-1}(\tau) \ := \inf\limits_{\Phi(t)\geq\tau} t \tag{2.15}$$

Here inf is equal to ∞ if the set of $t \in [0,\infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function Φ^{-1} is non-decreasing, too.

It is also evident immediately by the definition that $\Phi^{-1}(\Phi(t)) \le t$ for all $t \in [0,\infty]$ with the equality except intervals of constancy of the function $\Phi(t)$.

Recall connections between integral conditions, see e.g. Theorem 2.5 in [42].

Remark 3. Let $\Phi : [0,\infty] \to [0,\infty]$ be a non-decreasing function and set

$$H(t) \ = \ \log \Phi(t) \ . \tag{2.16}$$

Then the equality

$$\int\limits_{\Delta}^{\infty} H'(t)\, \frac{dt}{t} = \infty, \tag{2.17}$$

implies the equality

$$\int\limits_{\Delta}^{\infty} \frac{dH(t)}{t} = \infty\, , \tag{2.18}$$

and (2.18) is equivalent to

$$\int\limits_{\Delta}^{\infty} H(t)\, \frac{dt}{t^2} = \infty \tag{2.19}$$

for some $\Delta > 0$, and (2.19) is equivalent to each of the equalities

$$\int\limits_{0}^{\delta_*} H\left(\frac{1}{t}\right)\, dt = \infty \tag{2.20}$$

for some $\delta_* > 0$,

$$\int\limits_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} = \infty \tag{2.21}$$

for some $\Delta_* > H(+0)$ and to (2.13) for some $\delta > \Phi(+0)$.

Moreover, (2.17) is equivalent to (2.18) and to hence (2.17)–(2.21) as well as to (2.13) are equivalent to each other if Φ is in addition absolutely continuous. In particular, all the given conditions are equivalent if Φ is convex and non-decreasing.

Note that the integral in (2.18) is understood as the Lebesgue–Stieltjes integral and the integrals in (2.17) and (2.19)–(2.21) as the ordinary Lebesgue integrals. It is necessary to give one more explanation. From the right hand sides in the conditions (2.17)–(2.21) we have in mind $+\infty$. If $\Phi(t) = 0$ for $t \in [0, t_*$, then $H(t) = -\infty$ for $t \in [0, t_*]$ and we complete the definition $H'(t) = 0$ for $t \in [0, t_*]$. Note, the conditions (2.18) and (2.19) exclude that t_* belongs to the interval of integrability because in the contrary case the left hand sides in (2.18) and (2.19) are either equal to $-\infty$ or indeterminate. Hence we may assume in (2.17)–(2.20) that $\delta > t_0$, correspondingly, $\Delta < 1/t_0$ where $t_0 := \sup\limits_{\Phi(t)=0} t$, and set $t_0 = 0$ if $\Phi(0) > 0$. The most interesting condition (2.19) can be written in the form:

$$\int\limits_{\Delta}^{\infty} \log\,\Phi(t)\,\frac{dt}{t^2} \;=\; +\infty \qquad \text{for some } \Delta > 0\ . \tag{2.22}$$

3. The Dirichlet problem in simply connected domains

Recall that a mapping $f : D \to \mathbb{C}$ is called **discrete** if the preimage $f^{-1}(y)$ consists of isolated points for every $y \in \mathbb{C}$, and **open** if f maps every open set $U \subseteq D$ onto an open set in $\mathbb{C}$. If $\varphi(\zeta) \not\equiv \text{const}$, then the **regular solution** of the Dirichlet problem (1.5) for the Beltrami equation (1.1) is a continuous, discrete and open mapping $f : D \to \mathbb{C}$ of the Sobolev class $W^{1,1}_{\text{loc}}$ with its Jacobian $J_f(z) = |f_z|^2 - |f_{\bar z}|^2 \neq 0$ a.e. satisfying (1.1) a.e. and the condition (1.5). The regular solution of such a problem with $\varphi(\zeta) \equiv c$, $\zeta \in \partial D$, for the Beltrami equation (1.1) is the function $f(z) \equiv c$, $z \in D$.

In this section, we prove the existence of a regular solution to the Dirichlet problem (1.5) for every continuous function $\varphi : \partial D \to \mathbb{R}$ under some appropriate conditions on $\mu(z)$ in an arbitrary bounded simply connected domain D. Moreover, we show that every such solution can be represented as a composition of a regular homeomorphic solution of the Beltrami equation (1.1) with hydrodynamic normalization at the infinity and a holomorphic solution of the corresponding Dirichlet problem associated with it. The main criteria are formulated in terms of the tangent dilatation $K^T_\mu(z, z_0)$.

We assume further that the dilatations $K^T_\mu(z, z_0)$ and $K_\mu(z)$ are extended by 1 outside of the domain D.

Lemma 4. *Let D be a bounded simply connected domain in $\mathbb{C}$. Suppose that $\mu : D \to \mathbb{C}$ is a measurable function with $|\mu(z)| < 1$ a.e.,*

$K_\mu \in L^1(D)$ and

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu^T(z,z_0)\cdot\psi_{z_0,\varepsilon}^2(|z-z_0|)\,dm(z) = o(I_{z_0}^2(\varepsilon)) \text{ as } \varepsilon\to 0\ \forall\ z_0\in\overline{D} \tag{3.1}$$

for some $\varepsilon_0 = \varepsilon(z_0) > 0$ and a family of measurable functions $\psi_{z_0,\varepsilon}$: $(0,\varepsilon_0)\to(0,\infty)$ with

$$I_{z_0}(\varepsilon)\colon = \int\limits_{\varepsilon}^{\varepsilon_0}\psi_{z_0,\varepsilon}(t)\,dt < \infty \qquad \forall\ \varepsilon\in(0,\varepsilon_0)\,. \tag{3.2}$$

Then the Beltrami equation (1.1) *has a regular solution f of the Dirichlet problem* (1.5) *in D for each continuous function $\varphi:\partial D\to\mathbb{R}$.*

Moreover, such a solution f can be represented as the composition

$$f\ =\ h\circ g\ ,\qquad g(z)\ =\ z\,+\,o(1)\quad as\quad z\to\infty\ , \tag{3.3}$$

where $g:\mathbb{C}\to\mathbb{C}$ is a regular homeomorphic solution of the Beltrami equation (1.1) *in $\mathbb{C}$ with μ extended by zero outside of D and $h: D_*\to\mathbb{C}$, $D_* := g(D)$, is a holomorphic solution of the Dirichlet problem*

$$\lim_{\xi\to\zeta}\ \mathrm{Re}\,h(\xi)\ =\ \varphi_*(\zeta)\qquad\forall\ \zeta\in\partial D_*\ ,\qquad where\ \varphi_* := \varphi\circ g^{-1}. \tag{3.4}$$

Proof. Indeed, by Lemma 1 there is a regular homeomorphic solution with hydrodynamic normalization $g(z) := z + o(1)$ as $z\to\infty$ of the Beltrami equation (1.1) in $\mathbb{C}$ with μ extended by zero outside of D. Note that $D_* := g(D)$ is also a simply connected domain in $\mathbb{C}$ with no boundary component degenerated to a single point because of $g:\mathbb{C}\to\mathbb{C}$ is a homeomorphism. Consequently, by Theorem 4.2.1 and Corollary 4.1.8 in [33] there is a unique harmonic function $u: D_*\to\mathbb{R}$ that satisfies the Dirichlet boundary condition

$$\lim_{\xi\to\zeta}\ u(\xi)\ :=\ \varphi_*(\zeta)\qquad\forall\ \zeta\in\partial D_*\ ,\qquad \text{where } \varphi_* := \varphi\circ g^{-1}. \tag{3.5}$$

On the other hand, there is a conjugate harmonic function $v: D_*\to\mathbb{R}$ such that $h := u+iv : D_*\to\mathbb{C}$ forms a holomorphic function because of the domain D_* is simply connected, see e.g. arguments in the beginning of the book [22]. Thus, the function $f := h\circ g$ gives the desired solution of the Dirichlet problem (1.5) in D for the Beltrami equation (1.1). ☐

Remark 4. Note that if the family of the functions $\psi_{z_0,\varepsilon}(t) \equiv \psi_{z_0}(t)$ is independent on the parameter ε, then the condition (3.1) implies that $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. This follows immediately from arguments by contradiction, apply for it (1.4) and the condition $K_\mu \in L^1(D)$. Note also that (3.1) holds, in particular, if, for some $\varepsilon_0 = \varepsilon(z_0)$,

$$\int\limits_{|z-z_0|<\varepsilon_0} K_\mu^T(z,z_0) \cdot \psi_{z_0}^2(|z-z_0|)\, dm(z) < \infty \qquad \forall\, z_0 \in \overline{D} \tag{3.6}$$

and $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In other words, for the solvability of the Dirichlet problem (1.5) in D for the Beltrami equation (1.1) for all continuous boundary functions φ, it is sufficient that the integral in (3.6) converges for some nonnegative function $\psi_{z_0}(t)$ that is locally integrable over $(0,\varepsilon_0]$ but has a nonintegrable singularity at 0. The functions $\log^\lambda(e/|z-z_0|)$, $\lambda \in (0,1)$, $z \in \mathbb{D}$, $z_0 \in \overline{\mathbb{D}}$, and $\psi(t) = 1/(t\ \log(e/t))$, $t \in (0,1)$, show that the condition (3.6) is compatible with the condition $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Furthermore, the condition (3.1) shows that it is sufficient for the solvability of the Dirichlet problem even if the integral in (3.6) is divergent in a controlled way.

Choosing $\psi(t) = 1/\left(t \log\left(1/t\right)\right)$ in Lemma 4, we obtain by Lemma 2 the following result.

Theorem 1. *Let D be a bounded simply connected domain in $\mathbb{C}$ and $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1(D)$. Suppose that $K_\mu^T(z,z_0) \leqslant Q_{z_0}(z)$ a.e. in U_{z_0} for every point $z_0 \in \overline{D}$, a neighborhood U_{z_0} of z_0 and a function $Q_{z_0} : U_{z_0} \to [0,\infty]$ in the class* FMO(z_0). *Then the Beltrami equation* (1.1) *has a regular solution of the Dirichlet problem* (1.5) *in D with the representation* (3.3) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

In particular, by Proposition 1 the conclusion of Theorem 1 holds if every point $z_0 \in \overline{D}$ is the Lebesgue point of the function Q_{z_0}.

By Corollary 2 we obtain the following nice consequence of Theorem 1, too.

Corollary 4. *Let D be a bounded simply connected domain in $\mathbb{C}$ and $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1(D)$ and*

$$\overline{\lim_{\varepsilon\to 0}} \ \ -\!\!\!\!\!\!\int_{B(z_0,\varepsilon)} K_\mu^T(z,z_0)\, dm(z) < \infty \qquad \forall\, z_0 \in \overline{D}\,. \tag{3.7}$$

Then the Beltrami equation (1.1) *has a regular solution of the Dirichlet problem* (1.5) *in D with the representation* (3.3) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Since $K^T_\mu(z,z_0) \leqslant K_\mu(z)$ for all z and $z_0 \in \mathbb{C}$, we also obtain the following consequences of Theorem 1.

Corollary 5. *Let D be a bounded simply connected domain in $\mathbb{C}$ and $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1(D)$ have a majorant $Q : \mathbb{C} \to [1,\infty)$ in the class* $\mathrm{BMO}_{\mathrm{loc}}$. *Then the Beltrami equation* (1.1) *has a regular solution of the Dirichlet problem* (1.5) *in D with the representation* (3.3) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Remark 5. In particular, the conclusion of Corollary 5 holds if $Q \in \mathrm{W}^{1,2}_{\mathrm{loc}}$ because $W^{1,2}_{\mathrm{loc}} \subset \mathrm{VMO}_{\mathrm{loc}}$, see e.g. [9].

Corollary 6. *Let D be a bounded simply connected domain in $\mathbb{C}$ and $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ and $K_\mu(z) \leqslant Q(z)$ a.e. in D with a function Q in the class* $\mathrm{FMO}(\overline{D})$. *Then the Beltrami equation* (1.1) *has a regular solution of the Dirichlet problem* (1.5) *in D with the representation* (3.3) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Similarly, choosing in Lemma 4 the function $\psi(t) = 1/t$, we come to the next statement.

Theorem 2. *Let D be a bounded simply connected domain in $\mathbb{C}$ and $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1(D)$. Suppose that*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z,z_0)\,\frac{dm(z)}{|z-z_0|^2} = o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \quad as\ \varepsilon\to 0\ \forall\ z_0\in\overline{D} \tag{3.8}$$

for some $\varepsilon_0 = \varepsilon(z_0) > 0$. Then Beltrami equation (1.1) *has a regular solution of the Dirichlet problem* (1.5) *in D with the representation* (3.3) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Remark 6. Choosing in Lemma 4 the function $\psi(t) = 1/(t\log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (3.8) by

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{K^T_\mu(z,z_0)\,dm(z)}{\left(|z-z_0|\log\frac{1}{|z-z_0|}\right)^2} = o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right) \tag{3.9}$$

In general, we are able to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t\log 1/t\cdot\log\log 1/t\cdot \ldots\cdot\log\ldots\log 1/t)$.

Choosing in Lemma 4 the functional parameter $\psi_{z_0,\varepsilon}(t) \equiv \psi_{z_0}(t) := 1/[tk^T_\mu(z_0,t)]$, where $k^T_\mu(z_0,r)$ is the integral mean of $K^T_\mu(z,z_0)$ over the circle $S(z_0,r) := \{z\in\mathbb{C} : |z-z_0| = r\}$, we obtain one more important conclusion.

Theorem 3. *Let D be a bounded simply connected domain in $\mathbb{C}$ and $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1(D)$. Suppose that*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{r k_\mu^T(z_0, r)} = \infty \qquad \forall\ z_0 \in \overline{D} \tag{3.10}$$

for some $\varepsilon_0 = \varepsilon(z_0) > 0$. Then Beltrami equation (1.1) *has a regular solution of the Dirichlet problem* (1.5) *in D with the representation* (3.3) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Corollary 7. *Let D be a bounded simply connected domain in $\mathbb{C}$ and $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1(D)$ and*

$$k_\mu^T(z_0, \varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \qquad \text{as } \varepsilon \to 0 \qquad \forall\ z_0 \in \overline{D}\ . \tag{3.11}$$

Then the Beltrami equation (1.1) *has a regular solution of the Dirichlet problem* (1.5) *in D with the representation* (3.3) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Remark 7. In particular, the conclusion of Corollary 7 holds if

$$K_\mu^T(z, z_0) = O\left(\log \frac{1}{|z - z_0|}\right) \qquad \text{as} \quad z \to z_0 \quad \forall\ z_0 \in \overline{D}\,. \tag{3.12}$$

Moreover, the condition (3.11) can be replaced by the whole series of more weak conditions

$$k_\mu^T(z_0, \varepsilon) = O\left(\left[\log \frac{1}{\varepsilon} \cdot \log\log \frac{1}{\varepsilon} \cdot \ldots \cdot \log \ldots \log \frac{1}{\varepsilon}\right]\right) \qquad \forall\ z_0 \in \overline{D}\ . \tag{3.13}$$

Combining Theorems 3, Proposition 2 and Remark 3, we obtain the following result.

Theorem 4. *Let D be a bounded simply connected domain in $\mathbb{C}$ and $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1(D)$. Suppose that*

$$\int\limits_{U_{z_0}} \Phi_{z_0}\left(K_\mu^T(z, z_0)\right)\, dm(z) < \infty \qquad \forall\ z_0 \in \overline{D} \tag{3.14}$$

for a neighborhood U_{z_0} *of* z_0 *and a convex non-decreasing function* $\Phi_{z_0} : [0,\infty] \to [0,\infty]$ *with*

$$\int\limits_{\Delta(z_0)}^{\infty} \log\, \Phi_{z_0}(t)\, \frac{dt}{t^2} \;=\; +\infty \tag{3.15}$$

for some $\Delta(z_0) > 0$. *Then Beltrami equation* (1.1) *has a regular solution of the Dirichlet problem* (1.5) *in* D *with the representation* (3.3) *for each continuous function* $\varphi : \partial D \to \mathbb{R}$.

Corollary 8. *Let* D *be a bounded simply connected domain in* $\mathbb{C}$ *and* $\mu : D \to \mathbb{C}$ *be a measurable function with* $|\mu(z)| < 1$ *a.e.,* $K_\mu \in L^1(D)$ *and*

$$\int\limits_{U_{z_0}} e^{\alpha(z_0) K_\mu^T(z,z_0)}\, dm(z) < \infty \qquad \forall\ z_0 \in \overline{D} \tag{3.16}$$

for some $\alpha(z_0) > 0$ *and a neighborhood* U_{z_0} *of the point* z_0. *Then the Beltrami equation* (1.1) *has a regular solution of the Dirichlet problem* (1.5) *in* D *with the representation* (3.3) *for each continuous function* $\varphi : \partial D \to \mathbb{R}$.

Since $K_\mu^T(z,z_0) \leqslant K_\mu(z)$ for z and $z_0 \in \mathbb{C}$ and $z \in D$, we also obtain the following consequences of Theorem 4.

Corollary 9. *Let* D *be a bounded simply connected domain in* $\mathbb{C}$ *and* $\mu : D \to \mathbb{C}$ *be a measurable function with* $|\mu(z)| < 1$ *a.e. and* $K_\mu \in L^1(D)$. *Suppose that*

$$\int\limits_{D} \Phi\left(K_\mu(z)\right)\, dm(z) < \infty \tag{3.17}$$

for a convex non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$ *with*

$$\int\limits_{\delta}^{\infty} \log\, \Phi(t)\, \frac{dt}{t^2} \;=\; +\infty \tag{3.18}$$

for some $\delta > 0$. *Then Beltrami equation* (1.1) *has a regular solution of the Dirichlet problem* (1.5) *in* D *with the representation* (3.3) *for each continuous function* $\varphi : \partial D \to \mathbb{R}$.

Corollary 10. *Let* D *be a bounded simply connected domain in* $\mathbb{C}$ *and* $\mu : D \to \mathbb{C}$ *be a measurable function with* $|\mu(z)| < 1$ *a.e. and, for some* $\alpha > 0$,

$$\int\limits_{D} e^{\alpha K_\mu(z)}\, dm(z) \;<\; \infty\ . \tag{3.19}$$

Then the Beltrami equation (1.1) *has a regular solution of the Dirichlet problem* (1.5) *in D with the representation* (3.3) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Remark 8. By the Stoilow theorem, see e.g. [47], a regular solution f of the Dirichlet problem (1.5) in D for the Beltrami equation (1.1) with $K_\mu \in L^1_{\rm loc}(D)$ can be represented in the form $f = h \circ F$ where h is a holomorphic function and F is a homeomorphic regular solution of (1.1) in the class $W^{1,1}_{\rm loc}$. Thus, by Theorem 5.1 in [42] the condition (3.18) is not only sufficient but also necessary to have a regular solution of the Dirichlet problem (1.5) in D for arbitrary Beltrami equations (1.1) with the integral constraints (3.17) for all continuous functions $\varphi : \partial D \to \mathbb{R}$, see also Remark 3.

4. On the Dirichlet problem in general domains

In this section we obtain criteria for the existence of multi-valued solutions f of the Dirichlet problem to the Beltrami equations in the spirit of the theory of multi-valued analytic functions in arbitrary bounded domains D in $\mathbb{C}$ with no boundary component degenerated to a single point. Simple examples show that such domains form the most wide class of domains for which the problem is always solvable for any continuous boundary functions.

We say that a discrete open mapping $f : B(z_0, \varepsilon_0) \to \mathbb{C}$, where $B(z_0, \varepsilon_0) \subseteq D$, is a **local regular solution of the equation** (1.1) if $f \in W^{1,1}_{\rm loc}$, $J_f(z) \neq 0$ and f satisfies (1.1) a.e. in $B(z_0, \varepsilon_0)$. The local regular solutions $f_0 : B(z_0, \varepsilon_0) \to \mathbb{C}$ and $f_* : B(z_*, \varepsilon_*) \to \mathbb{C}$ of the equation (1.1) will be called extension of each to other if there is a finite chain of such solutions $f_i : B(z_i, \varepsilon_i) \to \mathbb{C}$, $i = 1, \ldots, m$, such that $f_1 = f_0$, $f_m = f_*$ and $f_i(z) \equiv f_{i+1}(z)$ for $z \in E_i := B(z_i, \varepsilon_i) \cap B(z_{i+1}, \varepsilon_{i+1}) \neq \emptyset$, $i = 1, \ldots, m-1$.

A collection of local regular solutions $f_j : B(z_j, \varepsilon_j) \to \mathbb{C}$, $j \in J$, will be called a **multi-valued solution** of the equation (1.1) in D if the disks $B(z_j, \varepsilon_j)$ cover the whole domain D and f_j are extensions of each to other through the collection and the collection is maximal by inclusion.

A multi-valued solution of the equation (1.1) will be called a **multi-valued solution of the Dirichlet problem** (1.5) in D if $u(z) = \operatorname{Re} f(z) = \operatorname{Re} f_j(z)$, $z \in B(z_j, \varepsilon_j)$, $j \in J$, is a single-valued function in D satisfying the condition $\lim\limits_{z \in \zeta} u(z) = \varphi(\zeta)$ for all $\zeta \to \partial D$.

As it was before, we assume further that the dilatations $K^T_\mu(z, z_0)$ and $K_\mu(z)$ are extended by 1 outside of the domain D.

Lemma 5. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1(D)$ and*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z,z_0)\cdot\psi^2_{z_0,\varepsilon}(|z-z_0|)\,dm(z) = o(I^2_{z_0}(\varepsilon)) \text{ as } \varepsilon\to 0\ \forall\ z_0\in\overline{D} \tag{4.1}$$

for some $\varepsilon_0 = \varepsilon(z_0) > 0$ and a family of measurable functions $\psi_{z_0,\varepsilon} : (0,\varepsilon_0) \to (0,\infty)$ with

$$I_{z_0}(\varepsilon) \colon = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\,dt < \infty\ \forall\ \varepsilon\in(0,\varepsilon_0)\,. \tag{4.2}$$

Then the Beltrami equation (1.1) *has a multi-valued solution f of the Dirichlet problem* (1.5) *in D in D for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Moreover, such a solution f can be represented as the composition

$$f\ =\ \mathcal{A}\circ g\ , \qquad g(z)\ =\ z\ +\ o(1) \quad as \quad z\to\infty\ , \tag{4.3}$$

where $g : \mathbb{C} \to \mathbb{C}$ is a regular homeomorphic solution of the Beltrami equation (1.1) in $\mathbb{C}$ with μ extended by zero outside of D and $\mathcal{A} : D_* \to \mathbb{C}$, $D_* := g(D)$, is a multi-valued analytic function with a single-valued harmonic function $\operatorname{Re}\mathcal{A}$ *satisfying the Dirichlet condition*

$$\lim_{\xi\to\zeta}\ \operatorname{Re}\mathcal{A}(\xi)\ =\ \varphi_*(\zeta) \qquad \forall\ \zeta\in\partial D_*\ , \qquad \textit{where } \varphi_* := \varphi\circ g^{-1}. \tag{4.4}$$

Proof. Indeed, by Lemma 1 there is a regular homeomorphic solution with hydrodynamic normalization $g(z) := z + o(1)$ as $z \to \infty$ of the Beltrami equation (1.1) in $\mathbb{C}$ with μ extended by zero outside of D. Note that $D_* := g(D)$ is also a simply connected domain in $\mathbb{C}$ with no boundary component degenerated to a single point because of $g : \mathbb{C} \to \mathbb{C}$ is a homeomorphism. Consequently, by Theorem 4.2.2 and Corollary 4.1.8 in [33] there is a unique harmonic function $u : D_* \to \mathbb{R}$ that satisfies the Dirichlet boundary condition

$$\lim_{\xi\to\zeta}\ u(\xi)\ :=\ \varphi_*(\zeta) \qquad \forall\ \zeta\in\partial D_*\ , \qquad \text{where } \varphi_* := \varphi\circ g^{-1}. \tag{4.5}$$

Let $B_0 = B(z_0, r_0)$ is a disk in the domain D. Then $\mathfrak{B}_0 = g(B_0)$ is a simply connected subdomain of the domain $D_* := g(D)$ where there is a conjugate function v determined up to an additive constant such that $h = u + iv$ is a single–valued analytic function. Let us denote through h_0 the holomorphic function corresponding to the choice of such a harmonic function v_0 in $\mathfrak{B}_0$ with the normalization $v_0(g(z_0)) = 0$. Thereby we have determined the initial element of a multi-valued analytic function. The function h_0 can be extended to, generally speaking multi-valued, analytic function $\mathcal{A}$ along any path in D_* because u is given in the whole domain D_*. Thus, $f = \mathcal{A} \circ g$ is a desired multi-valued solution of the Dirichlet problem (1.5) in D for Beltrami equation (1.1). □

Remark 9. Arguing perfectly as in the last section, one can obtain the corresponding similar criteria in terms of the dilatations K_μ and K_μ^T that, however, we do not formulate in the explicit form because they are the same.

5. Applications to the potential theory

The results of the last section seem too abstract, and therefore allegedly useless. However, we give here some their applications to one of the main equations of the mathematical physics in strongly anisotropic and inhomogeneous media associated with the degenerate Beltrami equation.

Namely, in this section we obtain criteria for the existence and representation of solutions u of the Dirichlet problem to the elliptic equations of the form

$$\operatorname{div} A\nabla\, u = 0 \tag{5.1}$$

with measurable matrix-valued function $A(z) = \{a_{ij}(z)\}$ in arbitrary bounded domains D in $\mathbb{C}$ with no boundary component degenerated to a single point. Our example in the end of the paper shows that such domains form the most wide class of domains for which the Dirichlet problem will be always solvable for each continuous boundary date $\varphi : \partial D \to \mathbb{R}$ at least for harmonic functions.

A continuous function $u : D \to \mathbb{R}$ is called **A−harmonic function**, see e.g. [18], if u satisfies (5.1) in the sense of distributions, i.e., if $u \in W^{1,1}_{\rm loc}(D)$ and

$$\int_D \langle A(z)\nabla u(z), \nabla\psi(z)\rangle \; d\,m(z) = 0 \qquad \forall\ \psi \in C_0^\infty(D)\ , \tag{5.2}$$

where $C_0^\infty(D)$ denotes the collection of all infinitely differentiable functions $\psi : D \to \mathbb{R}$ with compact support in D, $\langle a, b\rangle$ means the scalar product of vectors a and b in $\mathbb{R}^2$, and $d\,m(z)$ corresponds to the Lebesgue measure in the plane $\mathbb{C}$.

In this connection, let us describe the relevance of the Beltrami equations (1.1) and the equations (5.1). First of all, recall that the **Hodge operator** $\mathbb{H}$ is the counterclockwise rotation by the angle $\pi/2$ in $\mathbb{R}^2$:

$$\mathbb{H} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \; : \; \mathbb{R}^2 \to \mathbb{R}^2 \, , \qquad \mathbb{H}^2 \; = \; -\, I \; , \tag{5.3}$$

where I denotes the unit 2×2 matrix. Thus, the matrix $\mathbb{H}$ plays the role of an imaginary unit in the space of two-dimensional square matrices with real-valued elements.

By Theorem 16.1.6 in [3], if f is a $W^{1,1}_{\rm loc}$ solution of the Beltrami equation (1.1), then the functions $u := {\rm Re} f$ and $v := {\rm Im} f$ satisfy the equation:

$$\nabla\, v(z) \; = \; \mathbb{H}\, A(z) \nabla\, u(z) \; , \tag{5.4}$$

where the matrix-valued function $A(z)$ is calculated through $\mu(z)$ in the following way:

$$A \; = \; \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \; := \; \begin{bmatrix} \frac{|1-\mu|^2}{1-|\mu|^2} & \frac{-2{\rm Im}\,\mu}{1-|\mu|^2} \\ \frac{-2{\rm Im}\,\mu}{1-|\mu|^2} & \frac{|1+\mu|^2}{1-|\mu|^2} \end{bmatrix} . \tag{5.5}$$

The function v is called the $A-$**harmonic conjugate of** u or sometimes a **stream function of the potential** u. Note that by (5.3) the equation (5.4) is equivalent to the equation

$$A(z) \nabla\, u(z) \; = \; -\mathbb{H} \nabla\, v(z) \; . \tag{5.6}$$

As known, the curl of any gradient field is zero in the sense of distributions and the Hodge operator $\mathbb{H}$ transforms curl-free fields into divergence-free fields, and vice versa, see e.g. 16.1.3 in [3]. Hence (5.6) implies (5.1).

We see from (5.5) that the matrix A is symmetric and it is clear by elementary calculations that $\det A = 1$. Moreover, since $|\mu(z)| < 1$ a.e., from ellipticity of this matrix A follows that $\det\,(I + A) > 0$ a.e., which in terms of its elements means that $(1 + a_{11})(1 + a_{22}) > a_{12}a_{21}$ a.e. Further $\mathbb{S}^{2\times 2}$ denotes the collection of all such matrices. Thus, by Theorem 16.1.6 in [3], the Beltrami equation is the complex form of one of the main equations of mathematical physics, the potential equation (5.1) with the matrix-valued coefficient A in the class $\mathbb{S}^{2\times 2}$.

Note that the matrix identities in (5.5) can be converted a.e. to express the coefficient $\mu(z)$ of the Beltrami equation (1.1) through the elements of the matrices $A(z)$:

$$\mu \; = \; \mu_A \; := \; -\frac{a_{11} - a_{22} + i(a_{12} + a_{21})}{2 + a_{11} + a_{22}} \; . \tag{5.7}$$

Thus, we obtain the latter expression as a criterion for the solvability of the Dirichlet problem

$$\lim_{z\to\zeta} u(z) = \varphi(\zeta) \qquad \forall\, \zeta\in\partial D \tag{5.8}$$

to the potential equation (5.1). Namely, by the above arguments in this section as well as Lemma 5 we come to the following general criteria.

Lemma 6. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $A : D \to \mathbb{S}^{2\times 2}$ be a measurable function in D with $K_{\mu_A}\in L^1(D)$ and*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_{\mu_A}(z,z_0)\cdot\psi^2_{z_0,\varepsilon}(|z-z_0|)\,dm(z) = o(I^2_{z_0}(\varepsilon)) \text{ as } \varepsilon\to 0\ \forall\, z_0\in\overline{D} \tag{5.9}$$

for some $\varepsilon_0=\varepsilon(z_0)>0$ and a family of measurable functions $\psi_{z_0,\varepsilon}$: $(0,\varepsilon_0)\to(0,\infty)$ with

$$I_{z_0}(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0}\psi_{z_0,\varepsilon}(t)\,dt<\infty \qquad \forall\,\varepsilon\in(0,\varepsilon_0)\,. \tag{5.10}$$

Then the potential equation (5.1) *has $A-$harmonic solutions u of the Dirichlet problem* (5.8) *in D for each continuous function $\varphi : \partial D\to\mathbb{R}$.*

Moreover, such a solution u can be represented as the composition

$$u = \mathcal{H}\circ g\,, \qquad g(z) = z + o(1) \quad \text{as} \quad z\to\infty\,, \tag{5.11}$$

where $g : \mathbb{C}\to\mathbb{C}$ is a regular homeomorphic solution of the Beltrami equation (1.1) *in $\mathbb{C}$ with μ_A extended by zero outside of D and $\mathcal{H} : D_*\to\mathbb{C}$, $D_* := g(D)$, is a unique harmonic function satisfying the Dirichlet condition*

$$\lim_{\xi\to\zeta}\mathcal{H}(\xi) = \varphi_*(\zeta) \qquad \forall\,\zeta\in\partial D_*\,, \qquad \textit{where } \varphi_* := \varphi\circ g^{-1}. \tag{5.12}$$

As it was before, we assume here that the dilatations $K^T_{\mu_A}(z,z_0)$ and $K_{\mu_A}(z)$ are extended by 1 outside of the domain D.

Remark 10. Note that if the family of the functions $\psi_{z_0,\varepsilon}(t)\equiv\psi_{z_0}(t)$ is independent on the parameter ε, then the condition (5.9) implies that $I_{z_0}(\varepsilon)\to\infty$ as $\varepsilon\to 0$. This follows immediately from arguments by contradiction, apply for it (1.4) and the condition $K_{\mu_A}\in L^1(D)$. Note also that (5.9) holds, in particular, if, for some $\varepsilon_0=\varepsilon(z_0)$,

$$\int\limits_{|z-z_0|<\varepsilon_0} K^T_{\mu_A}(z,z_0)\cdot\psi^2_{z_0}(|z-z_0|)\,dm(z)<\infty \qquad \forall\, z_0\in\overline{D} \tag{5.13}$$

and $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In other words, for the existence of $A-$harmonic solutions of the Dirichlet problem (5.8) in D to the potential equation (5.1) with each continuous boundary functions φ, it is sufficient that the integral in (5.13) converges for some nonnegative function $\psi_{z_0}(t)$ that is locally integrable over $(0, \varepsilon_0]$ but has a nonintegrable singularity at 0. The functions $\log^{\lambda}(e/|z - z_0|)$, $\lambda \in (0, 1)$, $z \in \mathbb{D}$, $z_0 \in \overline{\mathbb{D}}$, and $\psi(t) = 1/(t \ \log(e/t))$, $t \in (0, 1)$, show that the condition (5.13) is compatible with the condition $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Furthermore, the condition (5.9) in Lemma 6 shows that it is sufficient for the existence of $A-$harmonic solutions of the Dirichlet problem (5.8) in D to the potential equation (5.1) even that the integral in (5.13) to be divergent in a controlled way.

Arguing similarly to Section 3, we derive from Lemma 6 the next series of results.

For instance, choosing $\psi(t) = 1/\left(t \log\left(1/t\right)\right)$ in Lemma 6, we obtain by Lemma 2 the following.

Theorem 5. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point and $A : D \to \mathbb{S}^{2\times 2}$ be a measurable function in D with $K_{\mu_A} \in L^1(D)$. Suppose that $K^T_{\mu_A}(z, z_0) \leqslant Q_{z_0}(z)$ a.e. in U_{z_0} for every point $z_0 \in \overline{D}$, a neighborhood U_{z_0} of z_0 and a function $Q_{z_0} : U_{z_0} \to [0, \infty]$ in the class* FMO(z_0). *Then the potential equation* (5.1) *has $A-$harmonic solutions of the Dirichlet problem* (5.8) *in D with the representation* (5.11) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

In particular, by Proposition 1 the conclusion of Theorem 5 holds if every point $z_0 \in \overline{D}$ is the Lebesgue point of a suitable dominant Q_{z_0}.

By Corollary 2 we obtain the following nice consequence of Theorem 5, too.

Corollary 11. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point and $A : D \to \mathbb{S}^{2\times 2}$ be a measurable function in D with $K_{\mu_A} \in L^1(D)$ and*

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{B(z_0,\varepsilon)} K^T_{\mu_A}(z, z_0)\, dm(z) < \infty \qquad \forall\ z_0 \in \overline{D}\,. \tag{5.14}$$

Then the potential equation (5.1) *has $A-$harmonic solutions of the Dirichlet problem* (5.8) *in D with the representation* (5.11) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Since $K^T_{\mu_A}(z, z_0) \leqslant K_{\mu_A}(z)$ for all z and $z_0 \in \mathbb{C}$, we also obtain the following consequences of Theorem 5.

Corollary 12. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $A : D \to \mathbb{S}^{2\times 2}$ be a measurable function in D and K_{μ_A} have a dominant $Q : \mathbb{C} \to [1, \infty)$ in the class* $\mathrm{BMO}_{\mathrm{loc}}$. *Then the potential equation* (5.1) *has $A-$harmonic solutions of the Dirichlet problem* (5.8) *in D with the representation* (5.11) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Remark 11. In particular, the conclusion of Corollary 12 holds if $Q \in \mathrm{W}^{1,2}_{\mathrm{loc}}$ because $W^{1,2}_{\mathrm{loc}} \subset \mathrm{VMO}_{\mathrm{loc}}$, see [9].

Corollary 13. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $A : D \to \mathbb{S}^{2\times 2}$ be a measurable function in D and $K_{\mu_A}(z) \leqslant Q(z)$ a.e. in D with a function Q in the class* $\mathrm{FMO}(\overline{D})$. *Then the potential equation* (5.1) *has $A-$harmonic solutions of the Dirichlet problem* (5.8) *in D with the representation* (5.11) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Similarly, choosing in Lemma 6 the function $\psi(t) = 1/t$, we come to the next statement.

Theorem 6. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $A : D \to \mathbb{S}^{2\times 2}$ be a measurable function in D with $K_{\mu_A} \in L^1(D)$. Suppose that*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_{\mu_A}(z, z_0)\,\frac{dm(z)}{|z-z_0|^2} = o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \quad \text{as } \varepsilon \to 0 \ \forall\ z_0 \in \overline{D} \tag{5.15}$$

for some $\varepsilon_0 = \varepsilon(z_0) > 0$. Then the potential equation (5.1) *has $A-$harmonic solutions of the Dirichlet problem* (5.8) *in D with representation* (5.11) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Remark 12. Choosing in Lemma 6 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (5.15) by

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{K^T_{\mu_A}(z, z_0)\, dm(z)}{\left(|z-z_0| \log \frac{1}{|z-z_0|}\right)^2} = o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right) \tag{5.16}$$

In general, we are able to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t \log 1/t \cdot \log\log 1/t \cdot \ldots \cdot \log\ldots\log 1/t)$.

Choosing in Lemma 6 the functional parameter $\psi_{z_0,\varepsilon}(t) \equiv \psi_{z_0}(t) := 1/[t k^T_{\mu_A}(z_0, t)]$, where $k^T_{\mu_A}(z_0, r)$ is the integral mean of $K^T_{\mu_A}(z, z_0)$ over the circle $S(z_0, r) := \{z \in \mathbb{C} : |z - z_0| = r\}$, we obtain one more important conclusion.

Theorem 7. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $A : D \to \mathbb{S}^{2\times 2}$ be a measurable function in D with $K_{\mu_A} \in L^1(D)$. Suppose that*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{r k^T_{\mu_A}(z_0, r)} = \infty \qquad \forall\ z_0 \in \overline{D} \tag{5.17}$$

for some $\varepsilon_0 = \varepsilon(z_0) > 0$. Then the potential equation (5.1) *has $A-$harmonic solutions of the Dirichlet problem* (5.8) *in D with representation* (5.11) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Corollary 14. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $A : D \to \mathbb{S}^{2\times 2}$ be a measurable function in D with $K_{\mu_A} \in L^1(D)$ and*

$$k^T_{\mu_A}(z_0, \varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \qquad \text{as } \varepsilon \to 0 \qquad \forall\ z_0 \in \overline{D}\ . \tag{5.18}$$

Then the potential equation (5.1) *has $A-$harmonic solutions of the Dirichlet problem* (5.8) *in D with the representation* (5.11) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Remark 13. In particular, the conclusion of Corollary 14 holds if

$$K^T_{\mu_A}(z, z_0) = O\left(\log \frac{1}{|z - z_0|}\right) \qquad \text{as} \quad z \to z_0 \quad \forall\ z_0 \in \overline{D}\,. \tag{5.19}$$

Moreover, the condition (5.18) can be replaced by the whole series of more weak conditions

$$k^T_{\mu_A}(z_0, \varepsilon) = O\left(\left[\log \frac{1}{\varepsilon} \cdot \log\log \frac{1}{\varepsilon} \cdot \ldots \cdot \log\ldots\log \frac{1}{\varepsilon}\right]\right) \qquad \forall\ z_0 \in \overline{D}\ . \tag{5.20}$$

Combining Theorems 7, Proposition 2 and Remark 3, we obtain the following result.

Theorem 8. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $A : D \to \mathbb{S}^{2\times 2}$ be a measurable function in D with $K_{\mu_A} \in L^1(D)$. Suppose that*

$$\int\limits_{U_{z_0}} \Phi_{z_0}\left(K^T_{\mu_A}(z, z_0)\right)\, dm(z) < \infty \qquad \forall\ z_0 \in \overline{D} \tag{5.21}$$

for a neighborhood U_{z_0} of z_0 and a convex non-decreasing function Φ_{z_0} : $[0,\infty] \to [0,\infty]$ with

$$\int_{\Delta(z_0)}^{\infty} \log \Phi_{z_0}(t) \frac{dt}{t^2} = +\infty \tag{5.22}$$

for some $\Delta(z_0) > 0$. Then the potential equation (5.1) *has $A-$harmonic solutions of the Dirichlet problem* (5.8) *in D with representation* (5.11) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Corollary 15. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $A : D \to \mathbb{S}^{2\times 2}$ be a measurable function in D with $K_{\mu_A} \in L^1(D)$ and*

$$\int_{U_{z_0}} e^{\alpha(z_0) K^T_{\mu_A}(z,z_0)} \, dm(z) < \infty \qquad \forall \; z_0 \in \overline{D} \tag{5.23}$$

for some $\alpha(z_0) > 0$ and a neighborhood U_{z_0} of the point z_0. Then the potential equation (5.1) *has $A-$harmonic solutions of the Dirichlet problem* (5.8) *in D with the representation* (5.11) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Since $K^T_{\mu_A}(z,z_0) \leqslant K_{\mu_A}(z)$ for z and $z_0 \in \mathbb{C}$ and $z \in D$, we also obtain the following consequences of Theorem 8.

Corollary 16. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $A : D \to \mathbb{S}^{2\times 2}$ be a measurable function in D with $K_{\mu_A} \in L^1(D)$. Suppose that*

$$\int_D \Phi\left(K_{\mu_A}(z)\right) \, dm(z) < \infty \tag{5.24}$$

for a convex non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$ with

$$\int_{\delta}^{\infty} \log \Phi(t) \frac{dt}{t^2} = +\infty \tag{5.25}$$

for some $\delta > 0$. Then the potential equation (5.1) *has $A-$harmonic solutions of the Dirichlet problem* (5.8) *in D with the representation* (5.11) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Remark 14. By the Stoilow theorem, see e.g. [47], a multi-valued solution $f = u + iv$ of the Dirichlet problem (5.8) in D for the Beltrami equation (1.1) with $K_{\mu_A} \in L^1_{\rm loc}(D)$ can be represented in the form $f = \mathcal{A} \circ F$ where $\mathcal{A}$ is a multi-valued analytic function and F is a homeomorphic solution of (1.1) with $\mu := \mu_A$ in the class $W^{1,1}_{\rm loc}$. Thus, by Theorem 5.1 in [42], see also Theorem 16.1.6 in [3], the condition (5.25) is not only sufficient but also necessary to have $A-$harmonic solutions u of the Dirichlet problem (5.8) in D to the potential equation (5.1) with the integral constraints (5.24) for all continuous functions $\varphi : \partial D \to \mathbb{R}$, see also Remark 3.

Corollary 17. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $A : D \to \mathbb{S}^{2\times 2}$ be a measurable function in D such that, for some $\alpha > 0$,*

$$\int\limits_{D} e^{\alpha K_{\mu_A}(z)}\, dm(z) \ < \ \infty \ . \tag{5.26}$$

Then the potential equation (5.1) *has $A-$harmonic solutions of the Dirichlet problem* (5.8) *in D with the representation* (5.11) *for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Thus, we have a number of effective criteria for solvability of the Dirichlet problem to the main equation (5.1) of the hydromechanics (fluid mechanics) in strongly anisotropic and inhomogeneous media.

Let us emphasize, the request on domains to have no boundary component degenerated to a single point is necessary. Indeed, consider the punctured unit disk $\mathbb{D}_0 := \mathbb{D} \setminus \{0\}$. Setting $\varphi(\zeta) \equiv 1$ on $\partial\mathbb{D}$ and $\varphi(0) = 0$, we see that φ is continuous on $\partial\mathbb{D}_0 = \partial\mathbb{D} \cup \{0\}$. Let us assume that there is a harmonic function u satisfying (5.8) with the given φ. Then u is bounded by the maximum principle for harmonic functions and by the classic Cauchy–Riemann theorem, see also Theorem V.4.2 in [29], the extended u is harmonic in $\mathbb{D}$. Thus, by contradiction with the Mean-Value-Property we disprove the above assumption, see e.g. Theorem 0.2.4 in [43].

In this connection, recall that a point $p \in \partial D$ for a domain D in $\mathbb{R}^n, n \geq 2$, is called a **regular point** if each solution of the Dirichlet problem for the Laplace equation in D, whose Dirichlet boundary date is continuous at p, is also continuous at p. The famous Wiener criterion for regularity of a boundary point, see [50], that has been formulated in terms of so-called barrier functions, generally speaking, has no satisfactory geometric interpretation. However, there is a very simple geometric criterion of regular points in the case of $\mathbb{C}$. Namely, a point $p \in \partial D$ is regular if p belongs to a component of ∂D that is not degenerated to a single point, see Theorem 4.2.2 in [33]. The example in the last item shows that this condition is not only sufficient but also necessary for regularity of a boundary point.

Thus, the results on the Dirichlet problem given in Section 5 were obtained for $A-$harmonic functions in the most general admissible domains D in $\mathbb{C}$.

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Contact information

Vladimir Gutlyanskii
Institute of Applied Mathematics
and Mechanics of the NAS of Ukraine,
Slavyansk, Ukraine
E-Mail: vgutlyanskii@gmail.com

Vladimir Ryazanov
Institute of Applied Mathematics
and Mechanics of the NAS of Ukraine,
Slavyansk, Ukraine,
Bogdan Khmelnytsky National
University of Cherkasy,
Lab. of Math. Phys.,
Cherkasy, Ukraine
E-Mail: Ryazanov@nas.gov.ua,
vl.ryazanov1@gmail.com

Evgeny Sevost'yanov
Zhytomyr Ivan Franko State University,
Zhytomyr, Ukraine
E-Mail: esevostyanov2009@gmail.com

Eduard Yakubov
Holon Institute of Technology, Holon, Israel
E-Mail: yakubov@hit.ac.il,
eduardyakubov@gmail.com