Український математичний вісник
Том 19 **(2022)**, № 3, 382 – 412
https://doi.org/10.32782/1810-3200-2022-19-3-6

УМВ

Two coefficient conjectures for nonvanishing Hardy functions, I

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Abstract. There are two eminent still open conjectures for nonvanishing holomorphic Hardy functions $f(z)$ on the unit disk. The Hummel–Scheinberg–Zalcman conjecture posed in 1977 extends Krzyz's conjecture of 1968 to H^p spaces with finite $p > 1$ and states that Taylor's coefficients of nonvanishing holomorphic functions $f \in H^p$ with norm $\|f\|_p \leq 1$ are sharply estimated by $|c_n| \leq (2/e)^{1-1/p}$, with appropriate extremal functions.

Both conjectures have been investigated by many authors; however still remain open. The desired Krzyz's estimate $|c_n| \leq 2/e$ for $f \in H^\infty$ with $\|f\|_\infty \leq 1$ was established only for the initial coefficients c_n with $n \leq 5$ The only known results for the Hummel–Scheinberg–Zalcman conjecture are that it is true for $n = 1$ and $n = 2$ as well as some results for special subclasses of H^p.

We prove here that the Hummel–Scheinberg–Zalcman conjecture is true for all spaces H^{2m} with $m \in \mathbb{N}$. In the limit as $m \to \infty$, this also provides the proof of Krzyz's conjecture.

Our approach involves deep results from Teichmüller space theory, especially the Bers isomorphism theorem for Teichmüller spaces of punctured Riemann surfaces, and special quasiconformal deformations of H^{2m} functions.

2010 MSC. Primary: 30C50, 30C55, 30H05, 30H10; Secondary 30F60.

Key words and phrases. Nonvanishing holomorphic functions, the Hardy spaces, Krzyz's conjecture, the Hummel–Scheinberg–Zalcman conjecture, Schwarzian derivative, quasiconformal extension, Teichmüller's spaces, Bers' isomorphism theorem.

1. Introductory remarks and main results

1.1. There are two eminent still open conjectures for nonvanishing holomorphic functions $f(z)$ on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ from the

Received 07.05.2022

Hardy spaces H^p with $1 < p \le \infty$. Recall that the norm in these spaces is defined by

$$\|f\|_p = \sup_{r<1} \Big(\frac{1}{2\pi} \int\limits_0^{2\pi} |f(re^{i\theta})|^p d\theta\Big)^{1/p}$$

for $p < \infty$ and $\|f\|_\infty$ coincides with L_∞- norm of this function.

The *Krzyz conjecture* [18] of 1968 states that Taylor's coefficients of nonvanishing holomorphic functions $f(z) = \sum\limits_0^\infty c_n z^n \in H^\infty$ with $\|f\|_\infty \le 1$ are sharply estimated by $|c_n| \le 2/e$, with equality only for the function $\kappa_\infty(z^n)$, where

$$\kappa_\infty(z) = \exp\Big(\frac{z-1}{z+1}\Big) = \frac{1}{e} + \frac{2}{e}z - \frac{2}{3e}z^3 + ...$$

(and its compositions with rotations about the origin).

Its deep generalization to spaces H^p with $p > 1$ is given by the *Hummel–Scheinberg–Zalcman conjecture* posed in 1977, which states for all nonvanishing $f \in H^p$ with $\|f\|_p \le 1$ the sharp estimate

$$|c_n| \le (2/e)^{1-1/p}, \tag{1}$$

and this bound is realized only by the functions $\epsilon_2 \kappa_{n,p}(\epsilon_1 z)$, where $|\epsilon_1| = |\epsilon_2| = 1$ and

$$\kappa_{n,p}(z) = \Big[\frac{(1+z^n)^2}{2}\Big]^{1/p} \Big[\exp\frac{z^n-1}{z^n+1}\Big]^{1-1/p}. \tag{2}$$

Both conjectures have been investigated by many authors; however, as was mentioned, still remain open. The desired Krzyz's estimate $|c_n| \le 2/e$ was established only for the initial coefficients c_n with $n \le 5$ (see, e.g., [12, 19, 21, 23, 25, 26] and the references cited there).

The only known results for the second conjecture are that the conjecture is true for $n = 1$ (proved by Brown) and $n = 2$ (proved by Suffridge) as well as some results for special subclasses of H^p, see [6, 7, 24]. Brown also showed that (1) is true for arbitrary $n \ge 2$, provided $c_m = 0$ for all $m,\ 1 \le m < (n+1)/2$.

The paper [3] provides some estimates for coefficients c_n of H^p functions, $1 \le p \le \infty$, with $\|f\|_p \le 1$, whose values at the origin are fixed.

In the present paper, we consider the spaces H^p with $p \ge 2$ and establish that the Hummel–Scheinberg–Zalcman conjecture is true for all even natural values of p.

Theorem 1. *The estimate* (1) *is valid for all spaces* H^{2m}, $m \in \mathbb{N}$; *that is, for any nonvanishing function* $f \in H^{2m}$ *and any* $n > 1$,

$$|c_n| \le |\kappa'_{1,2m}(0)| = (2/e)^{1-1/2m}. \tag{3}$$

The equality in (3) *is attained only on the function* $f(z) = \kappa_{n,2m}(z)$ *given by* (2) *and its compositions with pre and post rotations about the origin.*

On going to the limit as $m \to \infty$ one obtains, as a consequence of Theorem 1, that *the coefficients of all nonvanishing functions* $f(z) = \sum_0^\infty c_n z^n \in H^\infty$ *on the unit disk, with* $\|f\|_\infty \le 1$ *satisfy the inequality*

$$\begin{aligned} |c_n| &\le \inf\{|c_n(f)| : \ f \in \bigcap_{m\ge1} B_1^0(H^{2m})\} \\ &= \inf\{|c_n(f)| : \ f \in \bigcap_{p\ge1} B_1^0(H^p)\} = 2/e \end{aligned} \tag{4}$$

for all $n > 1$; *here* $B_1^0(H^{2m})$ *denotes the collection of nonvanishing functions from the unit ball in* H^{2m}.

This estimate $\max |c_n| = 2/e$ for $f \in B_1^0(H^\infty)$ is sharp being realized by the function $\kappa_\infty(z^n)$ and its compositions with rotations. Note that the function $\kappa_\infty(z)$ is the universal holomorphic covering map of the punctured disk $\mathbb{D} \setminus \{0\}$ by $\mathbb{D}$.

This proves the Krzyz conjecture. Some generalizations of this conjecture also can be proved in this way.

In fact the estimate (4) is valid on much broader set of functions, because there exist unbounded nonvanishing holomorphic functions $f(z)$ on the unit disk which belong to all spaces H^p, $p > 0$. For example, one can take the functions

$$f_a(z) = \log \frac{a}{1-z} \quad \text{with} \ \ |a-1| \ge 2$$

(it is a slight modification of a known function which belongs to all H^p with $p > 0$, see [20]).

1.2. To prove Theorem 1, we use a new approach to extremal coefficient problems in geometric complex analysis recently applied by the author in [16, 17] to univalent functions and their Schwarzian derivatives. This approach involves a deep result from Teichmüller space theory given by Bers's isomorphism theorem for Teichmüller spaces of punctured Riemann surfaces [5].

The aim of this paper is to extend this approach to holomorphic functions of different types related to nonvanishing Hardy functions. We embed the balls $B_\rho(H^p)$ from H^p with center at the origin and sufficiently small radii $\rho \le \rho_p$ into the universal Teichmüller space, and regard the functions $f \in B_\rho(H^p)$ as the Schwarzian derivatives of univalent functions in the unit disk.

Another basic tool applied in the proof is given by special quasiconformal deformations of H^{2m} functions preserving their L_{2m} norm up to a small distortion.

2. Digression to Teichmüller spaces

We briefly recall some needed results from Teichmüller space theory in order to prove our Theorem; the details can be found, for example, in [5, 10].

2.1. The universal Teichmüller space $\mathbf{T} = \mathrm{Teich}(\mathbb{D})$ is the space of quasisymmetric homeomorphisms of the unit circle $\mathbb{S}^1$ factorized by Möbius maps; all Teichmüller spaces have their isometric copies in $\mathbf{T}$.

The canonical complex Banach structure on $\mathbf{T}$ is defined by factorization of the ball of the Beltrami coefficients (or complex dilatations)

$$\mathbf{Belt}(\mathbb{D})_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu|\mathbb{D}^* = 0, \ \|\mu\| < 1\},$$

vanishing on the complementary disk $\mathbb{D}^* = \{z \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} : \ |z| > 1\}$.

The coefficients $\mu_1, \mu_2 \in \mathbf{Belt}(\mathbb{D})_1$ are called equivalent if the corresponding quasiconformal maps w^{μ_1}, w^{μ_2} (solutions to the Beltrami equation $\partial_{\overline{z}} w = \mu \partial_z w$ with $\mu = \mu_1, \mu_2$) coincide on the unit circle $\mathbb{S}^1 = \partial\mathbb{D}^*$ (hence, on $\overline{\mathbb{D}^*}$). Such μ and the corresponding maps w^μ are called **T**-*equivalent*. The equivalence classes $[w^\mu]_{\mathbf{T}}$ are in one-to-one correspondence with the Schwarzian derivatives

$$S_w(z) = \left(\frac{w''(z)}{w'(z)}\right)' - \frac{1}{2}\left(\frac{w''(z)}{w'(z)}\right)^2 \quad (w = w^\mu(z), \quad z \in \mathbb{D}^*).$$

The chain rule for the Schwarzian derivatives yields

$$S_{f_1 \circ f}(z) = (S_{f_1} \circ f) f'(z)^2 + S_f(z);$$

in particular, for the Möbius (fractional linear) maps $w = \gamma(z)$,

$$S_{f_1 \circ \gamma}(z) = (S_{f_1} \circ \gamma)\gamma'(z)^2, \quad S_{\gamma \circ f}(z) = S_f(z).$$

Note also that every solution $w(z)$ of the Schwarzian equation $S_w(z) = \varphi(z)$ with a given holomorphic φ is the ratio η_2/η_1 of two independent solutions of the linear equation

$$2\eta''(z) + \varphi(z)\eta(z) = 0$$

and vice versa.

For each univalent function $w(z)$ on a simply connected hyperbolic domain $D \subset \widehat{\mathbb{C}}$, its Schwarzian derivative belongs to the complex Banach space $\mathbf{B}(D)$ of hyperbolically bounded holomorphic functions on D with the norm

$$\|\varphi\|_{\mathbf{B}} = \sup_D \lambda_D^{-2}(z)|\varphi(z)|,$$

where $\lambda_D(z)|dz|$ is the hyperbolic metric on D of Gaussian curvature -4; hence $\varphi(z) = O(z^{-4})$ as $z \to \infty$ if $\infty \in D$. In particular, for the unit disk,

$$\lambda_{\mathbb{D}}(z) = 1/(1 - |z|^2).$$

The space $\mathbf{B}(D)$ is dual to the Bergman space $A_1(D)$, a subspace of $L_1(D)$ formed by integrable holomorphic functions (quadratic differentials $\varphi(z)dz^2$ on D), since every linear functional $l(\varphi)$ on $A_1(D)$ is represented in the form

$$l(\varphi) = \langle \psi, \varphi \rangle_D = \iint_D \lambda_D^{-2}(z)\overline{\psi(z)}\varphi(z)dxdy \tag{5}$$

with a uniquely determined $\psi \in \mathbf{B}(D)$.

The Schwarzians $S_{w^\mu}(z)$ with $\mu \in \mathbf{Belt}(\mathbb{D})_1$ range over a bounded domain in the space $\mathbf{B} = \mathbf{B}(\mathbb{D}^*)$. This domain models the space $\mathbf{T}$. It lies in the ball $\{\|\varphi\|_{\mathbf{B}} < 6\}$ and contains the ball $\{\|\varphi\|_{\mathbf{B}} < 2\}$. In this model, the Teichmüller spaces of all hyperbolic Riemann surfaces are contained in $\mathbf{T}$ as its complex submanifolds.

The factorizing projection

$$\phi_{\mathbf{T}}(\mu) = S_{w^\mu} : \ \mathbf{Belt}(\mathbb{D})_1 \to \mathbf{T}$$

is a holomorphic map from $L_\infty(\mathbb{D})$ to $\mathbf{B}$, and in view of holomorphy,

$$\|S_{w^{\mu_1}} - S_{w^{\mu_2}}\|_{\mathbf{B}} \le \text{const}\, \|\mu_1 - \mu_2\|_\infty.$$

This map is a split submersion, which means that $\phi_{\mathbf{T}}$ has local holomorphic sections (see, e.g., [10]).

Both equations $S_w = \varphi$ and $\partial_{\bar{z}} w = \mu \partial_z w$ (on $\mathbb{D}^*$ and $\mathbb{D}$, respectively) determine their solutions up to a Möbius transformation of $\widehat{\mathbb{C}}$. So appropriate normalization of solution $w^\mu(z)$ (for example, fixing the points $1, i, -1$ or other three points on the unit circle), provides uniqueness of solution of either equation, and moreover, then the values $w^\mu(z_0)$ at any point $z_0 \in \mathbb{C} \setminus \{1, i, -1\}$ and the Taylor coefficients $b_1, b_2, \dots$ of $w^\mu \in \Sigma_\theta$ depend holomorphically on $\mu \in \mathbf{Belt}(\mathbb{D})_1$ and on $S_{w^\mu} \in \mathbf{T}$. Later we shall use another normalization which also insures the needed uniqueness and holomorphy.

2.2. The points of Teichmüller space $\mathbf{T}_1 = \text{Teich}(\mathbb{D}_*)$ of the punctured disk $\mathbb{D}_* = \mathbb{D} \setminus \{0\}$ are the classes $[\mu]_{\mathbf{T}_1}$ of $\mathbf{T}_1$-*equivalent* Beltrami coefficients $\mu \in \mathbf{Belt}(\mathbb{D})_1$ so that the corresponding quasiconformal automorphisms w^μ of the unit disk coincide on both boundary components (unit circle $\mathbb{S}^1 = \{|z| = 1\}$ and the puncture $z = 0$) and are homotopic on $\mathbb{D} \setminus \{0\}$. This space can be endowed with a canonical complex structure of a complex Banach manifold and embedded into $\mathbf{T}$ using uniformization.

Namely, the disk $\mathbb{D}_*$ is conformally equivalent to the factor $\mathbb{D}/\Gamma$, where Γ is a cyclic parabolic Fuchsian group acting discontinuously on $\mathbb{D}$ and $\mathbb{D}^*$. The functions $\mu \in L_\infty(\mathbb{D})$ are lifted to $\mathbb{D}$ as the Beltrami $(-1, 1)$-measurable forms $\widetilde{\mu} d\overline{z}/dz$ in $\mathbb{D}$ with respect to Γ, i.e., via $(\widetilde{\mu} \circ \gamma)\overline{\gamma'}/\gamma' = \widetilde{\mu},\ \gamma \in \Gamma$, forming the Banach space $L_\infty(\mathbb{D}, \Gamma)$.

We extend these $\widetilde{\mu}$ by zero to $\mathbb{D}^*$ and consider the unit ball $\mathbf{Belt}(\mathbb{D}, \Gamma)_1$ of $L_\infty(\mathbb{D}, \Gamma)$. Then the corresponding Schwarzians $S_{w^{\widetilde{\mu}}|\mathbb{D}^*}$ belong to $\mathbf{T}$. Moreover, $\mathbf{T}_1$ is canonically isomorphic to the subspace $\mathbf{T}(\Gamma) = \mathbf{T} \cap \mathbf{B}(\Gamma)$, where $\mathbf{B}(\Gamma)$ consists of elements $\varphi \in \mathbf{B}$ satisfying $(\varphi \circ \gamma)(\gamma')^2 = \varphi$ in $\mathbb{D}^*$ for all $\gamma \in \Gamma$.

Due to the Bers isomorphism theorem, the space $\mathbf{T}_1$ is biholomorphically isomorphic to the Bers fiber space

$$\mathcal{F}(\mathbf{T}) = \{(\phi_{\mathbf{T}}(\mu), z) \in \mathbf{T} \times \mathbb{C} : \ \mu \in \mathbf{Belt}(\mathbb{D})_1, \ z \in w^\mu(\mathbb{D})\}$$

over the universal space $\mathbf{T}$ with holomorphic projection $\pi(\psi, z) = \psi$ (see [5]).

This fiber space is a bounded hyperbolic domain in $\mathbf{B} \times \mathbb{C}$ and represents the collection of domains $D_\mu = w^\mu(\mathbb{D})$ as a holomorphic family over the space $\mathbf{T}$. For every $z \in \mathbb{D}$, its orbit $w^\mu(z)$ in $\mathbf{T}_1$ is a holomorphic curve over $\mathbf{T}$.

The indicated isomorphism between $\mathbf{T}_1$ and $\mathcal{F}(\mathbf{T})$ is induced by the inclusion map $j : \mathbb{D}_* \hookrightarrow \mathbb{D}$ forgetting the puncture at the origin via

$$\mu \mapsto (S_{w^{\mu_1}}, w^{\mu_1}(0)) \quad \text{with} \ \ \mu_1 = j_* \mu := (\mu \circ j_0)\overline{j_0'}/j_0', \tag{6}$$

where j_0 is the lift of j to $\mathbb{D}$.

By Koebe's one-quarter theorem, for any univalent function $W(z) = z + b_0 + b_1 z^{-1} + \dots$ in $\mathbb{D}^*$, the boundary of domain $W(D*)$ is located in the disk $\{|w - b_0| \le 2\}$. If $W(z) \ne 0$ in $\mathbb{D}^*$, its inversion $w(z) = z + a_2 z^2 + \dots$ is univalent in $\mathbb{D}$, and $b_0 = -a_2$ satisfies $|b_0| \le 2$. Using the maps W with quasiconformal extensions, one gets by the Bers theorem that the indicated domains D_μ are filled by the admissible values of $W^\mu(0)$; all these domains are located in the disk $\{|W| \le 4\}$.

In the line with our goals, we slightly modified the Bers construction, applying quasiconformal maps F^μ of $\mathbb{D}_*$ admitting conformal extension to $\mathbb{D}^*$ (and accordingly using the Beltrami coefficients μ supported in the disk) (cf. [15]). These changes are not essential and do not affect the underlying features of the Bers isomorphism (giving the same space up to a biholomorphic isomorphism).

The Bers theorem is valid for Teichmüller spaces $\mathbf{T}(X_0 \setminus \{x_0\})$ of all punctured hyperbolic Riemann surfaces $X_0 \setminus \{x_0\}$ and implies that $\mathbf{T}(X_0 \setminus \{x_0\})$ is biholomorphically isomorphic to the Bers fiber space $\mathcal{F}(\mathbf{T}(X_0))$ over $\mathbf{T}(X_0)$.

Note that $\mathbf{B}(\Gamma_0)$ has the same elements as the space $A_1(\mathbb{D}^*, \Gamma_0)$ of integrable holomorphic forms of degree -4 with norm $\|\varphi\|_{A_1(\mathbb{D}^*,\Gamma_0)} = \iint_{\mathbb{D}^*/\Gamma_0} |\varphi(z)| dxdy$; and similar to (5), every linear functional $l(\varphi)$ on $A_1(\mathbb{D}^*, \Gamma_0)$ is represented in the form

$$l(\varphi) = \langle \psi, \varphi \rangle_{\mathbb{D}/\Gamma_0} := \iint\limits_{\mathbb{D}^*/\Gamma_0} (1 - |z|^2)^2 \, \overline{\psi(z)} \varphi(z) dxdy$$

with uniquely determined $\psi \in \mathbf{B}(\Gamma_0)$.

Every Teichmüller space $\mathbf{T}(X)$ is a complete metric space with intrinsic Teichmüller metric $\tau_{\mathbf{T}}(\cdot, \cdot)$ defined by quasiconformal maps. By the Royden–Gardiner theorem, this metric is equal the hyperbolic Kobayashi metric $d_{\mathbf{T}}(\cdot, \cdot)$ determined by the complex structure on this space (see, e.g., [9, 10, 22]). In other words, the Kobayashi–Teichmüller metric is the maximal invariant metric on $\mathbf{T}(X)$.

3. Proof of Theorem 1

We carry out the proof it in several stages and deduce the assertion of the theorem as a consequence of lemmas. With one exception, these stages are valid for all spaces H^p with $p \geq 2$.

Step 1: Four underlying lemmas. Denote the unit ball of H^p by $B_1(H^p)$ and its subset of nonvanishing functions by $B_1^0(H^p)$. It will be convenient to regard the free coefficients $c_0(f)$ also as elements of $B_1^0(H^p)$, which are constant on the disk $\mathbb{D}$. Let

$$\widehat{B}_1^0(H^p) = B_1^0(H^p) \cup \{f_0\},$$

where $f_0(z) \equiv 0$. The corresponding sets for the disks $\mathbb{D}_r = \{|z| < r\}$ will be denoted by $B_1^0(H^p(\mathbb{D}_r))$ and $\widehat{B}_1^0(H^p(\mathbb{D}_r))$.

We shall essentially use Brown's result quoted above and present it as

Lemma 1. [6] *For any* $f(z) = c_0 + c_1 z + c_2 z^2 + \cdots \in B_1^0(H^p)$, *we have*

$$|c_1| \le (2/e)^{1-1/p},$$

with equality only for the rotations of function $\kappa_{1,p}(z)$ *given by* (2).

The following lemma concerns the topological features of sets of nonvanishing holomorphic functions. Consider the subsets $\mathcal{B}_r$ of $B_1^0(H^p)$ defined by

$$\mathcal{B}_r = \{f \in B_1(H^p(\mathbb{D}_r)) : \ f(z) \ne 0 \text{ on the disk } \mathbb{D}_r = \{|z| < r\}\}, \ 1 < r < \infty;$$

then $\mathcal{B}_{r'} \Subset \mathcal{B}_r$ if $r' > r$. Put

$$B_*(H^p) = \bigcup_{r>1} \mathcal{B}_r$$

with topology of the inductive limit. All functions from $B_*(H^p)$ are zero free in $\mathbb{D}$.

It will be convenient to regard the free coefficients $c_0(f)$ also as elements of H^p, which are constant on each disk $\mathbb{D}_r$.

Any point $f_0 \in B_*(H^p)$ belongs to all sets $\mathcal{B}_r$ with $r \ge r_0$. Consider the intersection of $B_*(H^p)$ with the balls $\{f \in H^p(B_{r_0}) : \ \|f - f_0\|_p < \epsilon\}$ and denote their connected components containing f_0 by $U(f_0, \epsilon)$.

Lemma 2. *Each point* $f \in B_*(H^p)$ *has a neighborhood* $U(f, \epsilon)$ *in* $B_*(H^p)$ *filled by the functions which are zero free in the disk* $\mathbb{D}$. *Take the maximal neighborhoods* $U(f, \epsilon)$ *with such property. Then their union*

$$\mathcal{U}^p = \bigcup_{f \in B_*(H^p)} U(f, \epsilon)$$

is an open path-wise connective set, hence a domain, in the space $B_*(H^p)$.

Proof. (a) Openness. It suffices to show that for each $r > 1$ and $1 \le r' < r$, every $f \in \mathcal{B}_r$ has a neighborhood $U(f, \epsilon)$ in $B_*(H^p)$, which contains only the functions that are zero free on the disk $\mathbb{D}$.

This is trivial for $r > r' > 1$. Let $r' = 1$, and assume the contrary. Then for some $r > 1$ there exist a function $f_0 \in \mathcal{B}_r$, a sequence of functions $f_n \in B_1^0(H^p(\mathbb{D}_{r_n}))$ convergent to f_0 as $r_n \nearrow r$ so that

$$\lim_{n \to \infty} \|f_n - f_0\|_{H^p(\mathbb{D}_{r_n})} = 0, \tag{7}$$

and a sequence of points $z_n \in \mathbb{D}_{r_n}$ convergent to z_0 with $|z_0| \le 1$ such that $f_n(z_n) = 0$ $(n = 1, 2, \dots)$.

In the case $|z_0| < 1$ we immediately reach a contradiction, because then the uniform convergence of f_n on compact sets in $\mathbb{D}$ implies $f_0(z_0) = 0$, which is impossible.

The case $|z_0| = 1$ requires other arguments. Since f_0 is holomorphic and does not vanish in the disk $\mathbb{D}_r$ with $r > 1$,

$$\min_{|z|\le 1} |f_0(z)| = a > 0.$$

Hence, for each z_n,

$$|f_n(z_n) - f_0(z_n)| = |f_0(z_n)| \ge a,$$

and by continuity, there exists a neighborhood $\Delta(z_n, \delta_n) = \{|z - z_n| < \delta_n\}$ of z_n in $\mathbb{D}$, in which

$$|f_n(z) - f_0(z)| \ge a/2 \quad \text{for all} \quad z.$$

This immediately implies

$$\|f_n - f_0\|_{H^p(D_{r_n})} \ge \|f_n - f_0\|_{H^p} \ge C_p(a), \tag{8}$$

where $C_p(a)$ is a positive constant depending only on a (for given p), and the inequality (8) must hold for all n. But this contradicts to (7).

(b) Connectedness. We establish that the set $\mathcal{U}^p$ is path-wise connective. Pick two points

$$f_1(z) = c_0^1 + c_1^1 z + \dots\ , \quad f_2(z) = c_0^2 + c_1^2 z + \dots$$

from $\mathcal{U}^p$, which lie in the balls $U(f_1^0, \epsilon_1)$ and $U(f_2^0, \epsilon_2)$, respectively; these points can be connected with the centers f_1^0, f_2^0 by radial segments. Then take the homotopies

$$f_j^0(z, t) = f_j^0(tz) = c_0^{0,j} + c_1^{0,j} tz + \dots, \quad 0 \le t \le 1 \quad (j = 1, 2)$$

connecting these centers with the points c_0^j; clearly, $f_j^0(tz) \ne 0$ in $\mathbb{D}$. The general properties of integrals depending on parameters yield that these homotopies are extended to the complex holomorphic isotopies $\mathbb{D} \times \mathbb{D} \to B_1(H^p)$; hence the corresponding curves $t \mapsto f_j^0(\cdot, t)$ are continuous.

Finally, the points $c_0^{0,1}$ and $c_0^{0,2}$ can be joint by a continuous curve in $B_1^0(H^p)$ filled by the constant functions $c_0(f)$. The lemma follows.

The distinguished domain $\mathcal{U}^p \subset B_*(H^p)$ is dense weakly (in the topology of locally uniform convergence on $\mathbb{D}$) in $B_1^0(H^p)$; hence,

$$\sup_{U^p} |c_n| = \sup_{B_1^0(H^p)} |c_n|.$$

Now, let $\mathcal{P}_n$ be the linear space of polynomials of degree less than or equal to n, and

$$\mathcal{P} = \bigcup_n \mathcal{P}_n.$$

Lemma 3. *The intersection $\mathcal{U}^p \bigcap \mathcal{P}$ is dense in $\mathcal{U}^p$, which means that any f from the distinguished domain $\mathcal{U}^p$ is approximated in H^p by nonvanishing polynomials.*

Proof. Each $f \in \mathcal{U}^p$ can be approximated in H^p by holomorphic nonvanishing functions f_n on the closed disk $\overline{\mathbb{D}_r}$ with $r = r(f_n) > 1$. Their Taylor partial sums are convergent to f_n uniformly on $\overline{\mathbb{D}_r}$, hence do not vanish on D. This implies the conclusion of the lemma.

The following lemma ensures the existence of univalent functions in the disk with quasiconformal extension satisfying the prescribed normalization and some other conditions. It concerns the solutions w^μ of the Beltrami equation $\partial_{\overline{z}} w = \mu(z)\partial_z w$ on $\mathbb{C}$ with coefficients μ supported in the disk $\mathbb{D}$, i.e., from the ball

$$\mathbf{Belt}(\mathbb{D}^*)_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu|\mathbb{D} = 0, \ \|\mu\| < 1\}$$

(and hence the solutions of the corresponding Schwarzian equation $S_w(z) = \varphi$ in $\mathbb{D}$ with given $\varphi \in \mathbf{B}$).

We shall consider the compositions of homeomorphisms w^μ with the Möbius maps

$$\gamma_a(z) = (1 - az)/(z - a), \quad a \in \mathbb{D}^*, \tag{9}$$

and their inverse $\gamma_a^{-1}(z) = (1 + az)/(z + a)$ preserving either from disks $\mathbb{D}$ and $\mathbb{D}^*$.

The composed maps $w^\mu \circ \gamma_a$ and $\gamma_\alpha^{-1} \circ w^\mu \circ \gamma_a$ have the same Beltrami coefficient

$$\gamma_{a,*}\mu := \mu_{w \circ \gamma_a}(z) = \mu \circ \gamma_a(z) \gamma_a'(z)/\overline{\gamma_a'(z)}$$

and also are conformal in the unit disk.

Lemma 4. *For any Beltrami coefficient $\mu \in \mathbf{Belt}(\mathbb{D}^*)_1$ and any $\theta_0 \in [0, 2\pi]$, there exists a point $z_0 = e^{i\alpha}$ located on $\mathbb{S}^1$ so that $|e^{i\theta_0} - e^{i\alpha}| < 1$ and such that for any θ satisfying $|e^{i\theta} - e^{i\alpha}| < 1$ the equation $\partial_{\overline{z}} w =$*

$\mu(z)\partial_z w$ has a unique homeomorphic solution $w = w^\mu(z)$, which is holomorphic on the unit disk $\mathbb{D}$ and satisfies

$$w(0) = 0, \quad w'(0) = e^{i\theta}, \quad w(z_0) = z_0. \tag{10}$$

Hence, $w^\mu(z)$ is conformal and does not have a pole in $\mathbb{D}$ (so $w^\mu(z_*) = \infty$ at some point z_* with $|z_*| \geq 1$).

Proof. First we establish the assertion of the lemma for $\theta_0 = 0$ corresponding to $z = 1$ and start with the coefficients μ vanishing in a broader disk $\mathbb{D}_r = \{|z| < r\}$, $r > 1$ (so w^μ is conformal on $\mathbb{D}_r \Supset \overline{\mathbb{D}}$), and assume that $\mu \neq \mathbf{0}$ (the origin of $\mathbf{Belt}(\mathbb{D}^*)_1$).

Fix $a \in (1, r)$; then $1/a \in \mathbb{D}$. The generalized Riemann mapping theorem for the Beltrami equation $\partial_{\overline{z}} w = \mu(z)\partial_z w$ on $\widehat{\mathbb{C}}$ implies for a given $\theta \in [0, 2\pi]$ a homeomorphic solution $\widehat{w}$ to this equation satisfying

$$\widehat{w}(1/a) = 1/a, \quad \widehat{w}'(1/a) = e^{i\theta}\ , \quad \widehat{w}(\infty) = \infty. \tag{11}$$

Since, by the classical Schwarz lemma, for any holomorphic map $g : \mathbb{D} \to \mathbb{D}$ and any point $z_0 \in \mathbb{D}$,

$$|g'(z_0)| \leq (1 - |g(z_0)|^2)/(1 - |z_0|^2)$$

with equality only for appropriate Möbius automorphism of $\mathbb{D}$, the above normalization (10) and the assumption on μ yield for the constructed map $\widehat{w}(z)$ that the image $\widehat{w}(\mathbb{D})$ does not cover $\mathbb{D}$, and thus either $\widehat{w}(\mathbb{D})$ is a proper subdomain of $\mathbb{D}$ or it also contains the points z with $|z| > 1$ outer for $\mathbb{D}$.

Applying this to suitable rotated map $\widehat{w}_\alpha(z) = e^{-i\alpha}\widehat{w}(e^{i\alpha}z)$ having Beltrami coefficient $\mu_\alpha(z) = \mu(e^{i\alpha}z)e^{2i\alpha}$, one obtains that domain $\widehat{w}_\alpha(\mathbb{D})$ does not contain simultaneously both distinguished points a and $1/a$, at least sufficiently close to 1 (and the same is valid for the points $a' \in \mathbb{D}^*$ close to a). Now consider the map

$$w_{a,\alpha}(z) = \gamma_a^{-1} \circ \widehat{w}_\alpha \circ \gamma_a(z),$$

having the same Beltrami coefficient $\gamma_{a,*}\mu$. Since, by (9),

$$\gamma_a(\infty) = -a, \quad \gamma_a(a) = \infty, \quad \gamma_a(0) = -1/a, \quad \gamma_a(1/a) = 0$$

and accordingly,

$$\gamma_a^{-1}(\infty) = a,\ \gamma_a^{-1}(-a) = \infty,\ \gamma_a^{-1}(0) = 1/a,\ \gamma_a^{-1}(-1/a) = 0,$$

the map $w_{a,\alpha}$ satisfies

$$w_{a,\alpha}(0) = 0, \quad w'_{a,\alpha}(0) = \widehat{w}'_\alpha(1/a) = e^{i\theta}, \quad w_{a,\alpha}(a) = \gamma_a^{-1} \circ \widehat{w}_\alpha \circ \gamma_a(a) = a. \tag{12}$$

In view of our assumptions on the map $\widehat{w}_\alpha$, the point $\widehat{w}_\alpha^{-1}(a)$ does not lie in the unit disk $\mathbb{D}$, which provides that the function $w_{a,\alpha}$ is holomorphic in this disk.

Now we investigate the limit process as $a \to 1$. Any from the constructed maps $w_{a,a}$ is represented as a composition of a fixed solution $\widehat{w}$ to the equation $\partial_{\overline{z}} w = \mu(z)\partial_z w$ subject to (11) and some Möbius maps $\widehat{\gamma}_a$. The first two conditions in (11) imply that the restrictions of these $\widehat{\gamma}_a$ to $\widehat{w}(\mathbb{D}_r)$ form a (sequentially) compact set of $\widehat{\gamma}_a$ in the topology of convergence in the spherical metric on $\widehat{\mathbb{C}}$. Letting $a \to 1$, one obtains in the limit the map $\widehat{\gamma}_1(z) = \lim\limits_{a\to 1} \widehat{\gamma}_a(z)$, which also is a non-degenerate (nonconstant) Möbius map. Accordingly,

$$\lim_{a\to 1} w_{a,\alpha}(z) = \widehat{\gamma}_1 \circ \widehat{w}_\alpha \circ \gamma_1(z) =: \widehat{w}_1(z),$$

and this map satisfies (12) with $a = 1$, which is equivalent to (11).

Note that the relations (11) do not depend on r and that the normalization (12) (or (11)) also holds for the inverse rotation $e^{i\alpha}\widehat{w}_1(e^{-i\alpha}z)$ of the limit function. Letting

$$w(z) = e^{i\alpha}\widehat{w}_1(e^{-i\alpha}z), \quad z_0 = e^{i\alpha},$$

one obtains a weakened assertion of Lemma 4 for all Beltrami coefficients $\mu \neq \mathbf{0}$ supported in the disk $\mathbb{D}_r^* = \{|z| > r\}$ with $r > 1$ (without a restriction for $|e^{i\theta} - z_0|$).

To extend the obtained result to arbitrary $\mu \in \mathbf{Belt}(\mathbb{D}^*)_1$ ($\mu \neq \mathbf{0}$), we pass to the truncated coefficients

$$\mu_r(z) = \mu(rz), \quad |z| > 1,$$

which are equal to zero on the disk $\{|z| < r\}$. The compactness properties of the k-quasiconformal families (i.e., with $\|\mu\|_\infty \leq k < 1$) imply the convergence of maps $w^{\mu_r}(z)$ normalized by (10) to $w^\mu(\mathrm{z})$ as $r \to 1$ in the spherical metric on $\widehat{\mathbb{C}}$ (and hence everywhere on $\widehat{\mathbb{C}}$). Accordingly, one must now take $z_0 = \lim\limits_{r\to 1} w^{\mu_r}(e^{i\alpha})$.

It remains to estimate the lower bond for $|e^{i\theta} - z_0|$ and consider the case $\mu(z) \equiv 0$ omitted above. We consider for this the homotopy functions

$$w_t(z)w^{\mu_t}(z) = c(t)w(tz) : \ \mathbb{C} \times (0,1) \to \mathbb{C}$$

with $\mu_t(z) = \mu(tz)$, $0 < t < 1$. The factor $c(t)$ is determined by normalization (10) and is a fractional linear function of t.

For any t, the points $w_t(a)$ and $w_t(1/a)$ as well as 0 and ∞ are separated by the quasicircle $w_t(\mathbb{S}^1)$. Thus, arguing similar to above, one obtains for any of these w_t the same point $z_0 = e^{i\alpha}$, and this also holds for $t = 0$.

It remains to observe that if $\mu \to \mathbf{0}$ in L_∞ norm (or even $\mu(z) \to 0$ almost everywhere in $\mathbb{D}^*$), then the corresponding limit map $w(z)$ satisfying (10) must be an elliptic fractional linear transformation with fixed points 0 and z_0; hence,

$$\frac{w - z_0}{w} = e^{-i\theta}\ \frac{z - z_0}{z},$$

which implies

$$w = \frac{e^{i\theta} z}{(1 - e^{-i\theta}) z_0^{-1} z + 1}. \tag{13}$$

The value $w(z_*) = \infty$ occurs when

$$(1 - e^{-i\theta}) z_0^{-1} z_* + 1 = 0.$$

Then $|z_*| = 1/|e^{i\theta} - 1|$, and hence, $|z_*| < 1$ if $|e^{i\theta} - 1| > 1$, what is excluded by assumption.

This implies the assertion of Lemma 4 for $\theta_0 = 0$ and Beltrami coefficient μ_α. To get it for μ, one must conjugate w^{μ_α} by rotation $z \mapsto e^{-i\alpha} z$, replacing the fixed point $z_0 = 1$ by $e^{i\alpha}$.

Similarly, the case of arbitrary θ_0 satisfying $|e^{i\theta_0} - e^{i\alpha}| < 1$ is reduced to the previous one by compositions of w with pre and post rotations $z \mapsto e^{i\theta_0} z$, completing the proof of the lemma. [1]

It follows from Lemma 4 and from its proof that for any fixed $\theta_0 \in [-\pi, \pi]$ there is a point $z_0 = e^{i\alpha_0} \in \mathbb{S}^1$ such that for all θ with $|e^{i\theta} - z_0| < 1$ any two Beltrami coefficients $\mu_1, \mu_2 \in \mathbf{Belt}(\mathbb{D}^*)_1$ generate quasiconformal maps w^{μ_1} and w^{μ_2} normalized by (10) (hence, having the same fixed point z_0), unless these maps are conjugated by a rotation, or equivalently, $\mu_2(z) = \mu_1(e^{i\alpha} z) e^{-2i\alpha}$ with some $\alpha \in [-\pi, \pi]$.

Step 2: Holomorphic embedding of nonvanishing H^p functions into Teichmüller spaces. Consider the space $\mathbf{B} = \mathbf{B}(\mathbb{D})$ of hyperbolically bounded holomorphic functions $f(z)$ (regarded as holomorphic quadratic differentials $f(z)dz^2$) on the unit disk, with norm

$$\|f\|_{\mathbf{B}} = \sup_{\mathbb{D}} (1 - |z|^2)^2 |f(z)|$$

As was mentioned above, every $f \in \mathbf{B}$ is the Schwarzian derivative S_w of a locally univalent function $w(z)$ in the disk $\mathbb{D}$ determined (up to a Moebius map of the sphere $\widehat{\mathbb{C}}$) from the nonlinear differential equation

$$w'''/w' - 3(w''/w')^2/2 = f$$

[1] This lemma corrects the corresponding assertion in [17].

(or equivalently, as the ratio $w = \eta_2/\eta_1$ of two linearly independent solutions of the linear equation $2\eta'' + f\eta = 0$ in $\mathbb{D}$). This space is dual to the space $A(\mathbb{D})$ of integrable holomorphic functions on $\mathbb{D}$ with L_1 norm.

Accordingly, the Schwarzians of functions w univalent in the whole disk $\mathbb{D}$ and having quasiconformal extensions to $\widehat{\mathbb{C}}$ fill a path-wise bounded domain in $\mathbf{B}$, which models the universal Teichmüller space $\mathbf{T}$.

We consider also the Bergman spaces $A_p(\mathbb{D}), p > 2$, of holomorphic functions in $\mathbb{D}$ with norm

$$\|f\|_{A_p} = \Big(\frac{1}{\pi} \iint_{\mathbb{D}} |f(z)| dxdy\Big)^{1/p} \quad (z = x + iy).$$

The Hölder inequality yields that for any $f \in A_p$ with $p > 2$,

$$\|f\|_{A_1} = \frac{1}{\pi} \iint_{\mathbb{D}} |f(z)| dxdy \le \|f\|_{A_p};$$

so all such A_p are the subspaces of $A(\mathbb{D}) = A_1$.

Further, if $f \in H^p$, then its norm in A_p is estimated by

$$\|f\|_{A_p}^p = \frac{1}{\pi} \iint\limits_{\mathbb{D}} |f(z)|^p dxdy = \frac{1}{\pi} \int\limits_0^1 \Big(\int\limits_0^{2\pi} |f(re^{i\theta})|^p d\theta\Big) r dr \le \frac{1}{2} \|f\|_{H^p}^p. \tag{14}$$

Combining this with the well-known relations $A(\mathbb{D}) \subset \mathbf{B}$

$$\|f\|_{\mathbf{B}} \le \|f\|_{A(\mathbb{D})} \quad \text{if} \ \ f \in A(\mathbb{D}) \tag{15}$$

(see, e.g., [4]), one concludes that all functions $f \in H^p$ belong to the space $\mathbf{B}$, and hence these functions (more precisely, the corresponding quadratic differentials $f(z)dz^2$) can be regarded as the Schwarzian derivatives of locally univalent functions in $\mathbb{D}$. [2]

Now, noting that by the well-known Ahlfors–Weill theorem [2], any $g \in \mathbf{B}$ with norm $\|g\|_{\mathbf{B}} = k < 2$ is the Schwarzian derivative $S_w = g$ of a function w which is univalent on the disk $\mathbb{D}$ and admits k-quasiconformal extension across the unit circle $\{|z| = 1\}$ to $\widehat{\mathbb{C}}$ with Beltrami coefficient

$$\nu_{S_w}(\zeta) = \partial_{\overline{\zeta}} w / \partial_\zeta w = -\frac{1}{2}(|\zeta|^2 - 1)^2 \frac{\zeta^2}{\overline{\zeta}^2} S_w\Big(\frac{1}{\overline{\zeta}}\Big),$$

[2]One need to deal with quadratic differentials $\varphi = f(z)dz^2$ to insure the needed behaviour of these Schwarzian derivatives $\varphi(h(z))h'(z)^2 = \varphi(z)$ under conformal changes h of variables.

The H^p functions are moved to their equivalence classes so that f and f_1 are equivalent if $f_1(z) = \epsilon_1 f(\epsilon_2 z)$ for some constants $\epsilon_1, \ \epsilon_2$ with modulus 1. Such equivalence preserves H^p norm and moduli of coefficients.

one obtains from (14) and (15) that all functions f from the ball

$$B_\rho(H^p) = \{f \in H^p : \ \|f\| < \rho\}$$

with radius

$$\rho = 1/2^{1/p}$$

belong to the space $\mathbf{B}$, and hence can be regarded as the Schwarzian derivatives of univalent functions in $\mathbb{D}$ with quasiconformal extension, i.e., as the points of the universal Teichmüller space $\mathbf{T}$.

This implies a *holomorphic embedding* ι of the ball $B_\rho(H^p)$ and of its open subset

$$\frac{1}{2^{1/p}}\mathcal{U}^p = \{\frac{1}{2^{1/p}} f : \ f \in \mathcal{U}^p\}$$

into the space $\mathbf{T}$.

In view of linearity of the functional $J_n(f) = c_n$ on H^p, we have to establish that for all nonvanishing functions $f \in B_{1/2^{1/p}}(H^p)$,

$$|c_n| \le \frac{1}{2^{1/p}} \left(\frac{2}{e}\right)^{1-1/p}, \tag{16}$$

with the corresponding extremal functions (recall that we are concerned with $p = 2m$).

Step 3: Lifting the functional $J_n(f) = c_n$ into the universal Teichmüller space. Consider the family $\widehat{S}(1)$ of quasiconformally extendable to $\widehat{\mathbb{C}}$ holomorphic univalent functions

$$w(z) = a_1 z + a_2 z^2 + \dots, \quad z \in \mathbb{D},$$

with $|a_1| = 1$ and $w(z_0) = z_0$ for some point $z_0 \in \mathbb{S}^1$ (depending on w), completed in the topology of locally uniform convergence on $\mathbb{C}$. This collection is a disjunct union

$$\widehat{S}(1) = \bigcup_{-\pi \le \theta, \alpha < \pi} S_{\theta,\alpha},$$

where $S_{\theta,\alpha}$ consists of univalent functions $w(z) = e^{i\theta} z + a_2 z^2 + \dots$ with quasiconformal extensions to $\widehat{\mathbb{C}}$ satisfying $w(1) = 1$ and their rotations $w_{\alpha,\alpha}(z)$(also completed in the indicated weak topology).

The assertion of Lemma 4 is also valid for the limit functions of sequences $\{w_n\}$ of functions $w_n \in \widehat{S}(1)$ with quasiconformal extension, but in the general case the equality $w(z_0) = z_0$ must be understand in terms of the Carathéodory prime ends. As was indicated above, any function from $\widehat{S}(1)$ with θ chosen following Lemma 4 is holomorphic on the disk $\mathbb{D}$ (has there no pole).

This family $\widehat{S}(1)$ is closely related to the canonical class S of univalent functions $w(z)$ on $\mathbb{D}$ normalized by $w(0) = 0,\ w'(0) = 1$. Every $w \in S$ has its representative $\widehat{w}$ in $\widehat{S}(1)$ (not necessarily unique) obtained by pre and post compositions of w with rotations $z \mapsto e^{i\alpha}z$ about the origin, related by

$$w_{\tau,\theta}(z) = e^{-i\theta}w(e^{i\tau}z) \quad \text{with} \quad \tau = \arg z_0, \tag{17}$$

where z_0 is a point for which $w(z_0) = e^{i\theta}$ is a common point of the unit circle and the boundary of domain $w(\mathbb{D})$.

This is trivial for the identical map $w(z) \equiv z$ (then one can take $\theta = \tau = 0$). For any another $w(z)$ the existence of such a point z_0 follows from the Schwarz lemma.

This connection implies, in particular, that the functions conformal in the closed disk $\overline{\mathbb{D}}$ are dense in each class $S_{\theta,\alpha}$.

The relation (17) allows us to model the universal Teichmüller space $\mathbf{T}$ by the Schwarzians $S_w = \varphi$ of functions $w(z)$ from S_θ, α taking the admissible values of θ for a each $e^{i\alpha} \in \mathbb{S}^1$ with a fixed θ for all α (choosing θ in accordance with Lemma 4). In this case the base point $\varphi = \mathbf{0}$ of $\mathbf{T}$ corresponds to the function (13) with $z_0 = e^{i\alpha}$.

The prescribed normalizing conditions $w(0) = 0,\ w'(0) = e^{i\theta},\ w(e^{i\alpha}) = e^{i\alpha}$ are compatible with existence and uniqueness of the corresponding conformal and quasiconformal maps and the Teichmüller space theory, ensure holomorphy of their Taylor coefficients, etc. Actually we deal with the classical model of Teichmüller spaces via domains in the Banach spaces of Schwarzian dervatives S_w in $\mathbb{D}$ (or in the disk $\mathbb{D}^*$) of univalent holomorphic functions normalized either by fixing three boundary points on the unit circle S^1 or via $w(0) = 0,\ w'(0) = 1,\ w(z_0) = z_0$, where z_0 on $\mathbb{S}^1$. Often the disk is replaced by the half-plane.

The relation (17) and Lemma 4 imply that for *each fixed θ the Schwarzians S_w of functions $w \in S_{\theta,\alpha}$ run over the same domain in* $\mathbf{B}$ modeling the space $\mathbf{T}$.

It is more convenient technically to deal with univalent functions in the complementary disk $\mathbb{D}^*$. Lemma 4 allows us to model the space $\mathbf{T}$ by the Schwarzians S_W of the inverted functions $W(z) = 1/w(1/z)$ for $w \in S_{\theta,\alpha}$.

These functions form the corresponding classes $\Sigma_{\theta,\alpha}$ of nonvanishing univalent functions on the disk $\mathbb{D}^*$ with expansions

$$W(z) = e^{-i\theta}z + b_0 + b_1z^{-1} + b_2z^{-2} + \dots, \quad W(1/\alpha) = 1/\alpha,$$

and $\widehat{\Sigma}(1) = \bigcup_{\theta,\alpha} \Sigma_{\theta,\alpha}$.

Simple computations yield that the coefficients a_n of $f \in S_{\theta,\alpha}$ and the corresponding coefficients b_j of $W(z) = 1/f(1/z) \in \Sigma_{\theta,\alpha}$ are related by

$$b_0 + e^{2i\theta} a_2 = 0, \quad b_n + \sum_{j=1}^{n} \epsilon_{n,j} b_{n-j} a_{j+1} + \epsilon_{n+2,0} a_{n+2} = 0, \quad n = 1, 2, \ldots ,$$

where $\epsilon_{n,j}$ are the entire powers of $e^{i\theta}$. This successively implies the representations of a_n by b_j via

$$\begin{aligned} a_n = (-1)^{n-1} \epsilon_{n-1,0} b_0^{n-1} - (-1)^{n-1} (n-2) \epsilon_{1,n-3} b_1 b_0^{n-3} \\ + \text{lower terms with respect to } b_0. \end{aligned} \tag{18}$$

By abuse of notation, we shall denote the holomorphic embedding of H^p into the space $\mathbf{T}$ modelled by Schwarzians in $\mathbb{D}^*$ by the same letter ι. The image ιH^p is a non-complete linear subspace in $\mathbf{B}$, and the image of the distinguished domain $\frac{1}{2^p}\mathcal{U}^p$ is a complex submanifold in $\mathbf{T}$.

Note that the coefficients α_n of Schwarzians

$$S_w(z) = \sum_0^\infty \alpha_n z^n$$

are represented as polynomials of $n+2$ initial coefficients of $w \in S_{\theta,\alpha}$ and, in view of (18), as polynomials of $n+1$ initial coefficients of the corresponding $W \in \Sigma_{\theta,\alpha}$ (provided that θ and α are given and fixed and the number $e^{i\theta}$ is considered to be a constant).

We denote these polynomials by $J_n(w)$ and $\widetilde{J}_n(W)$, respectively, and will deal with these polynomial functionals only on the union of admissible classes $S_{\theta,\alpha}$ or $\Sigma_{\theta,\alpha}$.

Holomorphic dependence of normalized quasiconformal maps on complex parameters (first established by Ahlfors and Bers in [1] for maps with three fixed points on $\widehat{\mathbb{C}}$) is an underlying fact for the Teichmüller space theory and for many other applications.

Another somewhat equivalent proof of holomorphy involves the variational technique for quasiconformal maps. For the maps w from $S_{\theta,\alpha}$, this holomorphy is a consequence of the following lemma from [14], Ch. 5 combined with appropriate Möbius maps.

Lemma 5. *Let $w(z)$ be a quasiconformal map of the plane $\widehat{\mathbb{C}}$ with Beltrami coefficient $\mu(z)$ which satisfies $\|\mu\|_\infty < \varepsilon_0 < 1$ and vanishes in the disk $\{|z| < r\}$. Suppose that $w(0) = 0$, $w'(0) = 1$, and $w(1) = 1$.*

Then, for sufficiently small ε_0 *and for* $|z| \le R < r_0(\varepsilon_0, r)$ *we have the variational formula*

$$w(z) = z - \frac{z^2(z-1)}{\pi} \iint\limits_{|\zeta|>r} \frac{\mu(\zeta) d\xi d\eta}{\zeta^2(\zeta-1)(\zeta-z)} + \Omega_\mu(z),$$

where $\zeta = \xi + i\eta$; $\max_{|z|\le R} |\Omega_\mu(z) \le C(\varepsilon_0, r, R)\|\mu\|_\infty^2$; $r_0(\varepsilon_0, r)$ *is a well defined function of* ε_0 *and* r *such that* $\lim_{\varepsilon_0 \to 0}\ r_0(\varepsilon_0, r) = \infty$*, and the constant* $C(\varepsilon_0, r, R)$ *depends only on* ε_0, r *and* R.

Step 4: Lifting to covering space $\mathbf{T}_1$ and estimating the restricted plurisubharmonic functional. Our next step is to lift both polynomial functionals $J_n(w)$ and $\widetilde{J}_n(W)$ (equivalently c_n) onto the Teichmüller space $\mathbf{T}_1$, which covers $\mathbf{T}$. Letting

$$\widehat{J}_n(\mu) = \widetilde{J}_n(W^\mu), \tag{19}$$

we lift these functionals from the sets $S_{\theta,\alpha}$ and $\Sigma_{\theta,\alpha}$ onto the ball $\mathbf{Belt}(\mathbb{D})_1$. Then, under the indicated $\mathbf{T}_1$-equivalence, i.e., by the quotient map

$$\phi_{\mathbf{T}_1} : \ \mathbf{Belt}(\mathbb{D})_1 \to \mathbf{T}_1, \quad \mu \to [\mu]_{\mathbf{T}_1},$$

the functional $\widetilde{J}_n(W^\mu)$ is pushed down to a bounded holomorphic functional $\mathcal{J}_n$ on the space $\mathbf{T}_1$ with the same range domain.

Equivalently, one can apply the quotient map $\mathbf{Belt}(\mathbb{D})_1 \to \mathbf{T}$ (i.e., $\mathbf{T}$-equivalence) and compose the descended functional on $\mathbf{T}$ with the natural holomorphic map $\iota_1 : \ \mathbf{T}_1 \to \mathbf{T}$ generated by the inclusion $\mathbb{D}_* \hookrightarrow \mathbb{D}$ forgetting the puncture.

Note that since the coefficients $b_0, \ b_1, \dots$ of $W^\mu \in \Sigma_{\theta,\alpha}$ are uniquely determined by its Schwarzian S_{W^μ}, the values of $\mathcal{J}_n$ in the points $X_1, \ X_2 \in \mathbf{T}_1$ with $\iota_1(X_1) = \iota_1(X_2)$ are equal.

Now, using the Bers isomorphism theorem, we regard the points of the space $\mathbf{T}_1$ as the pairs $X_{W^\mu} = (S_{W^\mu}, W^\mu(0))$, where $\mu \in \mathbf{Belt}(\mathbb{D})_1$ obey $\mathbf{T}_1$-equivalence (hence, also $\mathbf{T}$-equivalence). Denote (for simplicity of notations) the composition of $\mathcal{J}_n$ with biholomorphism $\mathbf{T}_1 \cong \mathcal{F}(\mathbf{T})$ again by $\mathcal{J}_n$. In view of (6) and (19), it is presented on the fiber space $\mathcal{F}(\mathbf{T})$ by

$$\mathcal{J}_n(X_{W^\mu}) = \mathcal{J}_n(S_{W^\mu}, \ t), \quad t = W^\mu(0). \tag{20}$$

This yields a logarithmically plurisubharmonic functional $|\mathcal{J}_n(S_{W^\mu}, t)|$ on $\mathcal{F}(\mathbf{T})$.

Note that since the coefficients $b_0, \ b_1, \dots$ of $W^\mu \in \Sigma_\theta$ are uniquely determined by its Schwarzian S_{W^μ}, the values of $\mathcal{J}$ in the points $X_1, \ X_2 \in \mathbf{T}_1$ with $\iota_1(X_1) = \iota_1(X_2)$ are equal.

We have to estimate a smaller plurisubharmonic functional arising after restriction of $\mathcal{J}_n(S_{F^\mu},\ t)$ onto the images in these spaces of the distinguished above domain

$$\iota\Big(\frac{1}{2^{1/p}}\mathcal{U}^p\Big),$$

i.e., the restriction of functional (19) onto the corresponding set of pairs $(S_{W^\mu}, W^\mu(0))$ consisting of $S_{W^\mu} \in \iota\Big(\frac{1}{2^{1/p}}\mathcal{U}^p\Big)$ and of the values $W^\mu(0)$ filling some subdomain in the disk $\{|t| < 4\}$. We denote this subdomain by $D_{\rho,\theta}$ and this restricted functional by $\mathcal{J}_{n,0}(S_{W^\mu},\ W^\mu(0))$.

Now define on $D_{\rho,\theta}$ the function

$$u_\theta(t) = \sup_{S_{W^\mu}} |\mathcal{J}_{n,0}(S_{W^\mu}, t)|, \tag{21}$$

where the supremum is taken over all $S_{F^\mu} \in \iota\Big(\frac{1}{2^{1/p}}\mathcal{U}^p\Big)$ admissible for a given $t = W^\mu(0) \in D_{\rho,\theta}$, that means over the pairs $(S_{W^\mu}, t) \in \mathcal{F}(\mathbf{T})$ with $S_{F^\mu} \in \iota\Big(\frac{1}{2^{1/p}}\mathcal{U}^p\Big)$ and a fixed t.

Our goal is to establish that this function inherits subharmonicity of $\mathcal{J}$. This is given by the following basic lemma.

Lemma 6. *The function $u_\theta(t)$ is subharmonic on the set $\iota\Big(\frac{1}{2^{1/p}}\mathcal{U}^p\Big)$ (which is open and connected in the induced topology).*

Proof. Consider in $\iota\Big(\frac{1}{2^{1/p}}\mathcal{U}^p\Big)$ its s-dimensional analytic subsets

$$V_s = \iota\Big(\frac{1}{2^{1/p}}\mathcal{U}^p \cap \mathcal{P}_s\Big).$$

For any $f \in \iota\Big(\frac{1}{2^{1/p}}\mathcal{U}^p\Big)$, define the function

$$F(z) = f(1/z)/z^4,$$

holomorphic on the complementary disk $\mathbb{D}^*$, and take a univalent solution $W \in \Sigma_\theta$ of the Schwarzian equation $S_W(z) = F(z)$ in $\mathbb{D}^*$. Let W^μ be one of its quasiconformal extensions onto the unit disk $\mathbb{D}$.

Using Lemma 3, we approximate f by polynomials p_s, and let W_s and $W_s^{\mu_s}$ be the corresponding functions defined similarly by these polynomials. As $s \to \infty$, the domains $W_s(\mathbb{D}^*)$ and $W_m^{\mu_m}(\mathbb{D})$ approximate $W(\mathbb{D}^*)$ and $W^\mu(\mathbb{D})$ uniformly (in the spherical metric on $\widehat{\mathbb{C}}$), and the points $W_s^{\mu_s}(0)$ are close to $W^\mu(0)$.

One can replace the extensions $W_s^{\mu_s}$ by $\omega_s \circ W_s^{\mu_s}$, where ω_s is the extremal quasiconformal automorphism of domain $W_s^{\mu_s}(\mathbb{D})$ moving the point $W_m^{\mu_s}(0)$ into $W^\mu(0)$ and identical on the boundary of $W_s^{\mu_s}(\mathbb{D})$ (cf. [27]). This provides for a prescribed $t = W^\mu(0)$ the points $S_{W_s^{\mu_s}} \in \mathcal{F}(\mathbf{T})$ corresponding to given $p_s \in V_s$.

Now, maximizing the function $\log |\mathcal{J}_{n,0}(S_{W_s^{\mu_s}}, t)|$ over the manifold V_s, i.e., over the collection of the Schwarzians $S_{W_s^{\mu_s}}$ (with appropriate s), one obtains a logarithmically plurisubharmonic function

$$u_s(t) = \sup_{V_s} |\mathcal{J}_{n,0}(S_{W_s^{\mu_s}}, t)|, \quad t = W^\mu(0), \tag{22}$$

in the domain $D_{\rho,\theta}$ indicated above. We take the upper semicontinuous regularization of this function, given by

$$u_s(t) = \limsup_{t' \to t} u_s(t')$$

(by abuse of notation, we denote the regularization by the same letter as the original function).

The general properties of subharmonic functions in the Euclidean spaces imply that such a regularization also is logarithmically subharmonic in each connected component of V_s.

In the limit, as $s \to \infty$, the sets V_s approximate the distinguished subset $\iota\Big(\frac{1}{2^p}\mathcal{U}^p\Big)$, and similar to above, taking the limit function

$$u(t) = \limsup_{s \to \infty} u_s(t) \tag{23}$$

followed by its upper semicontinuous regularization, one obtains a logarithmically subharmonic function on the $\iota\Big(\frac{1}{2^p}\mathcal{U}^p\Big)$ (which coincides with function (21)). The proof of Lemma 6 is completed. □

Step 5: Range domain of $W^\mu(0)$. The next step in maximization of the functional J_n (equivalently, of the function (21)) is to establish the value domain of $W^\mu(0)$ for W^μ running over $\iota\Big(\frac{1}{2^{1/p}}\mathcal{U}^p\Big)$. This requires the corresponding covering estimate.

Let G be a domain in a complex Banach space $X = \{\mathbf{x}\}$ and let χ be a holomorphic map from G into the universal Teichmüller space $\mathbf{T}$ modeled as a bounded subdomain of $\mathbf{B}$ and suppose that the image set $\chi(G)$ admits the circular symmetry, which means that for every point $\varphi \in \chi(G)$ the circle $e^{i\theta}\varphi$ belongs entirely to this set. Consider in the unit disk the corresponding Schwarzian differential equations

$$S_w(z) = \chi(\mathbf{x}) \tag{24}$$

and pick their holomorphic univalent solutions $w(z)$ in $\mathbb{D}$ satisfying $w(0) = 0,\ w'(0) = 1$ (hence $w(z) = z + \sum_2^\infty a_n z^n$). Put

$$|a_2^0| = \sup\{|a_2| : \ S_w \in \chi(G)\}, \tag{25}$$

and let $w_0(z) = z + a_2^0 z^2 + \dots$ be one of the maximizing functions (its existence follows from compactness of the family of these $w(z)$ in topology of locally uniform convergence in $\mathbb{D}$).

Lemma 7. (a) *For every indicated solution* $w(z) = z + a_2 z^2 + \dots$ *of the differential equation* (24), *the image domain* $w(\mathbb{D})$ *covers entirely the disk* $\{|w| < 1/(2|a_2^0|)\}$.

The radius value $1/(2|a_2^0|)$ *is sharp for this collection of functions, and the circle* $\{|w| = 1/(2|a_2^0|)\}$ *contains points not belonging to* $w(\mathbb{D})$ *if and only if* $|a_2| = |a_2^0|$ *(i.e., when* w *is one of the maximizing functions).*

(b) *The inverted functions*

$$W(\zeta) = 1/w(1/\zeta) = \zeta - a_2 + b_1\zeta^{-1} + b_2\zeta^{-2} + \dots$$

maps the disk $\mathbb{D}^*$ *onto a domain whose boundary is entirely contained in the disk* $\{|W + a_2| \le |a_2^0|\}$.

Note that the collection of solutions $w(z)$ with the indicated normalization preserves the rotational symmetry.

The proof of this lemma follows the classical lines of Koebe's 1/4 theorem (cf. [11]).

(a) Suppose that the point $w = c$ does not belong to the image of $\mathbb{D}$ under the map $w(z)$ defined above. Then $c \ne 0$, and the function

$$w_1(z) = cw(z)/(c - w(z)) = z + (a_2 + 1/c)z^2 + \dots$$

also belongs to this class, and hence by (25), $|a_2 + 1/c| \le |a_2^0|$, which implies

$$|c| \ge 1/(2|a_2^0|)$$

(hence, $1/|c| \le 2|a_2^0|$). Noting also that the values a_2 and c both admit the rotational symmetries, one concludes that the equality holds only when

$$|a_2 + 1/c| = |1/c| - |a_2| = |a_2^0| \quad \text{and} \quad |a_2| = |a_2^0|.$$

(b) If a point $\zeta = c$ does not belong to the image $W(\mathbb{D}^*)$, then the function

$$W_1(z) = 1/[W(1/z) - c] = z + (c + a_2)z^2 + \dots$$

is holomorphic and univalent in the disk $\mathbb{D}$, and therefore, $|c + a_2| \le |a_2^0|$. The lemma follows.

In particular, the domain $\iota\Big(\frac{1}{2^{1/p}} \mathcal{U}^p\Big)$ admits the required rotational symmetry (it is created by rotations (17)). Applying Lemma 7, one derives that the range domain of admissible values of $W^\mu(0)$ is contained in the disk

$$\mathbb{D}_{2|a_2^0|} = \{W : \ |W| < 2|a_2^0|\}, \tag{26}$$

and the boundary of this domain for $W^\mu(0)$ touches from inside the circle $\{|W| = 2|a_2^0|\}$ at the points corresponding to extremal functions maximizing $|a_2|$ on the closure of domain $\iota\Big(\frac{1}{2^{1/p}} \mathcal{U}^p\Big)$.

Step 6: Symmetrization and quasiconformal deformations of H^{2m} functions. One of the crucial points in the proof of Theorem 1 is to establish that in the case of nonvanishing H^{2m} functions this radius $2|a_2^0|$ is naturally connected with the extremal function $\kappa_{1,2m}(z)$ maximizing the coefficient $|c_1|$. This will be done only for $p = 2m$.

First, we apply the following generalization of the above construction (valid for all $p \ge 2$). We select a dense countable subset

$$\Theta = \{\theta_1, \theta_2, \dots, \theta_s, \dots\} \subset [-\pi, \pi],$$

and find by Lemma 4 a corresponding dense subset

$$\mathcal{A} = \{\alpha_1, \alpha_2, \dots, \alpha_s, \dots\}$$

(also in $[-\pi, \pi]$). Similar to Step 3, we consider for each pair (θ_j, α_j) and the admissible rotation angles θ the corresponding collections Σ_{θ,α_j} of nonvanishing univalent holomorphic functions

$$W_{\theta,j}(z) = e^{-i\theta} z + b_{0,j} + b_{1,j} z^{-1} + \cdots : \ \mathbb{D}^* \to \widehat{\mathbb{C}}$$

whose quasiconformal extensions $W_{\theta,j}^{\mu_j}$ to $\widehat{\mathbb{C}}$ satisfy $W_\theta^{\mu_j}(1/\alpha_j) = 1/\alpha_j$.

Using these functions, we construct in the similar fashion the increasing unions of the quotient spaces

$$\mathcal{T}_s = \bigcup_{j=1}^{s} \ \Sigma_{\theta_j,\alpha_j}/\sim \ = \bigcup_{j=1}^{s} \{(S_{W_{\theta_j,\alpha_j}}, W_\theta^\mu(0))\} \ \simeq \mathbf{T}_1 \cup \cdots \cup \mathbf{T}_1, \tag{27}$$

where the equivalence relation $\sim$ means $\mathbf{T}_1$-equivalence on the union $\widehat{\Sigma}(1)$ of all collections $\Sigma_{\theta,1/\alpha_j}$ consisting of the indicated functions $W_{\theta,j}$ and their rotations $e^{-i\alpha} W_{\theta,j}(e^{i\alpha} z), \ \alpha \in [-\pi, \pi]$; accordingly,

$$\mathbf{W}_\theta^\mu(0) := (W_{\theta,1}^{\mu_1}(0), \dots, W_{\theta,s}^{\mu_s}(0)).$$

The Beltrami coefficients $\mu_j \in \mathbf{Belt}(\mathbb{D})_1$ are chosen here independently. The corresponding collection

$$\beta = (\beta_1, \dots, \beta_s)$$

of the Bers isomorphisms

$$\beta_j : \ \{(S_{W_{\theta,j}}, W_{\theta,j}^{\mu_j}(0))\} \to \mathcal{F}(\mathbf{T})$$

determines a holomorphic surjection of the space $\mathcal{T}_s$ onto $\mathcal{F}(\mathbf{T})$.

Taking out in each union (27) the corresponding collection $\iota_s\Big(\frac{1}{2^{1/p}}\mathcal{U}^p\Big)$ covering $\Big(\frac{1}{2^{1/p}}\mathcal{U}^p\Big)$, one obtains similar to the above the maximal function

$$u(t) = \sup_{\Theta} u_{\theta_s}(t) = \sup \big\{|\mathcal{J}_{n,0}(S_{W^\mu}, t)| : \ S_{W^\mu} \in \bigcup_s \iota_s\Big(\frac{1}{2^{1/p}}\mathcal{U}^p\Big)\big\}, \quad (28)$$

which is defined and subharmonic in domain

$$D_\rho = \bigcup_{\Theta} D_{\rho,\theta_s}.$$

The density of both sets Θ and $\mathcal{A}$ in $[-\pi, \pi]$ implies that the union of all spaces (27) and the limit subharmonic function (28) both admit the circular symmetry (invariance under rotations upon any angle $\theta \in [-\pi, \pi]$). Therefore, the domain D_ρ must coincide with the disk (26) determined by Lemma 7.

From now on, we take $p = 2m$ with natural $m \in \mathbb{N}$. Our first goal is to show that in this case the range domain of $W^\mu(0)$ is determined by the following basic lemma, which reveals the specific features of nonvanishing H^{2m} functions.

Consider the collection $\mathcal{N}_{2m}$ of all nonvanishing H^{2m} functions located in the ball $\{\|\varphi\| < 2\}$ in $\mathbf{B}$ and denote the minimal radius of the balls in H^{2m} containing these functions by $r(m)$; that is

$$r(m) = \sup\{\|f\|_{2m} : \ f \in B_1^0(H^{2m}), \ \|f\|_{\mathbf{B}} \le 2\}.$$

For any such f, the solutions $w(z)$ of the equation $S_w = f$ are univalent holomorphic functions on the disk $\mathbb{D}$. The set $\frac{1}{2^{1/(2m)}}\mathcal{U}^{2m}$ applied earlier is a proper subset of $\mathcal{N}_{2m}$.

An essential point is that the extremal covering function $w_0(z)$ of this maximal collection is intrinsically connected with the function (2).

Lemma 8. *For any space H^{2m} and its subset $\mathcal{N}_{2m}$, we have the equality*

$$S_{w_0}(z) = r(m) \ \kappa_{1,2m}(z), \quad (29)$$

where $\kappa_{1,2m}$ is given by (2) *for $n = 1$, $p = 2m$. This means that the Schwarzian of the extremal univalent function $w_0(z)$ maximizing the second coefficient a_2 on the set $\mathcal{N}_{2m}$ equals the extremal function for c_1.*

Proof. It is enough to establish that

$$S'_{w_0}(0) = c_1^0 \neq 0,$$

This yields that the zero set of the functional $J_1(f) = c_1$ on $\widehat{S}(1)$ is separated from the set of rotations (17) of the function w_0, and therefore the corresponding function (28), constructed by maximization of functional $|J_1(f)| = |c_1|$, is defined and logarithmically subharmonic on the whole disk $\mathbb{D}_{2|a_2^0|}$ (determined by w_0), attaining its maximum on the boundary circle. Then the equality (29) follows from Lemma 1 giving the explicit representation of the extremal of $J_1(f)$ and its uniqueness (the linearity of coefficient c_1 on H^{2m} extends the assertion of Lemma 1 to any ball centered at the origin of H^{2m}).

We apply the special quasiconformal deformations of functions $f \in \mathcal{N}_{2m}$ preserving their L_{2m} norm and with some prescribed properties. The existence of such deformations was established in [15] for $p = 2m$; its origin goes back to the local existence theorems in [14] for Riemann surfaces with finitely generated fundamental groups.

Consider the functions $f(z) \in H^{2m} \cap L_\infty(\mathbb{D})$, with

$$\sup_{\mathbb{D}} |f(z)| = M > \|f\|_{2m}.$$

Let E be a ring domain bounded by a closed curve $L \subset \mathbb{D}$ containing inside the origin and by the unit circle $S^1 = \partial\mathbb{D}$. The degenerated cases $E = \mathbb{D} \setminus \{0\}$ and $E = S^1$ correspond to the Bergman space B^p and to the Hardy space H^p.

Let, in addition,

$$\mathbf{d}^0 = (0, 1, 0, \dots, 0) =: (d_k^0) \in \mathbb{R}^{n+1},$$

and $|\mathbf{x}|$ denote the Euclidean norm in $\mathbb{R}^l$.

Lemma 9. *For any holomorphic function $f(z) = \sum\limits_{k=j}^{\infty} c_k^0 z^k \in L_{2m}(E) \cap L_\infty(E)$ (with $c_j^0 \neq 0$, $0 \leq j < n$ and $m \in \mathbb{N}$), which is not a polynomial of degree $n_1 \leq n$, there exists a positive number ε_0 such that for every point*

$$\mathbf{d}' = (d'_{j+1}, \dots, , d'_n) \in \mathbb{C}^{n-j}$$

and every $a \in \mathbb{R}$ satisfying the inequalities

$$|\mathbf{d}'| \leq \varepsilon, \quad |a| \leq \varepsilon, \quad \varepsilon < \varepsilon_0,$$

there exists a quasiconformal automorphism h of the complex plane $\widehat{\mathbb{C}}$, which is conformal in the disk

$$D_0 = \{w: \ |w - c_0^0| < \sup_{\mathbb{D}} |f_0(z)| + |c_0^0| + 1\}$$

(hence also outside of E) and satisfies the conditions:

(i) $h^{(k)}(c_0^0) = k! d_k = k!(d_k^0 + d_k'), \ k = j+1, \dots, n$ *(i.e., $d_1 = 1 + d_1'$ and $d_k = d_k'$ for $k \geq 2$);*

(ii) $\|h \circ f\|_{2m}^{2m} = \|f\|_{2m}^{2m} + a$.

The proof of this lemma in [15] essentially involves the assumption $m = p/2 \in \mathbb{N}$ and does not extend to arbitrary $p \geq 2$.

Using the appropriately thin rings E adjacent to the unit circle, one derives that for any bounded in $\mathbb{D}$ function $f \in H^{2m}$ there exists a $O(\varepsilon)$-quasiconformal automorphism h of $\widehat{\mathbb{C}}$ satisfying the conditions (i) and

$$\|h \circ f\|_{2m}^{2m} = \|f\|_{2m}^{2m} + O(\varepsilon).$$

We proceed to the proof of Lemma 8 and consider the restrictions of the functional $\mathcal{J}_{1,0}$ corresponding to $J_1(f)$ onto the intersections

$$B_{1,M}^0(H^{2m}) = B_1^0(H^{2m}) \cap \{f \in L_\infty(\mathbb{D}): \ \|f\|_\infty < M\} \quad (M < \infty).$$

This implies, similar to (21) and (28), for each M the subharmonic function

$$u_M(t) = \sup\{|\mathcal{J}_{1,0}(S_{W^\mu}, t)|: \ S_{W^\mu} \in (\mathcal{N}_{2m} \cap \mathcal{P}) \cap B_{1,M}^0(H^{2m})\},$$

on the corresponding disk $\mathbb{D}_{2|a_{2,M}^0|}$. The above construction also yields that the collections of functions $u_M(t)$ and of radii $2|a_{2,M}^0|$ both are monotone increasing with M, i.e., for $M' > M$,

$$u_{M'}(t) \geq u_M(t), \quad |a_{2,M'}^0| \geq |a_{2,M}^0|.$$

Note also that all $|a_{2,M}^0| < |a_2^0|$.

The corresponding collection $\{f_M\}$ of the points $f_M = S_{w_{0,M}}$, maximizing $|a_{2,M}|$ on these sets, is weakly compact in H^{2m} (in the weak topology on H^{2m} defined by linear functionals) as well as in the topology of locally uniform convergence of continuous functions on the unit disk. Hence, as $M \to \infty$, the maximal values $c_{1,M}^0$ and $a_{2,M}^0$ on $(\mathcal{N}_{2m} \cap \mathcal{P}) \cap B_{1,M}^0(H^{2m})$ approach the corresponding maximal values $|c_1^0|$ and $|a_2^0|$ of the coefficients c_1 and a_2 on $\mathcal{N}_{2m}$. The limit subharmonic function $u(t) = \lim_{M \to \infty} u_M(t)$ must coincide with (28).

Now fix a sufficiently small $\varepsilon = \varepsilon(M) > 0$ and choose for the extremal on $\mathcal{N}_{2m} \cap B^0_{1,M}(H^{2m})$ function

$$f_{0,M}(z) = S_{w_{0,M}}(z) = c^0_M + c^0_1 z + \dots$$

a large $M > \|f_{0,M}\|_{2m}$, so that

$$|c^0_{1,M}| \ge |c^0_1| - \varepsilon, \quad |a^0_{2,M}| \ge |a^0_2| - \varepsilon.$$

Since the image $f_{0,M}(\mathbb{D})$ is a proper subset of the disk $\mathbb{D}_M = \{|w| < M\}$, one can vary arbitrarily the extremal value $|c^0_{1,M}|$ by applying Lemma 9, which provides $O(\varepsilon)$-quasiconformal automorphisms h_ε of $\widehat{\mathbb{C}}$ conformal in the domain $f_{0,M}(\mathbb{D})$ and such that $\|S_{h_\varepsilon \circ f_{0,M}}\|_{2m} = \|S_{f_{0,M}}\|_{2m} + O(\varepsilon)$, and

$$|a_2(h_\varepsilon \circ w_{0,M})| = |a_2(w_{0,M})|, \quad |c_1(h_\varepsilon \circ f_{0,M})| = |c^0_{1,M}| + O(\varepsilon) > |c^0_{1,M}|.$$

Therefore, as $M \to \infty$, the circles $\{|t| = 2|a^0_{2,M}| + O(\varepsilon)\}$ approach from inside the circle $\{|t| = 2|a^0_2|\}$. Hence, the corresponding maximal function (28) for the functional $|J_1(f)| = |c_1|$ also is defined and subharmonic on the disk $\mathbb{D}_{2|a^0_2|}$ determined by $w_0(z)$.

The uniqueness of the extremal function for $J_1(f)$ on $B^0_1(H^{2m})$ stated by Lemma 1 implies the desired relation (29). The proof of Lemma 8 is completed. □

Step 7: Finishing the proof. Now we can finish the proof of the theorem.

Let $f(z) \in H^{2m}$. Take $n = 2$ and, letting $f_2(z) = f(z^2)$, consider on $B^0_1(H^{2m})$ the functional

$$I_2(f) = \max\,(|J_2(f)|, |J_2(f_2)|) = \max\,(|c_2(f)|, |c_2(f_2)|). \tag{30}$$

Since the correspondence $f(z) \mapsto f_2(z)$ is linear, it is holomorphic in H^{2m} norm; thus this functional is plurisubharmonic.

One can repeat for this functional the above construction, lifting both functionals $J_2(f)$ and $J_2(f_2)$ onto the space $\mathbf{T}_1$ similar to above, and obtain in the same way the corresponding nonconstant radial subharmonic function of type (28) on the disk (26). Again, this function is logarithmically convex, hence monotone increasing, and attains its maximal value at $|t| = 2|a^0_2|$.

Now observe that by Parseval's equality for the boundary functions

$$f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}), \quad f \in H^{2m},$$

we have

$$1 \geq \frac{1}{2\pi} \int\limits_{\pi}^{\pi} |f(e^{i\theta})|^2 d\theta = \sum_{1}^{\infty} |c_n|^2.$$

Applying it to the function

$$f(z) = \kappa_{1,m}(z) = \sum_{0}^{\infty} c_n^0 z^n$$

and noting that by (2),

$$|c_1^0|^2 = (2/e)^{2(1-1/(2m))} = 0.5041...^{1-1/(2m)} > 0.5041... ,$$

one obtains that for this function,

$$\sum_{2}^{\infty} |c_n^0|^2 < 0.5 < |c_1^0|^2. \tag{31}$$

This implies (in view of the indicated by Lemma 8 connection of the radius $2|a_2^0|$ with extremal function $\kappa_{1,2m}$ for $J_1(f)$) that the maximal value of the functional (30) on $B_1^0(H^{2m})$ also is attained on the circle $\{|t| = 2|a_2^0|\}$. Therefore, only the functions

$$\kappa_{1,2m}(z), \quad \kappa_{1,2m;2}(z) := \kappa_{1,2m}(z^2)$$

and their rotations can be extremal functions for $I_2(f)$.

Since (31) yields $|c_2^0| < |c_1^0|$, we have

$$\max_{B_1^0(H^2 m)} I_2(f) = \max\ (|c_1^0|, |c_2^0|) = |c_1^0| = (2/e)^{1-1/2m}.$$

This implies the desired estimate (2) for $n = 2$, and the extremal function is unique, up to pre and post rotations of $\kappa_{1,2m}$.

Now take $n = 3$ and, letting $f_3(z) = f(z^3)$, consider the functional

$$I_3(f) = \max\ (|J_3(f)|, |J_3(f_3)|) = \max(|c_3(f)|, |c_3(f_3)|).$$

Similar to the previous cases of $J_1(f)$ and $I_2(f)$, we lift the functional $I_3(f_3)$ together with $J_3(f_3)$ onto the Teichmüller space $\mathbf{T}_1$ and applying the above lemmas, construct for I_3 the corresponding dominant circularly symmetric subharmonic function (28) defined on the same disk $\mathbb{D}_{2|a_2^0|}$.

This implies the bound

$$\max_{B_1^0(H^{2m})} I_3(f) = \max\ (|c_1^0|, |c_3^0|),$$

which estimates the functional $I_3(f)$ by coefficients of the function (2) with $n = 3$. The inequality (31) yields that $|c_3^0| < |c_1^0|$; therefore,

$$\max (|c_1^0|, |c_3^0|) = |c_1^0| = (2/e)^{1-1/2m},$$

which provides the desired estimate (3) for $n = 3$, with a similar description of the extremal functions.

Taking subsequently the functions $f_4(z) = f(z^4)$, $f_5(z) = f(z^5), \dots$ and the corresponding functionals

$$I_4(f) = \max (|J_4(f)|, |J_4(f_4)|) = \max(|c_4(f)|, |c_4(f_4)|),$$
$$I_5(f) = \max (|J_5(f)|, |J_5(f_5)|) = \max(|c_5(f)|, |c_5(f_5)|), \dots ,$$

one obtains by the same arguments that the estimate (3) is valid for all $n \in \mathbb{N}$. This completes the proof of Theorem 1. □

4. Additional remark

4.1. Another proof of Theorem 1. Actually there is not necessary to use Lemma 4 in the above proof. It can by replaced by applying the relation (17) (cf. [16]).

Indeed, the universal Teichmüller space $\mathbf{T}$ can be modeled by the Schwarzian derivatives S_f of univalent holomorphic functions $f(z) = z + a_2 z^2 + \dots$ in $\mathbb{D}$ with quasiconformal extensions preserving some fixed inner point $z_0 \in \mathbb{D}^*$ (take $z_0 = \infty$ for all $f(z)$); equivalently, one can use nonvanishing univalent functions $F(z) = z + b_0 + b_1 z^{-1} + \cdots : \mathbb{D}^* \to \widehat{\mathbb{C}}$ with $F(1/z_0) = 1/z_0$. These derivatives fill a bounded domain in the space $\mathbf{B}$ containing the origin $\varphi = \mathbf{0}$.

The equality (17) shows that in the case of functions from the classes $S_{\theta,\alpha}$ (with admissible θ), one has to deal with the Schwarzians $S_w(e^{i\tau})e^{2i\tau}$ which fill the same domain in $\mathbf{B}$. The following symmetrization similar to Step 6 and passing to the union of all spaces (27) provide the same result as in the above proof.

In fact, the needed circular symmetry to get the disk (26) and $\mathbb{C}$-holomorphy of appropriate univalent functions $w_{\tau,\theta}$ in $\mathbb{D}$ also are assured by (17).

4.2. Beyond Theorem 1. Theorem 1 relates to some results of [17] which are relevant to the distortion functionals of classical geometric function theory.

4.3. Connection with over-normalized maps. The classes $S_{\theta,\alpha}$ applied in the proof of Theorem 1 naturally relate to over-normalized univalent functions on the unit disk $\mathbb{D}$ with quasiconformal extension satisfying, for example,

$$w(0) = 0; \quad w'(0) = e^{i\alpha}, \quad w(z_0) = z_0 \ (|z_0| = 1); \quad w(\infty) = \infty \qquad (32)$$

which arise in the distortion theory and other questions. This reveals the intrinsic features of quasiconformality.

The existence of such nontrivial maps $w(z)$ is insured by a local existence theorem from [14]. Its special case for simply connected planar domains states:

Lemma 10. *Let D be a simply connected domain on the Riemann sphere $\widehat{\mathbb{C}}$. Assume that there are a set E of positive two-dimensional Lebesgue measure and a finite number of points $z_1, z_2, ..., z_m$ distinguished in D. Let $\alpha_1, \alpha_2, ..., \alpha_m$ be non-negative integers assigned to $z_1, z_2, ..., z_m$, respectively, so that $\alpha_j = 0$ if $z_j \in E$.*

Then, for a sufficiently small $\varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$, and for any given collection of numbers $w_{sj}, s = 0, 1, ..., \alpha_j, \ j = 1, 2, ..., m$ which satisfy the conditions $w_{0j} \in D$,

$$|w_{0j} - z_j| \le \varepsilon, \quad |w_{1j} - 1| \le \varepsilon, \quad |w_{sj}| \le \varepsilon \ (s = 0, 1, \ldots \alpha_j, \ j = 1, ..., m),$$

there exists a quasiconformal self-map h of D which is conformal on $D \setminus E$ and satisfies

$$h^{(s)}(z_j) = w_{sj} \quad \text{for all } s = 0, 1, ..., \alpha_j, \ j = 1, ..., m.$$

Moreover, the Beltrami coefficient $\mu_h(z) = \partial_{\bar{z}} h / \partial_z h$ of h on E satisfies $\|\mu_h\|_\infty \le M\varepsilon$. The constants ε_0 and M depend only upon the sets D, E and the vectors $(z_1, ..., z_m)$ and $(\alpha_1, ..., \alpha_m)$.

If the boundary ∂D is Jordan or is $C^{l+\delta}$-smooth, where $0 < \delta < 1$ and $l \ge 1$, we can also take $z_j \in \partial D$ with $\alpha_j = 0$ or $\alpha_j \le l$, respectively.

The finite compositions of such maps $w^{\mu_s} \circ w^{\mu_{s-1}} \circ \cdots \circ w^{\mu_1}$ (conformal, respectively, on domains $w^{\mu_1}(\mathbb{D})$, $w^{\mu_2} \circ w^{\mu_1}(\mathbb{D})$, $\ldots$) provide the maps w^μ with arbitrary $\|\mu\|_\infty < 1$ satisfying (32).

Such maps can be regarded as the limit case of quasiconformal automorphisms of the sphere $\widehat{\mathbb{C}}$ conformal on $\mathbb{D}$ (or on a more general quasidisk) moving the ordered quadruples $(0, \ z_*, \ z_0, \ \infty)$ onto $(0, \ e^{i\alpha} z_*, \ z_0, \ \infty)$ as $z_* \to z_0$.

More generally, one obtains on this way a function $w(z)$ mapping conformally the disk $\mathbb{D}$ onto a domain $D \subset \mathbb{C}$ with quasiconformal boundary L and containing the origin, so that $w(0) = 0, \ w'(0) = e^{i\alpha}, \ w(\infty) = \infty$ and $w(z_j) = z_j$ on a given common finite set of points $\{z_1, \ldots, z_s\} \subset \mathbb{S}^1 \cap L$.

The requirement $w(\infty) = \infty$ can be omitted without breaking holomorphy of w on $\mathbb{D}$ (applying, for example, Teichmüller's Verschiebungssatz [27]).

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