Український математичний вісник
Том 19 **(2022)**, № 4, 478 – 488
https://doi.org/10.37069/1810-3200-2022-19-4-3

On the boundary behavior of weak (p, q)-quasiconformal mappings

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Abstract. Let Ω and $\widetilde{\Omega}$ be domains in the Euclidean space $\mathbb{R}^n$. We study the boundary behavior of weak (p, q)-quasiconformal mappings $\varphi : \Omega \to \widetilde{\Omega}$, $n - 1 < q \leqslant p < n$. The suggested method is based on the capacitary distortion properties of the weak (p, q)-quasiconformal mappings.

2010 MSC. 30C65, 46E35.

Key words and phrases. Quasiconformal mappings, Sobolev spaces.

1. Introduction

Let Ω and $\widetilde{\Omega}$ be domains in the Euclidean space $\mathbb{R}^n$, $n \geqslant 2$. We consider the boundary behavior of weak (p, q)-quasiconformal mappings $\varphi : \Omega \to \widetilde{\Omega}$, $n - 1 < q \leqslant p < n$. The weak (p, q)-quasiconformal mappings represent generalizations of (quasi)conformal mappings and have significant applications in the geometric analysis of PDE (see, for example, [6, 9–11]). In the frameworks of the boundary value problems for elliptic equations becomes important the boundary behavior of this class of mappings. From the historic point of view this arises due to the boundary behavior of conformal mappings (univalent analytic functions) [3] and due to the boundary behavior of quasiconformal mappings [2]. In series of works [12,19,20,26,28,30,31,41] the boundary behavior of space quasiconformal mappings and their generalizations in the terms of capacitory (moduli) inequalities was studied.

The theory of weak (p, q)-quasiconformal mappings arose in the Sobolev embedding theory [8,22] and was founded in the series of works [5,32, 38–40]. Recent applications of the weak (p, q)-quasiconformal mappings

Received 21.11.2022

theory to Sobolev extension operators can be found in [17,18]. Recall that a mapping $\varphi : \Omega \to \widetilde{\Omega}$ is called a weak (p,q)-quasiconformal mapping if $\varphi \in W^1_{1,\mathrm{loc}}(\Omega)$, has finite distortion and

$$K_{p,q}^{\frac{pq}{p-q}}(\varphi;\Omega) = \int\limits_{\Omega} \left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|}\right)^{\frac{q}{p-q}}\, dx < \infty,$$

for $1 < q < p < \infty$ [32,38] and

$$K^p_{p,p}(\varphi;\Omega) = \operatorname*{ess\,sup}_{\Omega} \frac{|D\varphi(x)|^p}{|J(x,\varphi)|} < \infty,$$

for $1 < q = p < \infty$ [5,35]. In the case $p = q = n$ we have quasiconformal mappings [34] and in the case $1 < q < p = n$ we have mappings of integrable distortion [15].

The main result of the article states: *Let $\varphi : \Omega \to \widetilde{\Omega}$, $\varphi(\Omega) = \widetilde{\Omega}$, be a weak (p,q)-quasiconformal mapping, $n-1 < q \leqslant p < n$, where Ω has a strongly accessible boundary with respect to q-capacity and $\widetilde{\Omega}$ has locally connected boundary. Then the inverse mapping φ^{-1} can be extended to a continuous mapping*

$$\overline{\varphi^{-1}} : \overline{\widetilde{\Omega}} \to \overline{\Omega}.$$

The definition of a strongly accessible boundary in the terms of the q-capacity will be given in Section 3.

The suggested method is based on the capacitary distortion properties of the weak (p,q)-quasiconformal mappings. Let us note that the weak (p,q)-quasiconformal mappings are closely connected with mappings defined by weighted capacitary (moduli) inequalities [7,25].

2. Weak quasiconformal mappings

Let Ω be a domain in the Euclidean space $\mathbb{R}^n$, $n \geqslant 2$. The Sobolev space $W^1_p(\Omega)$, $1 \leqslant p \leqslant \infty$, is defined as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W^1_p(\Omega)\| = \|f \mid L_p(\Omega)\| + \|\nabla f \mid L_p(\Omega)\|,$$

where ∇f is the weak gradient of the function f.

The homogeneous seminormed Sobolev space $L^1_p(\Omega)$, $1 \leqslant p \leqslant \infty$, is defined as a space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$\|f \mid L^1_p(\Omega)\| = \|\nabla f \mid L_p(\Omega)\|.$$

In accordance with the non-linear potential theory [24] we consider elements of Sobolev spaces $W^1_p(\Omega)$ as equivalence classes up to a set of p-capacity zero [23].

Let Ω and $\widetilde{\Omega}$ be domains in $\mathbb{R}^n$, $n \geqslant 2$. We say that a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ induces a bounded composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega),\ \ 1 \leqslant q \leqslant p \leqslant \infty,$$

by the composition rule $\varphi^*(f) = f \circ \varphi$, if for any function $f \in L^1_p(\widetilde{\Omega})$, the composition $\varphi^*(f) \in L^1_q(\Omega)$ is defined quasi-everywhere in Ω [23] and there exists a constant $K_{p,q}(\Omega) < \infty$ such that

$$\|\varphi^*(f) \mid L^1_q(\Omega)\| \leqslant K_{p,q}(\Omega)\|f \mid L^1_p(\widetilde{\Omega})\|.$$

In the geometric function theory composition operators on Sobolev spaces arise in the work [36] and have numerous applications in the geometric analysis of PDE. The characterization of composition operators on Sobolev spaces is given in the following theorem [32, 38] ([35] for the case $p = q > n$ and [5] for the case $n - 1 < q = p < n$).

Theorem 2.1. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a homeomorphism between two domains Ω and $\widetilde{\Omega}$. The homeomorphism φ induces a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega),\ \ 1 \leqslant q < p < \infty,$$

if and only if φ is the weak (p, q)-quasiconformal mapping.

Recall the notion of a variational p-capacity [37]. The condenser in a domain $\Omega \subset \mathbb{R}^n$ is a pair (E, F) of connected disjoint closed relatively to Ω sets $E, F \subset \Omega$. A continuous function $u \in L^1_p(\Omega)$ is called an admissible function for the condenser (E, F), if the set $E \cap \Omega$ is contained in some connected component of the set $\operatorname{Int}\{x|u(x) = 0\}$ and the set $F \cap \Omega$ is contained in some connected component of the set $\operatorname{Int}\{x|u(x) = 1\}$. We call as the p-capacity of the condenser (E, F) relatively to domain Ω the following quantity:

$$\operatorname{cap}_p(E, F; \Omega) = \inf \|u|L^1_p(\Omega)\|^p.$$

Here the greatest lower bond is taken over all functions admissible for the condenser $(E, F) \subset \Omega$. If the condenser has no admissible functions we put the capacity is equal to infinity.

The following capacitory properties of weak (p, q)-quasiconformal mappings were established in [32, 38] (for the case $p = \infty$ see in [33]).

Theorem 2.2. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a weak (p,q)-quasiconformal mapping, $1 \leqslant q \leqslant p \leqslant \infty$. Then for every condenser $(E,F) \subset \widetilde{\Omega}$ the inequality*

$$\operatorname{cap}_q^{1/q}(\varphi^{-1}(E), \varphi^{-1}(F); \Omega) \leqslant K_{p,q}(\varphi; \Omega) \operatorname{cap}_p^{1/p}(E, F; \widetilde{\Omega})$$

holds.

By using these capacitory distortion properties we consider boundary behavior of weak (p,q)-quasiconformal mappings.

3. On boundaries correspondence

Recall that the boundary $\partial\Omega$ of a domain $\Omega \subset \mathbb{R}^n$ is called *strongly accessible at a point $x_0 \in \partial\Omega$ with respect to the p-capacity* if for each neighborhood U such that $\partial U \cap \Omega \neq \emptyset$ of x_0 there exist a compact set $E \subset \Omega$, a neighborhood $V \subset U$ of x_0 and $\delta > 0$ such that

$$\operatorname{cap}_p(E, F; \Omega) \geqslant \delta \tag{3.1}$$

for each continuum F in Ω that intersects ∂U and ∂V. The boundary $\partial\Omega$ of a domain Ω is called *strongly accessible at a point $x_0 \in \partial\Omega$*, if it is strongly accessible at a point $x_0 \in \partial\Omega$ with respect to the n-capacity (n-modulus).

Remark 3.1. *The notion of a strongly accessible boundary was introduced in [21, section 3.8] and it is very close to the notion of a uniform domain which was introduced by Näkki [27] Theorem 6.2.*

This notion also coincides, up to some (not too essential) details, with the concept of a quasiconformally accessible boundary [26, section 1.7]. Note that both of these concepts were formulated in terms of the modulus of families of paths. In this connection, we recall the concept of a modulus of a family of locally rectifiable curves (paths) Γ.

Let $\rho : \mathbb{R}^n \to [0, \infty]$ be a Borel function. Then ρ is called *admissible* for Γ (i. e. $\rho \in \operatorname{adm}\Gamma$), if the inequality $\int\limits_\gamma \rho(x)\, ds \geqslant 1$ holds for any locally rectifiable curve $\gamma \in \Gamma$. Let $p \geqslant 1$, then the quantity

$$M_p(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)\, dx \tag{3.2}$$

is called the p-modulus of the family of curves of Γ.

For a given domain Ω in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, and sets E and F in Ω we denote by the symbol $\Gamma(E, F, \Omega)$ the family of all locally rectifiable curves $\gamma : [0,1] \to \overline{\mathbb{R}^n}$ joining the sets E and F in Ω, that is: $\gamma(0) \in E$, $\gamma(1) \in F$ and $\gamma(t) \in \Omega$ for all $t \in (0,1)$.

We say that the boundary $\partial\Omega$ of a domain Ω is *strongly accessible at a point $x_0 \in \partial\Omega$ with respect to the p-modulus* if for each neighborhood U of x_0 there exist a compact set $E \subset \Omega$, a neighborhood $V \subset U$ of x_0 and $\delta > 0$ such that

$$M_p(\Gamma(E, F, \Omega) \geqslant \delta \tag{3.3}$$

for each continuum F in Ω that intersects ∂U and ∂V.

Note that (3.1) is equivalent to (3.3). Indeed,

$$M_p(\Gamma(E, F, \Omega) = \mathrm{cap}_p(E, F; \Omega)$$

by Hesse and Shlyk equalities (see [16, Theorem 5.5] and [29, Theorem 1]).

Based on the definition of domains with strongly accessible boundaries and taking Remark 3.1 into account, we give some examples of such domains.

1. By Theorem 6.2 and Corollary 6.8 in [27], the planar domain with finitely many boundary components has a strongly accessible boundary whenever it is finitely connected on the boundary.

2. Following [27], a domain Ω is said to be *quasiconformally collared* on the boundary if each point of $\partial\Omega$ has arbitrarily small neighborhoods U such that $U \cap \Omega$ can be mapped quasiconformally onto a ball $B \subset \mathbb{R}^n$. Let Ω be a domain which can be mapped quasiconformally onto some quasiconformally collared domain. If Ω is finitely connected on the boundary, then the boundary of Ω is strongly accessible (see Corollary 6.7 in [27]).

3. The next example gives domains with a strongly accessible boundary for $p \neq n$. Recall the notion of the upper gradient [13, 14]. Let (X, μ) be a metric measure space. A Borel function $g \colon X \to [0, \infty]$ is said to be an *upper gradient* of a function $u : X \to \mathbb{R}$ if

$$|u(x) - u(y)| \leqslant \int\limits_{\gamma} g \, ds$$

for any rectifiable curve γ joining x and y in X. Let $1 \leqslant p < \infty$. We say that X, μ admits *an* $(1;p)$*-Poincare inequality* if there exists a constant $1 \leqslant C_p < \infty$ such that

$$\frac{1}{\mu(B)} \int\limits_{B} |u - u_B| d\mu(x) \leqslant C_p \cdot (\operatorname{diam} B) \left(\frac{1}{\mu(B)} \int\limits_{B} g^p d\mu(x) \right)^{1/p}$$

for all balls B in X and for all bounded continuous functions u on B, where g is the upper gradient of u. Metric measure spaces where exists a number n such that the inequalities

$$\frac{1}{K}R^n \leqslant \mu(B(x_0,R)) \leqslant KR^n$$

hold with some constant $1 \leqslant K < \infty$, for every $x_0 \in X$ and all $R < \operatorname{diam} X$ are called *Ahlfors n-regular.* Let $\Omega \subset B(0,R)$ for some $R > 0$ be an Ahlfors n-regular domain satisfying the $(1;p)$-Poincare inequality for $n-1 < p \leqslant n$. Assume that E and F are some continua $E, F \subset \Omega$. By [1, Proposition 4.7] and due to Remark 3.1,

$$\operatorname{cap}_p(E,F;\Omega) \geqslant \frac{1}{C}\frac{\min\{\operatorname{diam} E, \operatorname{diam} F\}}{R^{1+p-n}}\,, \tag{3.4}$$

where $C > 0$ is some constant.

Let us prove that Ω has a strongly accessible boundary at any point $x_0 \in \partial\Omega$ with respect to p-capacity. Suppose that $x_0 \in \partial\Omega$ and that U is an arbitrary neighborhood of x_0. Choose a small enough $\varepsilon_1 > 0$ such that for $V := B(x_0,\varepsilon_1)$, we have $\overline{V} \subset \overline{U}$. Because $\partial U \cap D \neq \varnothing$ we can set $\varepsilon_2 := dist(\partial U, \partial V) > 0$. Note that for arbitrary continua F_1 and F_2 in Ω satisfying $F_1 \cap \partial U \neq \varnothing \neq F_1 \cap \partial V$ and $F_2 \cap \partial U \neq \varnothing \neq F_2 \cap \partial V$ we have $\operatorname{diam}(F_1) \geqslant \varepsilon_2$ and $\operatorname{diam}(F_2) \geqslant \varepsilon_2$. Therefore, by (3.4), we obtain $\operatorname{cap}_p(\Gamma(F_1,F_2;G_0)) \geqslant \varepsilon_2$, as required.

Given $x_0 \in \mathbb{R}^n$, we set

$$B(x_0,r) = \{x \in \mathbb{R}^n : |x-x_0| < r\}\,,$$

$$S(x_0,r) = \{x \in \mathbb{R}^n : |x-x_0| = r\}\,, \tag{3.5}$$

$$A = A(x_0,r_1,r_2) = \{x \in \mathbb{R}^n : r_1 < |x-x_0| < r_2\}\,. \tag{3.6}$$

The following statement holds.

Theorem 3.1. *Let $\varphi : \Omega \to \widetilde{\Omega}$, $\varphi(\Omega) = \widetilde{\Omega}$, be a weak (p,q)-quasiconformal mapping, $n-1 < q \leqslant p < n$, Ω has a strongly accessible boundary with a respect to q-capacity and $\widetilde{\Omega}$ has locally connected boundary. Then the inverse mapping φ^{-1} can be extended by continuity to the continuous mapping*

$$\overline{\varphi^{-1}} : \overline{\widetilde{\Omega}} \to \overline{\Omega}.$$

Proof. Suppose the contrary, namely, there exists such point $b \in \partial\widetilde{\Omega}$, that φ^{-1} has no the continuous extension to the point b. It means that there exist two sequences x_i, $x_i' \in \widetilde{\Omega}$, $i = 1, 2, \ldots$, such that $x_i \to b$, $x_i' \to b$ as $i \to \infty$, and $\varphi^{-1}(x_i) \to y$, $\varphi^{-1}(x_i') \to y'$ as $i \to \infty$, while $y' \neq y$. Note that y and $y' \in \partial\Omega$, because $C(\varphi^{-1}, \partial\widetilde{\Omega}) \subset \partial\Omega$ for any homeomorphism φ^{-1}, see [21, Proposition 13.5].

Here

$$C(\varphi^{-1}, \partial\widetilde{\Omega}) = \bigcup_{x_0 \in \partial\widetilde{\Omega}} C(\varphi^{-1}, x_0)\,,$$

where

$$C(\varphi^{-1}, x_0) = \{y \in \overline{\mathbb{R}^n} :\ \exists\, x_k \in \widetilde{\Omega}, x_k \to x_0 : \varphi^{-1}(x_k) \to y, k \to \infty\}.$$

By the definition of a strongly accessible boundary at the point $y \in \partial\Omega$ with respect to the q-capacity, for any neighborhood U of this point there exists a compact set $C_0' \subset \Omega$, a neighborhood V of a point y, $V \subset U$, and a number $\delta > 0$ such that

$$\operatorname{cap}_q(C_0', F; \Omega) \geqslant \delta > 0 \tag{3.7}$$

for any continua F, intersecting ∂U and ∂V. Since $C(\varphi^{-1}, \partial\widetilde{\Omega}) \subset \partial\Omega$, we obtain that the condition $C_0 \cap \partial\widetilde{\Omega} = \varnothing$ holds for $C_0 := \varphi(C_0')$. Now suppose $\varepsilon_0 > 0$ is such that $C_0 \cap \overline{B(b, \varepsilon_0)} = \varnothing$.

Since $\widetilde{\Omega}$ is locally connected at b, we can join the points x_i and x_i' by a path γ_i lying in $V \cap \widetilde{\Omega}$. We may consider that $\gamma_i \in \overline{B(b, 2^{-i})} \cap \widetilde{\Omega}$. Since $\varphi^{-1}(x_i) \in V$ and $\varphi^{-1}(x_i') \in \widetilde{\Omega} \setminus \overline{U}$ for sufficiently large $i \in \mathbb{N}$ and due to (5.8), we may find $i_0 \in \mathbb{N}$ such that

$$\operatorname{cap}_q(C_0', \varphi^{-1}(\gamma_i); \Omega)) \geqslant \delta > 0 \tag{3.8}$$

for any $i \geqslant i_0 \in \mathbb{N}$. Immerse the compact C_0 into some the continuum C_1, still completely lying in the domain $\widetilde{\Omega}$, see [30, Lemma 1]. By reducing $\varepsilon_0 > 0$, we may again assume that $C_1 \cap \overline{B(b, \varepsilon_0)} = \varnothing$. By Theorem 2.2

$$\begin{aligned} &\operatorname{cap}_q^{1/q} C_0', \varphi^{-1}(\gamma_i); \Omega) \leqslant \\ &\qquad \operatorname{cap}_q^{1/q}(\varphi^{-1}(\gamma_i), \varphi^{-1}(C_1); \Omega) \leqslant K_{p,q}(\varphi; \Omega) \operatorname{cap}_p^{1/p}(\gamma_i, C_1; \widetilde{\Omega})\,. \end{aligned} \tag{3.9}$$

Let us prove that $\operatorname{cap}_p(\gamma_i, C_1; \widetilde{\Omega}) \to 0$ as $i \to \infty$. Indeed, by the definition of the capacity

$$\operatorname{cap}_p(\gamma_i, C_1; \widetilde{\Omega}) \leqslant \operatorname{cap}_p(S(b, 2^{-i}), S(b, \varepsilon_0); A(b, 2^{-i}, \varepsilon_0))\,. \tag{3.10}$$

However, by [4, relation (2)] and due to the Remark 3.1

$$\mathrm{cap}_n(S(b,2^{-i}),S(b,\varepsilon_0);A(b,2^{-i},\varepsilon_0))=\frac{\omega_{n-1}}{\log^{n-1}\frac{\varepsilon_0}{2^{-i}}}\to 0\,,\qquad i\to\infty\,,$$

and

$$\mathrm{cap}_p(S(b,2^{-i}),S(b,\varepsilon_0);A(b,2^{-i},\varepsilon_0))$$
$$=\left(\frac{n-p}{p-1}\right)^{p-1}\frac{\omega_{n-1}}{\left((2^{-i})^{\frac{p-n}{p-1}}-\varepsilon_0^{\frac{p-n}{p-1}}\right)^{p-1}}\to 0\,,\quad i\to\infty\quad p\ne n\,.$$

Now, $\mathrm{cap}_p(\gamma_i,C_1;\widetilde{\Omega})\to 0$ as $i\to\infty$, as required. In this case, the relation (3.9) contradicts with (3.8). This contradiction proves the theorem. □

In the case $n<q\leqslant p<\infty$ we use the following composition duality theorem [32, 38]:

Theorem 3.2. *Let $\varphi:\Omega\to\widetilde{\Omega}$ be a weak (p,q)-quasiconformal mapping, $n-1<q\leqslant p<\infty$. Then the inverse mapping $\varphi^{-1}:\widetilde{\Omega}\to\Omega$ generates a bounded composition operator*

$$\left(\varphi^{-1}\right)^{*}:L^1_{q'}(\Omega)\to L^1_{p'}(\widetilde{\Omega}),$$

where $p'=p/(p-n+1)$, $q'=q/(q-n+1)$.

By using this composition duality theorem we obtain the capacitory distortion estimate:

Theorem 3.3. *Let $\varphi:\Omega\to\widetilde{\Omega}$ be a weak (p,q)-quasiconformal mapping, $n<q\leqslant p<(n-1)^2/(n-2)$. Then for every condenser $(F_0,F_1)\subset\Omega$ the inequality*

$$\mathrm{cap}_{p'}^{1/p'}(\varphi(F_0),\varphi(F_1);\widetilde{\Omega})\leqslant K_{q',p'}(\varphi^{-1};\widetilde{\Omega})\,\mathrm{cap}_{q'}^{1/q'}(F_0,F_1;\Omega),$$

$$n-1<p'\leqslant q'<n,$$

holds, where $p'=p/(p-(n-1))$ and $q'=q/(q-(n-1))$.

The condition $p<(n-1)^2/(n-2)$ provides that $p'>n-1$. Hence, by using Theorem 3.1 and Theorem 3.3 we obtain:

Theorem 3.4. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a weak (p,q)-quasiconformal mapping, $n < q \leqslant p < (n-1)^2/(n-2)$. Suppose that Ω has a locally connected boundary and $\widetilde{\Omega}$ has strongly accessible boundary with a respect to p'-capacity, $p' = p/(p-n+1)$. Then the mapping φ can be extended by continuity to the continuous mapping*

$$\overline{\varphi} : \overline{\Omega} \to \overline{\widetilde{\Omega}}.$$

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