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Existence of solitary traveling waves in Fermi–Pasta–Ulam type systems on 2D-lattice with saturable nonlinearities

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Abstract. The article deals with the Fermi–Pasta–Ulam type systems with saturable nonlinearities that describes an infinite systems of particles on a two dimensional lattice. The main result concerns the existence of solitary traveling waves solutions with vanishing relative displacement profiles. By means of critical point theory, we obtain sufficient conditions for the existence of such solutions.

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1. Introduction

Recently, considerable attention has been paid to models that are discrete in the spatial variables. Among the equations that describe such models, the most famous are the Discrete Nonlinear Shrödinger type equations, the Discrete Sine–Gordon type equations, the equations of chains of oscillators and the Fermi–Pasta–Ulam type systems. Such equations are of interest in view of numerous applications in physics [2, 16–18, 22]. Among the solutions of such systems, traveling waves deserve special attention. In papers [6, 8, 9, 14, 19, 20] traveling waves for infinite systems of linearly and nonlinearly coupled oscillators on 2D–lattice are studied, while [26] deal with periodic in time solutions for such systems. Papers [3, 7, 13] is devoted to the existence of homoclinic and heteroclinic traveling waves for the discrete sine–Gordon type equations on 2D–lattice. In papers [1,21,23,24] periodic and solitary traveling waves in Fermi–Pasta–Ulam system on 1D–lattice are studied. While [4,5,10,12] deal with traveling waves for such systems on 2D–lattice.

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In the present paper we study the Fermi–Pasta–Ulam type systems that describes the dynamics of an infinite systems of nonlinearly coupled particles on a two dimensional lattice. Let $q_{n,m} = q_{n,m}(t)$ be a coordinate of the (n,m)-th particle at time t. It is assumed that each particle interacts nonlinearly with its four nearest neighbors. The equations of motion of the system considered are of the form

$$\ddot{q}_{n,m} = W_1'(q_{n+1,m} - q_{n,m}) - W_1'(q_{n,m} - q_{n-1,m}) +$$

$$+W_2'(q_{n,m+1} - q_{n,m}) - W_2'(q_{n,m} - q_{n,m-1}), (n,m) \in \mathbb{Z}^2, \tag{1.1}$$

where W_1 and W_2 are the potentials of interaction. Equations (1.1) form an infinite system of ordinary differential equations.

In contrast to the previous results (see [10] and [12]), in this paper we study system (1.1) with saturable nonlinearities which means that at infinity $W_i'(r)$ growth as $const \cdot r$, i.e. $W_i(r)$ are asymptotically quadratic at infinity ($i = 1, 2$). Note that in [11] and [23] such nonlinearities are considered.

2. Statement of a problem

A traveling wave solution of Eq. (1.1) is a function of the form

$$q_{n,m}(t) = u(n\cos\varphi + m\sin\varphi - ct),$$

where the profile function $u(s)$ of the wave, or simply profile, satisfies the equation

$$c^2u''(s) = W_1'(u(s+\cos\varphi) - u(s)) - W_1'(u(s) - u(s-\cos\varphi))$$

$$+W_2'(u(s+\sin\varphi) - u(s)) - W_2'(u(s) - u(s-\sin\varphi)), \tag{2.1}$$

where $s = n\cos\varphi + m\sin\varphi - ct$.

In what follows, a solution of Eq. (2.1) is understood as a function $u(s)$ from the space $C^2(\mathbb{R};\mathbb{R})$ satisfying Eq. (2.1) for all $s \in \mathbb{R}$.

We consider two types of solutions:

— periodic traveling waves;

— solitary traveling waves.

In the first case profile satisfies the following periodicity condition

$$u'(s+2k) = u'(s),\ s \in \mathbb{R}, \tag{2.2}$$

where $k > 0$ is a real number. Note that the profile of such wave is not necessarily periodic. But its relative displacement profiles $r_i^{\pm}$ are periodic:

$$r_1^{\pm}(s) = \int\limits_s^{s\pm\cos\varphi} u'(\tau)d\tau,\ r_2^{\pm}(s) = \int\limits_s^{s\pm\sin\varphi} u'(\tau)d\tau.$$

Therefore, such waves are also called periodic (see [24]).

In the second case profile satisfies the following condition

$$\lim_{s\to\pm\infty} u'(s) = u'(\pm\infty) = 0, \tag{2.3}$$

i.e., the relative displacement profiles vanish at infinity.

We always assume that

(*i*) $W_i(r) = \frac{c_i^2}{2}r^2 + f_i(r)$, *where* $c_i \in \mathbb{R}$, $f_i \in C^1(\mathbb{R})$, $f_i(0) = f_i'(0) = 0$ *and* $f_i'(r) = o(r)$ *as* $r \to 0$, $i = 1, 2$;

(*ii*) *there exists a finite limit* $\lim\limits_{r\to\pm\infty} \frac{f_i'(r)}{r} = l$, *and the functions* $g_i(r) = f_i'(r) - lr$ *are bounded* ($i = 1, 2$);

(*iii*) $f_i(r) \geq 0$ *for all* $r \in \mathbb{R}$ *and for every* $r_0 > 0$ *there exists* $\delta_0 = \delta_0(r_0) > 0$ *such that*

$$\frac{1}{2}rf_i'(r) - f_i(r) \geq \delta_0$$

for $|r| \geq r_0$ ($i = 1, 2$).

To simplify notation, we denote

$$h_i(r) := f_i'(r) = lr + g_i(r),\ i = 1, 2,$$

and

$$G_i(r) := \int\limits_0^r g_i(\rho)d\rho,\ i = 1, 2,$$

and additionally assume that one of two conditions is satisfied:

(*iv*) $G_i(r) \to -\infty$ *as* $r \to \pm\infty$ ($i = 1, 2$);

or

(*v*) $c^2\left(\frac{\pi n}{k}\right)^2 - 4(c_1^2 + l)\sin^2\left(\frac{\pi n}{2k}\cos\varphi\right) - 4(c_2^2 + l)\sin^2\left(\frac{\pi n}{2k}\sin\varphi\right) \neq 0$ *for all* $n \in \mathbb{N}$.

Remark 2.1. Assumption (*iii*) implies, in particular, that the functions $f_i(r)$ are increasing for $r \geq 0$ and descending for $r \leq 0$, and $G_i(r) < 0$ for all $r \neq 0$, $i = 1, 2$.

The important role is played by the quantity defined by the equality

$$c_0(\varphi) := \sqrt{c_1^2\cos^2\varphi + c_2^2\sin^2\varphi}.$$

3. Periodic waves

Let E_k be the Hilbert space defined by

$$E_k = \{u \in H^1_{loc}(\mathbb{R}) : \ u'(s+2k) = u'(s), \ u(0) = 0\}$$

with the scalar product

$$(u,v)_k = \int\limits_{-k}^{k} u'(s)v'(s)ds$$

and corresponding norm $\|u\|_k = (u,u)^{\frac{1}{2}}$. In fact, E_k is 1-codimensional subspace of the Hilbert space

$$\widetilde{E}_k = \{u \in H^1_{loc}(\mathbb{R}) : u'(s+2k) = u'(s)\}$$

with

$$\int_{-k}^{k} u'(s)v'(s)ds + u(0)v(0)$$

as the scalar product.

On $\widetilde{E}_k$ we define operators $\widetilde{E}_k \to \widetilde{E}_k$:

$$(Au)(s) := u(s+\cos\varphi) - u(s) = \int_s^{s+\cos\varphi} u'(\tau)d\tau,$$

$$(Bu)(s) := u(s+\sin\varphi) - u(s) = \int_s^{s+\sin\varphi} u'(\tau)d\tau.$$

We introduce the functional

$$J_k(u) = \int\limits_{-k}^{k} \left[\frac{c^2}{2}(u'(s))^2 - \frac{c_1^2}{2}(Au(s))^2 - \frac{c_2^2}{2}(Bu(s))^2 - \right.$$

$$\left. - f_1(Au(s)) - f_2(Bu(s))\right] ds$$

defined on the space E_k. Any critical point of the functional J_k is a solution of Eq. (2.1) satisfying (2.2). Thus, to establish the existence of solutions to Eq. (2.1) satisfying (2.2), it is suffice to prove the existence of nontrivial critical points of the functional J_k. This requires a special form of the mountain pass theorem (see [24, 25]).

Let $I : H \to \mathbb{R}$ be a C^1-functional on a Hilbert space H with the norm $\|\cdot\|$. We say that I satisfies the *Palais-Smale condition*, if the following condition is satisfied:

(PS) *Let* $\{u_n\} \subset H$ *be a such sequence that* $\{I(u_n)\}$ *is bounded and* $I'(u_n) \to 0$, $n \to \infty$. *Then* $\{u_n\}$ *contains a convergent subsequence.*

If there exist $e \in H$ and $r > 0$ such that $\|e\| > r$ and

$$\beta := \inf_{\|u\|=r} I(u) > I(0) \geq I(e),$$

then we say that the functional I possesses the *mountain pass geometry.*

The following theorem of the mountain pass type can be found in [15] (Theorem 10).

Theorem 3.1. *Suppose that the* C^1*-functional* $I : H \to \mathbb{R}$ *satisfies the Palais–Smale condition and possesses the mountain pass geometry. Let* $P : H \to H$ *be a continuous mapping such that*

$$I(Pu) \leq I(u)$$

for all $u \in H$, $P(0) = 0$ *and* $P(e) = e$. *Then there exists a critical point* $u \in \overline{PH}$ *(the closure of* PH*) of the functional* I *with the critical value*

$$I(u) = b := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \geq \beta,$$

where $\Gamma := \{\gamma \in C([0,1], H) : \gamma(0) = 0, \gamma(1) = e\}$.

The following theorem is obtained in [5] (Theorem 3.1) with the aid of the mountain pass theorem.

Theorem 3.2. *Assume* (i)*–*(iii) *and either* (iv_2) *or* (v_2). *If* $\varphi \in [\pi n, \frac{\pi}{2} + \pi n]$, $n \in \mathbb{Z}$, $k > 0$ *and* $c_0^2 < c^2 < c_0^2 + l$, *then Eq.* (2.1) *has a non-constant non-decreasing and non-increasing solutions satisfying* (2.2).

Note that from a physical point of view, the increasing waves are *expansion waves*, and the decreasing waves are *compression waves.*

The following lemma gives a uniform upper bound for mountain pass value of the functional J_k (see [23], Lemma 3).

Lemma 3.1. *Let assumptions* (i)*–*(iii) *are satisfied and* $c_0^2 < c^2 < c_0^2 + l$. *Then there exists positive constatnt* K *such that for the mountain pass value* b_k *of* J_k *we have*

$$b_k \leq K \tag{3.1}$$

for all $k > 1$.

4. Solitary waves

In a sense, the case of solitary waves is a limit case of the periodic waves. Therefore, solitary waves will be constructed by considering critical points of the functional J_k and then passing to the limit as $k \to \infty$.

4.1. Variational setting

Let E be the Hilbert space defined by

$$E = \{u \in H^1_{loc}(\mathbb{R}) : u' \in Ł^2(\mathbb{R}), u(0) = 0\}$$

with the scalar product

$$(u, v) = \int_{-\infty}^{+\infty} u'(s)v'(s)ds$$

and corresponding norm $\|u\| = (u, u)^{\frac{1}{2}}$. Note that the condition $u' \in Ł^2(\mathbb{R})$ in the definition of E corresponds to the condition (2.3) and the condition $u(0) = 0$ is meaningful because every element of $H^1_{loc}(\mathbb{R})$ is a continuous function. By $\| \cdot \|_*$ we denote the dual norm on E^*, the dual space to E. Actually, E is 1-codimensional subspace of the Hilbert space

$$\widetilde{E} = \{u \in H^1_{loc}(\mathbb{R}) : u' \in Ł^2(\mathbb{R})\}$$

with

$$\int_{-\infty}^{+\infty} u'(s)v'(s)ds + u(0)v(0)$$

as the scalar product.

On $\widetilde{E}$ we define operators $\widetilde{E} \to \widetilde{E}$:

$$(Au)(s) := u(s + \cos\varphi) - u(s) = \int_s^{s+\cos\varphi} u'(\tau)d\tau,$$

$$(Bu)(s) := u(s + \sin\varphi) - u(s) = \int_s^{s+\sin\varphi} u'(\tau)d\tau.$$

We introduce the functional

$$J(u) := \int_{-\infty}^{+\infty} \{\frac{c^2}{2}|u'(s)|^2 - U(Au(s)) - U(Bu(s))\}ds$$

defined on the space E.

The proof of the following simple lemma can be found in [10] (Lemma 2 and Lemma 3).

Lemma 4.1. *Under assumption* (i) *the functional* J *is* C^1 *on* E *and*

$$\langle J'(u), h\rangle = \int\limits_{-\infty}^{+\infty} \left[c^2u'(s)h'(s) - c_1^2Au(s)Ah(s) - c_2^2Bu(s)Bh(s)\right.$$
$$\left. -f_1'(Au(s))Ah(s) - f_2'(Bu(s))Bh(s)\right] ds$$

for $u, h \in E$*. Moreover, any critical point of the functional* J *is a solution of Eq.* (2.1) *satisfying* (2.3).

4.2. Main result

The functional J satisfies a part of conditions of the mountain pass theorem. However, the Palais-Smale condition for this functional is not satisfied. Therefore, in this case, critical points of J will be constructed in a different way, namely, by passing to the limit as $k \to \infty$ in the critical points of J_k.

To get the main result we need the following concentration compactness lemma (see [24], Lemma 3.6).

Lemma 4.2. *Let $\{u_n\} \subset E_{k_n}$, $k_n \to \infty$, be a sequence such that $\|u_n\|_{k_n}$ is bounded. Then one of the two following possibilities holds:*

(*a*) *(non-vanishing) for any $\sigma > 0$ there exist $\eta > 0$, a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $\{\zeta_n\} \subset \mathbb{R}$ such that*

$$\int\limits_{\zeta_n-\sigma}^{\zeta_n+\sigma} \left(|Au_n(s)|^2 + |Bu_n(s)|^2\right) ds \geq \eta; \tag{4.1}$$

or

(*b*) *(vanishing) $\|Au_n\|_{L^p(-k_n,k_n)} + \|Bu_n\|_{L^p(-k_n,k_n)} \to 0$ for all $p > 2$.*

Suppose in addition that assumption (*i*) *is satisfied, $c > c_0$ and $\|J_{k_n}(u_n)\|_{k_n,*} \to 0$, then in case* (*b*) *we have $\|u_n\|_{k_n} \to 0$.*

The main result of the paper is the following theorem.

Theorem 4.1. *Assume (i)–(iii). If $\varphi \in [\pi n, \frac{\pi}{2} + \pi n]$, $n \in \mathbb{Z}$, and $c_0^2 < c^2 < c_0^2 + l$, then Eq. (2.1) has a non-constant non-decreasing and non-increasing solutions satisfying* (2.3).

Proof. First, we fix any sequence $\{k_n\}$, $k_n \to \infty$, and choose $\{c_n\}$, $c_n \to c$ such that Theorem 3.2 guaranties the existence of a non-decreasing solution $u_n \in E_{k_n}$ with the speed c_n ($c_n = c$ in case of (*iv*)). We denote by $\tilde{J}_{k_n}$ the functional J_{k_n} with c replaced by c_n.

Step 1. By contradiction we show that the sequence $\{\|u_n\|_{k_n}\}$ is bounded. Suppose the opposite. Then, passing to a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), we can assume that $\|u_n\|_{k_n} \to \infty$. By Lemma 4.2, for $v_n = \frac{u_n}{\|u_n\|_{k_n}}$ we have one of two possibilities: *non-vanishing* or *vanishing*.

Case (*a*). Suppose that *non-vanishing* holds. Due to the translation invariance of the problem, we can assume that in (4.1): $\zeta_n = 0$. Since $\|v_n\|_{k_n} = 1$, after passing to a subsequence, we can assume that $v_n \to v$

weakly in $H^1_{loc}(\mathbb{R})$ and uniformly on every finite interval. Moreover, $v \in E$ and $\|v\| \le 1$, and from (a) we obtain that v is non-constant.

Let $h \in C_0^\infty(\mathbb{R})$. Then for all n large enough: $2k_n$-periodization h_n of h is well-defined and belongs to E_{k_n}, and

$$0 = \frac{1}{\|u_n\|_{k_n}} \langle \tilde{J}'_k(u_n), h_n \rangle$$

$$= \int\limits_{-\infty}^{+\infty} \left[c_n^2 v'_n(s) h'(s) - (c_1^2 + l) A v_n(s) A h(s) - (c_2^2 + l) B v_n(s) B h(s)\right] ds$$

$$- \frac{1}{\|u_n\|_{k_n}} \int\limits_{-\infty}^{+\infty} [g_1(A u_n(s)) + g_2(B u_n(s))] \, ds.$$

Since the functions g_i are bounded, the second integral in the right hand part above tends to zero. Therefore, passing to the limit as $n \to \infty$, we obtain

$$\int\limits_{-\infty}^{+\infty} \left[c^2 v'(s) h'(s) - (c_1^2 + l) A v(s) A h(s) - (c_2^2 + l) B v(s) B h(s)\right] ds = 0.$$

This implies that

$$(Lv)(s) := -c^2 v''(s) + (c_1^2 + l)(v(s + \cos\varphi) + v(s - \cos\varphi) - 2v(s))$$
$$+ (c_2^2 + l)(v(s + \sin\varphi) + v(s - \sin\varphi) - 2v(s)) = 0.$$

The operator L is a pseudo differential operator with the symbol

$$\sigma(\xi) := c^2 \xi^2 - 4(c_1^2 + l) \sin^2\left(\frac{\xi}{2}\cos\varphi\right) - 4(c_2^2 + l) \sin^2\left(\frac{\xi}{2}\sin\varphi\right).$$

Obviously that $Lv' = 0$ and $v' \in L^2(\mathbb{R}) \setminus \{0\}$. On the other hand, passing to the Fourier transform, we obtain that $\sigma(\xi)\widehat{v'}(\xi) = 0$ and, hence, $v' = 0$. We got a contradiction that excludes *non-vanishing*.

Case (b). Now we suppose that *vanishing* holds. In this case we have that $\|Av_n\|_{L^p(-k_n,k_n)} + \|Bv_n\|_{L^p(-k_n,k_n)} \to 0$ $(n \to \infty)$ for all $p > 2$. We fix any such p. We have that

$$0 = \frac{1}{\|u_n\|^2_{k_n}} \langle \tilde{J}'_k(u_n), u_n \rangle$$

$$= \int\limits_{-k_n}^{k_n} \left[c_n^2 (v'_n(s))^2 - c_1^2 (A v_n(s))^2 - c_2^2 (B v_n(s))^2\right] ds -$$

$$-\int_{-k_n}^{k_n}\left[\frac{h_1(Au_n(s))}{Au_n(s)}(Av_n(s))^2+\frac{h_2(Bu_n(s))}{Bu_n(s)}(Bv_n(s))^2\right]ds.$$

Since $c_n \to c$, we have that $2\alpha_0 := \inf(c_n - c_0) > 0$ and, hence,

$$2\alpha_0 = 2\alpha_0\|v_n\|_{k_n}^2$$

$$\leq \int_{-k_n}^{k_n}\left[\frac{h_1(Au_n(s))}{Au_n(s)}(Av_n(s))^2+\frac{h_2(Bu_n(s))}{Bu_n(s)}(Bv_n(s))^2\right]ds. \tag{4.2}$$

By assumption (i), there is $r_0 > 0$ such that $\frac{h_i(r)}{r} \leq \alpha_0$ as $|r| \leq r_0$. Let $D_n = \{s \in [-k_n, k_n] : \max\{|Au_n(s)|, |Bu_n(s)|\} \leq r_0\}$ and $CD_n = [-k_n, k_n] \setminus D_n$. Then

$$\int_{D_n}\left[\frac{h_1(Au_n(s))}{Au_n(s)}(Av_n(s))^2+\frac{h_2(Bu_n(s))}{Bu_n(s)}(Bv_n(s))^2\right]ds$$

$$\leq \alpha_0\int_{D_n}\left[(Av_n(s))^2+(Bv_n(s))^2\right]ds$$

$$\leq \left[\|Av_n\|_{L^2(-k_n,k_n)}^2+\|Bv_n\|_{L^2(-k_n,k_n)}^2\right] \leq \alpha_0\|v_n\|_{k_n}^2 = \alpha_0.$$

Then, by Eq. (4.2), we have that

$$\liminf_{n\to\infty}\int_{CD_n}\left[\frac{h_1(Au_n(s))}{Au_n(s)}(Av_n(s))^2+\frac{h_2(Bu_n(s))}{Bu_n(s)}(Bv_n(s))^2\right]ds \geq \alpha_0. \tag{4.3}$$

On the other hand, by assumption (ii), there exists $\alpha_1 > 0$ such that $|h_i(r)| \leq \alpha_1|r|$ for all r. Thus, by the Hölder inequality, we have

$$\int_{CD_n}\left[\frac{h_1(Au_n(s))}{Au_n(s)}(Av_n(s))^2+\frac{h_2(Bu_n(s))}{Bu_n(s)}(Bv_n(s))^2\right]ds$$

$$\leq \alpha_1(\mathrm{meas}(CD_n))^{\frac{p-2}{p}}\left(|Av_n\|_{L^p(-k_n,k_n)}^{\frac{2}{p}}+|Bv_n\|_{L^p(-k_n,k_n)}^{\frac{2}{p}}\right), \tag{4.4}$$

where meas stands for the Lebesgue measure. From Eq. (4.3) and Eq. (4.2) we obtain that $\mathrm{meas}(CD_n) \to \infty$. Then, due to assumption (iii), we have

$$b_{k_n} = \tilde{J}_{k_n}(u_n) = \tilde{J}_{k_n}(u_n) - \frac{1}{2}\langle \tilde{J}_k'(u_n), u_n\rangle$$

$$= \int\limits_{-k_n}^{k_n} \left[\frac{1}{2}h_1(Au_n(s))Au_n(s) - f_1(Au_n(s))\right] ds$$

$$+ \int\limits_{-k_n}^{k_n} \left[\frac{1}{2}h_2(Bu_n(s))Bu_n(s) - f_2(Bu_n(s))\right] ds$$

$$= \int\limits_{CD_n} \left[\frac{1}{2}h_1(Au_n(s))Au_n(s) - f_1(Au_n(s))\right] ds$$

$$+ \int\limits_{CD_n} \left[\frac{1}{2}h_2(Bu_n(s))Bu_n(s) - f_2(Bu_n(s))\right] ds \geq 2\delta_0 \text{meas}(CD_n) \to \infty.$$

But, by Lemma 3.1, b_k is bounded from above. And we got a contradiction again. Hence, *vanishing* also impossible, which means that the sequence $\{\|u_n\|_{k_n}\}$ is bounded.

Step 2. The boundedness of $\{\|u_n\|_{k_n}\}$ implies that, passing to a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), there exist $\zeta_n \in \mathbb{R}$ and $u \neq 0$ such that $u_n(\cdot + \zeta_n) \to u$ weakly in $H^1_{loc}(\mathbb{R})$ and uniformly on every finite interval. Moreover, the boundedness of $\{\|u_n\|_{k_n}\}$ implies that $\|u\|$ is finite and, hence, $u \in E$. It is easy to see that u is a non-decreasing critical point of J and, hence, u is a solution of Eq. (2.1) that satisfy (2.3).

The proof in the case of non-increasing solutions is similar. The proof is complete. $\square$

Conclusion

Thus, in the present paper we obtain a result on the existence of non-constant monotone solitary traveling waves with vanishing relative displacement profiles in Fermi-Pasta-Ulam type systems with saturable nonlinearities on a two-dimensional lattice.

References

[1] Arioli, G., Gazzola, F. (1996). Periodic motion of an infinite lattice of particles with nearest neighbor interaction, *Nonlin. Anal., 26*(6), 1103–1114.

[2] Aubry, S. (1997). Breathers in nonlinear lattices: Existence, linear stability and quantization. *Physica D, 103*, 201–250.

[3] Bak, S.M. (2014). Existence of heteroclinic traveling waves in a system of oscillators on a two-dimensional lattice. *Mat. Metody ta Fizyko-Mekhanichni Polya, 57*(3), 45–52. Transl. in: (2016). *J. Math. Sci., 217*(2), 187–197.

[4] Bak, S.M. (2012). Existence of periodic traveling waves in Fermi-Pasta-Ulam system on 2D-lattice. *Mat. Stud., 37*(1), 76–88.

[5] Bak, S.M., Kovtonyuk, G.M. (2021). Existence of periodic traveling waves in Fermi–Pasta–Ulam type systems on 2D-lattice with saturable nonlinearities. *Ukr. Math. Bull., 18*(4), 466–478. Transl. in: (2022). *J. Math. Sci., 260* (5), 619–629.

[6] Bak, S.M. (2017). Existence of the solitary traveling waves for a system of nonlinearly coupled oscillators on the 2d-lattice. *Ukr. Mat. Zh., 69*(4), 435–444. Transl. in: (2017). *Ukr. Math. J., 69*(4), 509–520.

[7] Bak, S.M. (2019). Homoclinic traveling waves in discrete sine-Gordon equation with nonlinear interaction on 2D lattice. *Mat. Stud., 52*(2), 176–184.

[8] Bak, S. (2022). Periodic traveling waves in the system of linearly coupled nonlinear oscillators on 2D lattice. *Archivum Mathematicum, 58*(1), 1–13.

[9] Bak, S. (2022). Periodic traveling waves in a system of nonlinearly coupled nonlinear oscillators on a two-dimensional lattice. *Acta Mathematica Universitatis Comenianae, 91*(3), 1–10.

[10] Bak, S.M., Kovtonyuk, G.M. (2018). Existence of solitary traveling waves in Fermi–Pasta–Ulam system on 2D-lattice. *Mat. Stud., 50*(1), 75–87.

[11] Bak, S., Kovtonyuk, G. (2019). Existence of standing waves in DNLS with saturable nonlinearity on 2D-lattice. *Communications in Mathematical Analysis, 22*(2), 18–34.

[12] Bak, S.M., Kovtonyuk, G.M. (2020). Existence of traveling waves in Fermi–Pasta–Ulam type systems on 2D-lattice. *Ukr. Math. Bull., 17*(3), 301–312. Transl. in: (2021). *J. Math. Sci., 252*(4), 453–462.

[13] Bak, S. (2018). The existence of heteroclinic traveling waves in the discrete sine-Gordon equation with nonlinear interaction on a 2D-lattice. *J. Math. Phys., Anal., Geom., 14*(1), 16–26.

[14] Bak, S.N., Pankov, A.A. (2010). Traveling waves in systems of oscillators on 2D-lattices. *Ukr. Math. Bull., 7*(2), 154–175. Transl. in: (2011). *J. Math. Sci., 174*(4), 916–920.

[15] Berestycki, H., Capuzzo-Dolcetta, I., Nirenberg L. (1995). Variational methods for indefinite superlinear homogeneous elliptic problems. *Nonlin. Diff. Eq. and Appl., 2*, 553–572.

[16] Braun, O.M., Kivshar, Y.S. (1998). Nonlinear dynamics of the Frenkel–Kontorova model. *Physics Repts, 306*, 1–108.

[17] Braun, O.M., Kivshar, Y.S. (2004). *The Frenkel-Kontorova Model, Concepts, Methods and Applications.* Springer, Berlin.

[18] Butt, I.A., Wattis, J.A.D. (2006). Discrete breathers in a two-dimensional Fermi–Pasta–Ulam lattice. *J. Phys. A. Math. Gen., 39*, 4955–4984.

[19] Fečkan, M., Rothos, V. (2007). Traveling waves in Hamiltonian systems on 2D lattices with nearest neighbour interactions. *Nonlinearity, 20*, 319–341.

[20] Friesecke, G., Matthies, K. (2003). Geometric solitary waves in a 2D math-spring lattice. *Discrete and continuous dynamical systems, 3*(1), 105–114.

[21] Friesecke, G., Wattis, J.A.D. (1994). Existence theorem for solitary waves on lattices. *Commun. Math. Phys., 161*, 391–418.

[22] Henning, D., Tsironis, G. (1999). Wave transmission in nonliniear lattices. *Physics Repts., 309*, 333–432.

[23] Pankov, A., Rothos, V. (2011). Traveling waves in Fermi–Pasta–Ulam lattices with saturable nonlinearities. *Discr. Cont. Dyn. Syst., 30*(3), 835–840.

[24] Pankov, A. (2005). *Traveling Waves and Periodic Oscillations in Fermi-Pasta-Ulam Lattices*. Imperial College Press, London–Singapore.

[25] Rabinowitz, P. (1986). *Minimax methods in critical point theory with applications to differential equations*. American Math. Soc., Providence, R. I.

[26] Srikanth, P. (1998). On periodic motions of two-dimentional lattices. *Functional analysis with current applications in science, technology and industry, 377*, 118–122.

[27] Willem, M. (1996). *Minimax theorems*. Birkhäuser, Boston.

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