Український математичний вісник
Том 19 **(2022)**, № 4, 489 – 516
https://doi.org/10.37069/1810-3200-2022-19-4-4

On the Hilbert problem for semi-linear Beltrami equations

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Abstract. The present paper is devoted to the study of the well-known Hilbert boundary-value problem for semi-linear Beltrami equations with arbitrary boundary data that are measurable with respect to logarithmic capacity.

Namely, we prove here the corresponding results on existence, regularity and representation of its nonclassical solutions with geometric interpretation of boundary values as angular (along nontangential paths) limits in comparison with the classical approach in PDE.

For this purpose, we apply completely continuous operators by Ahlfors–Bers first of all to obtain solutions of semi-linear Beltrami equations, generally speaking with no boundary conditions, and then to derive their representation through solutions of the Vekua type equations and the so-called generalized analytic functions with sources.

Moreover, we obtain similar results on nonclassical solutions of the Poincare boundary-value problem on the directional derivatives and, in particular, of the Neumann problem with arbitrary measurable data to semi-linear equations of the Poisson type.

As consequences, it is given a series of applications of these results to some problems of mathematical physics describing such phenomena as diffusion with physical and chemical absorption, plasma states and stationary burning in anisotropic and inhomogeneous media.

2020 MSC. Primary 30C62, 35Q15, 35J61; Secondary 30E25, 31A15, 35Q35.

Key words and phrases. Dirichlet, Hilbert, Neumann and Poincare boundary-value problems, generalized analytic and generalized harmonic functions with sources, semi-linear Beltrami equations, semi-linear equations of the Poisson type.

1. Introduction

Recall that boundary-value problems for analytical functions and generalizations originated from the famous Riemann dissertation (1851)

Received 05.09.2022

and were further investigated in works of Hilbert (1904, 1912, 1924) and Poincare (1910).

The theory of boundary-value problems of mathematical physics is actively developed in many modern leading world schools, and the Hilbert boundary-value problem belongs to the most important of them because of its numerous applications to the hydromechanics, elasticity etc.

The classic formulation of the **Hilbert boundary value problem**, see [30], was as follows: To find an analytic function $f(z)$ in a domain D of the complex plane $\mathbb{C}$ bounded by a rectifiable Jordan contour C that satisfies the boundary condition

$$\lim_{z\to\zeta, z\in D} \mathrm{Re}\,\{\overline{\lambda(\zeta)}\; f(z)\} \;=\; \varphi(\zeta) \qquad\qquad \forall\, \zeta\in C\ , \tag{1.1}$$

where the **coefficient** λ and the **boundary date** φ of the problem were assumed continuously differentiable with respect to the natural parameter s and $\lambda\neq 0$ everywhere on C. The latter allows to consider that $|\lambda|\equiv 1$ on C. Note that the quantity $\mathrm{Re}\,\{\overline{\lambda}\, f\}$ in (1.1) means a projection of f into the direction λ interpreted as vectors in $\mathbb{R}^2$.

The reader can find a comprehensive treatment of its theory in excellent books [7,8,31,58]. We also recommend to make familiar with historic surveys in monographs [16, 45, 59] on the topic with an exhaustive bibliography. Recall here only that the first approach to its solution was proposed by Hilbert himself and it was based on the theory of singular integral equations. This approach made possible to prove the existence of its solutions for Hölder continuous data.

The research of boundary-value problems with arbitrary measurable data is due to the cornerstone dissertation of Luzin where he has studied the corresponding Dirichlet problem for harmonic functions in the unit disk $\mathbb{D}$.

In this connection, recall his lemma on antiderivatives that was one of the main results of his dissertation, see e.g. his paper [39], dissertation [40], p. 35, and its reprint [41], p. 78: *For any measurable function* $\varphi:[0,1]\to\mathbb{R}$*, there exists a continuous function* $\Phi:[0,1]\to\mathbb{R}$ *such that* $\Phi'=\varphi$ *a.e. on* $[0,1]$.

On this basis, Luzin just proved, see e.g. [41], p. 87: *For any measurable function* $\varphi:\partial\mathbb{D}\to\mathbb{R}$*, there exists a harmonic function* $u:\mathbb{D}\to\mathbb{R}$ *such that* $u(z)\to\varphi(\zeta)$ *for a.e.* $\zeta\in\partial\mathbb{D}$ *as* $z\to\zeta$ *along any nontangential path.*

Note that the Luzin dissertation was later on published only in Russian language as book [41] with comments of his students Bari and Men'shov already after his death. A part of its results was also printed in Italian [42]. However, his lemma was published in English in book [56] as

Theorem VII(2.3). Hence Frederick Gehring in [17] has rediscovered the Luzin theorem, and his proof on the basis of the lemma in fact coincided with the original proof of Luzin.

The Luzin theorem was strengthened by the statement that *the space of the Luzin harmonic functions has infinite dimension for each measurable function* $\varphi : \partial\mathbb{D} \to \mathbb{R}$, see e.g. Corollary 5.1 in [50] and also Corollary 3.1 [51]. The latter was key to establish that *the space of analytic functions* $f : D \to \mathbb{C}$ *with the angular (along nontangential paths) limits (1.1) for a.e.* $\zeta \in \partial D$ *has infinite dimension for any measurable functions* $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, *and* $\varphi : \partial D \to \mathbb{R}$, see Theorems 3.1 and Remark 5.2 in [50].

In turn, on the base of the latter result, the corresponding theorems on the existence of nonclassical solutions to the Poincare boundary-value problem on the directional derivatives and, in particular, to the Neumann problem for harmonic functions were derived in [52], Theorems 3, 4 and 5. Moreover, similar results were obtained for the so–called generalized analytic functions and generalized harmonic functions with sources, see [54]. In a series of further works, see e.g. [20]- [29], these results were extended to the Beltrami equations and analogs of the Poisson equation in anisotropic and inhomogeneous media with a replacement of measure of length by logarithmic capacity in the case of domains with nonrectifiable boundaries.

In this connection, let us recall that the known monograph [59] was devoted to the **generalized analytic functions**, i.e., continuous complex valued functions $h(z)$ of one complex variable $z = x + iy$ of class $W^{1,1}_{\rm loc}$ satisfying equations

$$\partial_{\bar z} h \;+\; ah \;+\; bh \;=\; c\;, \qquad \partial_{\bar z} \;:=\; \frac{1}{2}\left(\frac{\partial}{\partial x} \;+\; i\cdot\frac{\partial}{\partial y}\right)\;, \tag{1.2}$$

where it was assumed that the complex valued functions a, b and c belong to class $L^p(D)$ with some $p > 2$ in the corresponding domain $D \subseteq \mathbb{C}$.

In particular, paper [54] contained Theorem 1 on the existence of nonclassical solutions of the Hilbert boundary-value problem with arbitrary boundary data that were measurable with respect to the length measure for **generalized analytic functions with sources g**, when $a \equiv 0 \equiv b$,

$$\partial_{\bar z} h(z) \;=\; g(z)\;, \tag{1.3}$$

where g is of class $L^p(D)$, $p > 2$, in domains with rectifiable boundaries. A similar result, Theorem 1 in [26], was also proved on the Hilbert boundary-value problem with arbitrary boundary data that were measurable with respect to the logarithmic capacity in domains with nonrectifiable boundaries.

Moreover, paper [54] included Theorem 6 (Corollary 6) on the existence of continuous solutions of class $W^{2,p}_{\rm loc}$ to the Poincare (Neumann) boundary-value problem with arbitrary boundary data that were measurable with respect to the length measure in domains with rectifiable boundaries for **generalized harmonic functions with sources G** in $L^p(D)$, $p>2$, satisfying the Poisson equations

$$\triangle U(z) \ = \ G(z) \ . \tag{1.4}$$

Note that by the Sobolev embedding theorem, see Theorem I.10.2 in [57], such functions U belong to the class C^1. And again, similar results, Theorem 5 (Corollary 4) in [26], were also proved on the Poincare (Neumann) boundary-value problem with arbitrary boundary data that were measurable with respect to the logarithmic capacity in domains with nonrectifiable boundaries.

These results have been extended then to the corresponding Hilbert problem with arbitrary measurable boundary data for the semi-linear Vekua type equations of the form

$$\partial_{\bar z} h(z) \ = \ g(z)\cdot q(h(z)) \qquad \text{a.e. in } D\ , \tag{1.5}$$

with $g\in L^p(D)$, $p>2$, and continuous $q:\mathbb{C}\to\mathbb{C}$, see Theorem 2 in [27] and Theorem 2 in [55], and to the corresponding Poincare (Neumann) boundary-value problem with arbitrary measurable boundary data for the nonlinear Poisson equations of the form

$$\triangle U(z) \ = \ G(z)\cdot Q(U(z)) \qquad \text{a.e. in } D \tag{1.6}$$

with $G\in L^p(D)$, $p>2$, and continuous $Q:\mathbb{R}\to\mathbb{R}$, see Theorem 4 (Corollary 5) in [27] and Theorem 4 (Corollary 4) in [55] for the cases of nonrectifiable and rectifiable boundaries, correspondingly.

Recall also that the **Beltrami equation** is the equation of the form

$$f_{\bar z} = \mu(z) f_z \tag{1.7}$$

a.e. in D, where $\mu : D\to\mathbb{C}$ is a Lebesgue measurable function with $|\mu(z)|<1$ a.e., $f_{\bar z}=(f_x+if_y)/2$, $f_z=(f_x-if_y)/2$, $z=x+iy$, f_x and f_y are partial derivatives of the function f in x and y, respectively. Note that continuous functions with generalized derivative $f_{\bar z}=0$ are analytic functions, see e.g. Lemma 1 in [2].

Equation (1.7) is said to be **nondegenerate** if $||\mu||_\infty<1$ that we will assume later on. Homeomorphic solutions f of nondegenerate (1.7) in $W^{1,2}_{\rm loc}$ are called **quasiconformal mappings** or sometimes $\mu-$**conformal mappings**. Its continuous solutions in $W^{1,2}_{\rm loc}$ are called

$\mu-$**conformal functions**. On the corresponding existence theorems for nondegenerate (1.7), see e.g. [1, 9] and [37].

The **nonhomogeneous Beltrami equations**

$$\omega_{\bar{z}} \ = \ \mu(z) \cdot \omega_z \ + \ \sigma(z) \tag{1.8}$$

have been introduced and studied in the known Ahlfors-Bers paper [2], see also the Ahlfors monograph [1]. Boundary value problems for (1.8) have been researched in our preprint [25].

The present paper is devoted to the study of the Hilbert (Dirichlet) boundary value problem with arbitrary boundary data that are measurable with respect to the logarithmic capacity for the **semi-linear Beltrami equations** of the form

$$\omega_{\bar{z}} \ = \ \mu(z) \cdot \omega_z \ + \ \sigma(z) \cdot q(\omega(z)) \ , \tag{1.9}$$

where $\sigma : D \to \mathbb{C}$ belongs to class $L_p(D)$, $p > 2$, $q : \mathbb{C} \to \mathbb{C}$ is continuous and

$$\lim_{w\to\infty} \frac{q(w)}{w} \ = \ 0 \ , \tag{1.10}$$

as well as to the Poincare (Neumann) boundary-value problem with arbitrary boundary data that are also measurable with respect to the logarithmic capacity for the **semi-linear Poisson type equations**

$$\operatorname{div} A(z) \, \nabla \, U(z) \ = \ \Sigma(z) \cdot Q(U(z)) \ , \tag{1.11}$$

where $A(z)$ is a matrix valued function that is relevant to μ, $\Sigma : D \to \mathbb{R}$ belongs to class $L_p(D)$, $p > 2$, $Q : \mathbb{R} \to \mathbb{R}$ is continuous and

$$\lim_{t\to\infty} \frac{Q(t)}{t} \ = \ 0 \ . \tag{1.12}$$

2. On solutions with no boundary conditions

We start here from a theorem on existence of solutions for semi-linear Beltrami equations (1.9) without any boundary conditions.

Following [2], see also monograph [1], we assume that the **source** $\sigma : \mathbb{C} \to \mathbb{C}$ in equation (1.8) belongs to class $L_p(\mathbb{C})$ for some $p > 2$ with

$$k \, C_p \ < \ 1 \ , \quad k \ := \ \|\mu\|_\infty \ < \ 1 \ , \tag{2.1}$$

where C_p is the norm of the known operator $T : L_p(\mathbb{C}) \to L_p(\mathbb{C})$ defined through the Cauchy principal limit of the singular integral

$$(Tg)(\zeta) \ := \ \lim_{\varepsilon\to 0} \left\{ -\frac{1}{\pi} \int\limits_{|z-\zeta|>\varepsilon} \frac{g(z)}{(z-\zeta)^2} \, dxdy \right\} \ , \quad z = x + iy \ . \tag{2.2}$$

As known, $\|Tg\|_2 = \|g\|_2$, i.e. $C_2 = 1$, and by the Riesz convexity theorem $C_p \to 1$ as $p \to 2$. Thus, there are such p, whatever the value of k in (1.8).

Let us denote by B_p the Banach space of functions ω, defined on the whole plane $\mathbb{C}$, which satisfy a global Hölder condition of order $1 - 2/p$, which vanish at the origin, and whose generalized derivatives ω_z and $\omega_{\bar{z}}$ exist and belong to $L_p(\mathbb{C})$. The norm in B_p is defined by

$$\|\omega\|_{B_p} := \sup_{\substack{z_1,z_2\in\mathbb{C},\\ z_1\neq z_2}} \frac{|\omega(z_1) - \omega(z_2)|}{|z_1 - z_2|^{1-2/p}} + \|\omega_z\|_p + \|\omega_{\bar{z}}\|_p . \tag{2.3}$$

The principal result in [2], Theorem 1, is the following statement:

Theorem A. *Let condition* (2.1) *hold and* $\sigma \in L_p(\mathbb{C})$ *for* $p > 2$. *Then the equation* (1.8) *has a unique solution* $\omega^{\mu,\sigma} \in B_p$. *This is the only solution with* $\omega(0) = 0$ *and* $\omega_z \in L_p(\mathbb{C})$.

Its following consequence holds, see Theorem 4 and Lemma 8 in [2].

Theorem B. *Let* $\mu : \mathbb{C} \to \mathbb{C}$ *be in* $L_\infty(\mathbb{C})$ *with compact support and* $\|\mu\|_\infty < 1$. *Then there exists a unique* $\mu-$*conformal mapping* f^μ *in* $\mathbb{C}$ *which vanishes at the origin and satisfies condition* $f^\mu_z - 1 \in L_p(\mathbb{C})$ *for any* $p > 2$ *with* (2.1). *Moreover,* $f^\mu(z) = z + \omega^{\mu,\mu}(z)$.

To proceed to the semi-linear Beltrami equations, recall also that a **completely continuous** mapping from a metric space M_1 into a metric space M_2 is defined as a continuous mapping on M_1 which takes bounded subsets of M_1 into relatively compact subsets of M_2, i.e., with compact closures in space M_2. When a continuous mapping takes M_1 into a relatively compact subset of M_1, it is nowadays said to be **compact** on M_1.

Note that the notion of completely continuous (compact) operators is due essentially to Hilbert in a special space that, in reflexive spaces, is equivalent to Definition VI.5.1 for the Banach spaces in [14] which is due to F. Riesz, see also further comments of Section VI.12 in [14].

Recall some further definitions and the fundamental result of the celebrated paper [38]. Leray and Schauder extend as follows the Brouwer degree to compact perturbations of the identity I in a Banach space B, i.e., a complete normed linear space. Namely, given an open bounded set $\Omega \subset B$, a compact mapping $F : B \to B$ and $z \notin \Phi(\partial\Omega)$, $\Phi := I - F$, the **(Leray–Schauder) topological degree** $\deg[\Phi, \Omega, z]$ of Φ in Ω over z is constructed from the Brouwer degree by approximating the mapping F over Ω by mappings F_ε with range in a finite-dimensional subspace B_ε

(containing z) of B. It is showing that the Brouwer degrees $\deg\,[\Phi_\varepsilon, \Omega_\varepsilon, z]$ of $\Phi_\varepsilon := I_\varepsilon - F_\varepsilon$, $I_\varepsilon := I|_{B_\varepsilon}$, in $\Omega_\varepsilon := \Omega \cap B_\varepsilon$ over z stabilize for sufficiently small positive ε to a common value defining $\deg\,[\Phi, \Omega, z]$ of Φ in Ω over z.

This topological degree "algebraically counts" the number of fixed points of $F(\cdot) - z$ in Ω and conserves the basic properties of the Brouwer degree as additivity and homotopy invariance. Now, let a be an isolated fixed point of F. Then the **local (Leray–Schauder) index** of a is defined by $\text{ind}\,[\Phi, a] := \deg[\Phi, B(a,r), 0]$ for small enough $r > 0$. $\text{ind}\,[\Phi, 0]$ is called by **index** of F. In particular, if $F \equiv 0$, correspondingly, $\Phi \equiv I$, then the index of F is equal to 1.

Let us formulate the main result in [38], Theorem 1, see also the survey [44].

Proposition 1. *Let B be a Banach space, and let $F(\cdot, \tau) : B \to B$ be a family of operators with $\tau \in [0,1]$. Suppose that the following hypotheses hold:*

(H1) *$F(\cdot, \tau)$ is completely continuous on B for each $\tau \in [0,1]$ and uniformly continuous with respect to the parameter $\tau \in [0,1]$ on each bounded set in B;*

(H2) *the operator $F := F(\cdot, 0)$ has finite collection of fixed points whose total index is not equal to zero;*

(H3) *the collection of all fixed points of the operators $F(\cdot, \tau)$, $\tau \in [0,1]$, is bounded in B.*

Then the collection of all fixed points of the family of operators $F(\cdot, \tau)$ contains a continuum along which τ takes all values in $[0,1]$.

Remark 1. By Lemma 5 in [2] the mapping $\sigma \to \omega^{\mu,\sigma}$ from Theorem A is a bounded linear operator from $L_p(\mathbb{C})$ to $B_p(\mathbb{C})$ with a bound that depends only on k and p in (2.1). In particular, this is a bounded linear operator from $L_p(\mathbb{C})$ to $C(\mathbb{C})$. Namely, by (15) in [2] we have that $\omega^{\mu,\sigma}$ is Hölder continuous:

$$|\omega^{\mu,\sigma}(z_1) \;-\; \omega^{\mu,\sigma}(z_2)| \;\le\; c \cdot \|\sigma\|_p \cdot |z_1 - z_2|^{1-2/p} \qquad \forall\, z_1 \;\, z_2 \;\in \mathbb{C}\;, \tag{2.4}$$

where the constant c may depend only on k and p in (2.1). Moreover, $\omega^{\mu,\sigma}(z)$ is locally bounded because $\omega^{\mu,\sigma}(0) = 0$. Thus, the linear operator $\sigma \to \omega^{\mu,\sigma}|_S$ is completely continuous for each compact set S in $\mathbb{C}$ by Arzela–Ascoli theorem, see e.g. Theorem IV.6.7 in [14].

Theorem 1. *Let $\mu : \mathbb{C} \to \mathbb{C}$ belong to class $L_\infty(\mathbb{C})$ with $k := \|\mu\|_\infty < 1$ and $\sigma : \mathbb{C} \to \mathbb{C}$ be with compact support and of class $L_p(\mathbb{C})$ for some*

$p > 2$ satisfying (2.1). *Suppose that $q : \mathbb{C} \to \mathbb{C}$ is a continuous function with condition* (1.10). *Then the semi-linear Beltrami equation* (1.9) *has a solution ω of class $B_p(\mathbb{C})$.*

Proof. If $\|\sigma\|_p = 0$ or $\|q\|_C = 0$, then Theorem A above gives the desired solution $\omega := \omega^{\mu,0}$ of equation (1.9). Thus, we may assume that $\|\sigma\|_p \neq 0$ and $\|q\|_C \neq 0$. Set $q_*(t) = \max\limits_{|w|\le t} |q(w)|$, $t \in \mathbb{R}^+ := [0,\infty)$. Then the function $q_* : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing and, moreover, by (1.10)

$$\lim_{t\to\infty} \frac{q_*(t)}{t} \;=\; 0\ . \tag{2.5}$$

Let us show that the family of operators $F(g;\tau) : L^p_\sigma(\mathbb{C}) \to L^p_\sigma(\mathbb{C})$,

$$F(g;\tau) \;:=\; \tau\sigma \cdot q(\omega^{\mu,g}) \qquad \forall\, \tau \in [0,1]\ , \tag{2.6}$$

where $L^p_\sigma(\mathbb{C})$ consists of functions $g \in L^p(\mathbb{C})$ with supports in the support S of σ, satisfies hypotheses H1–H3 of Theorem 1 in [38], see Proposition 1 above. Indeed:

H1). First of all, the function $F(g;\tau) \in L^p_\sigma(\mathbb{C})$ for all $\tau \in [0,1]$ and $g \in L^p_\sigma(\mathbb{C})$ because the function $q(\omega^{\mu,g})$ is continuous and, furthermore, the operators $F(\cdot;\tau)$ are completely continuous for each $\tau \in [0,1]$ and even uniformly continuous with respect to parameter $\tau \in [0,1]$ by Theorem A and Remark 1.

H2). The index of the operator $F(g;0)$ is obviously equal to 1.

H3). Let us assume that the collection of all solutions of the equations $g = F(g;\tau)$, $\tau \in [0,1]$, is not bounded in $L^p_\sigma(\mathbb{C})$, i.e., there is a sequence of functions $g_n \in L^p_\sigma(\mathbb{C})$ with $\|g_n\|_p \to \infty$ as $n \to \infty$ such that $g_n = F(g_n;\tau_n)$ for some $\tau_n \in [0,1]$, $n = 1,2,\ldots$.

However, then by Remark 1 we have that

$$\|g_n\|_p \;\le\; \|\sigma\|_p\, q_*\left(\|\omega^{\mu,g_n}|_S\|_C\right) \;\le\; \|\sigma\|_p\, q_*\left(M\,\|g_n\|_p\right)$$

for some constant $M > 0$ and, consequently,

$$\frac{q_*(M\,\|g_n\|_p)}{M\,\|g_n\|_p} \;\ge\; \frac{1}{M\,\|\sigma\|_p} \;>\; 0\ . \tag{2.7}$$

The latter is impossible by condition (2.5). The obtained contradiction disproves the above assumption.

Thus, by Theorem 1 in [38], see Proposition 1 above, there is a function $g \in L^p_\sigma(\mathbb{C})$ with $F(g;1) = g$, and then by Theorem A the function $\omega := \omega^{\mu,g}$ gives the desired solution of (1.9). □

3. Factorization for Beltrami semi-linear equations

Let us first start in this section from the following factorization lemma for linear inhomogeneous Beltrami equations (1.8).

Lemma 1. *Let D be a bounded domain in $\mathbb{C}$, $\mu : D \to \mathbb{C}$ be in class $L_\infty(D)$ with $k := \|\mu\|_\infty < 1$, $\sigma : D \to \mathbb{C}$ be in class $L_p(D)$, $p > 2$, with condition* (2.1)*. Suppose that $f^\mu : \mathbb{C} \to \mathbb{C}$ is the $\mu-$conformal mapping from Theorem B with an arbitrary extension of μ onto $\mathbb{C}$ keeping compact support and condition* (2.1)*.*

Then each continuous solution ω of equation (1.8) *in D of class $W^{1,p}(D)$ has the representation as a composition $h \circ f^\mu|_D$, where h is a generalized analytic function in $D_* := f^\mu(D)$ with the source $g \in L_{p_*}(D_*)$, $p_* := p^2/2(p-1) \in (2,p)$,*

$$g \ := \ \left(f^\mu_z \cdot \frac{\sigma}{J}\right) \circ (f^\mu)^{-1} \ , \tag{3.1}$$

where J is the Jacobian of f^μ.

Vice versa, if h is a generalized analytic function with the source $g \in L_{p_*}(D_*)$, $p_* > 2$, in (3.1), then $\omega := h \circ f^\mu$ is a solution of (1.8) of class $C^\alpha_{\text{loc}} \cap W^{1,s}_{\text{loc}}(D)$, where $\alpha = 1 - 2/s$ and $s := p_*^2/2(p_*-1) \in (2, p_*)$.

Proof. To be short, let us apply here the notation f instead of f^μ. Let us consider the function $h := \omega \circ f^{-1}$. First of all, note that by point (iii) of Theorem 5 in [2] $f^* := f^{-1}|_{D^*}$, $D^* := f(D)$, is of class $W^{1,p}(D^*)$. Then, arguing as under the proof of Lemma 10 in [2], we obtain that $h \in W^{1,p_*}(D^*)$, where $p_* := p^2/2(p-1) \in (2,p)$. Since $\omega = h \circ f$, we get also, see e.g. formulas (28) in [2] or formulas I.C(1) in [1], that

$$\omega_z \ = \ (h_\zeta \circ f) \cdot f_z \ + \ (h_{\overline{\zeta}} \circ f) \cdot \overline{f_{\bar z}} \ ,$$

$$\omega_{\bar z} \ = \ (h_\zeta \circ f) \cdot f_{\bar z} \ + \ (h_{\overline{\zeta}} \circ f) \cdot \overline{f_z} \ ,$$

and, thus,

$$\sigma(z) \ = \omega_{\bar z} \ - \ \mu(z)\omega_z \ = \ (h_{\overline{\zeta}} \circ f)\overline{f_z}(1 - |\mu(z)|^2) \ = \ (h_{\overline{\zeta}} \circ f)J(z)/f_z \ ,$$

where $J(z) = |f_z|^2 - |f_{\bar z}|^2 = |f_z|^2(1 - |\mu(z)|^2)$ is the Jacobian of f, i.e.,

$$h_{\overline{\zeta}} \ = \ g(\zeta) \ := \left(f_z \frac{\sigma}{J}\right) \circ f^{-1}(\zeta) \ .$$

Similarly, applying Lemma 10 in [2] and the Sobolev embedding theorem, see Theorem I.10.2 in [57], we come to the inverse conclusion. □

Remark 2. Note that if h is a generalized analytic function with the source g in the domain D_*, then $H = h+\mathcal{A}$ is so for any analytic function $\mathcal{A}$ in D_* but $|\mathcal{A}'|^{p_*}$ can be integrable only locally in D_*. By Lemma 1, the source in (3.1) is always in class $L_{p_*}(D_*)$, $p_* := p^2/2(p-1) \in (2,p)$, in view of Theorem A with σ extended onto $\mathbb{C}$ by zero outside of D. Here we may assume that μ is extended onto $\mathbb{C}$ by zero outside of D. However, any other extension of μ keeping condition (2.1) is suitable here, too. Moreover, we may apply here as f^μ any $\mu-$conformal mappings with different normalizations, in particular, with the hydrodynamic normalization $f^\mu(z) = z + o(1)$ as $z \to \infty$.

The next lemma makes it is possible to reduce boundary value problems for semi-linear Beltrami equations (1.9) to semi-linear Vekua type equations (1.5).

Lemma 2. *Let D be a bounded domain in $\mathbb{C}$, $\mu : D \to \mathbb{C}$ be measurable with $\|\mu\|_\infty < 1$, $\sigma : D \to \mathbb{C}$ be in class $L_p(D)$, $p > 2$. Suppose that $q : \mathbb{C} \to \mathbb{C}$ is continuous and $f^\mu : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping from Theorem B with an arbitrary extension of μ onto $\mathbb{C}$ keeping compact support and condition* (2.1)*.*

Then each continuous solution ω of equation (1.9) *in D of class $W^{1,p}(D)$ has the representation as a composition $h \circ f^\mu|_D$, where h is a continuous solution of* (1.5) *in class $W^{1,p_*}_{\text{loc}}(D_*)$, where $D_* := f^\mu(D)$, $p_* := p^2/2(p-1) \in (2,p)$, with the multiplier g in* (1.5) *of class $L_{p_*}(D_*)$ defined by formula* (3.1)*.*

Vice versa, if h is a continuous solution in class $W^{1,p_*}_{\text{loc}}(D_*)$ of (1.5) *with multiplier $g \in L_{p_*}(D_*)$, $p_* > 2$, given by* (3.1)*, then $\omega := h \circ f^\mu$ is a solution of* (1.9) *in class $C^\alpha_{\text{loc}} \cap W^{1,s}_{\text{loc}}(D)$, where $\alpha = 1 - 2/s$ and $s := p_*^2/2(p_* - 1) \in (2, p_*)$.*

Proof. Indeed, if ω is a continuous solution of (1.9) in D of class $W^{1,p}(D)$, then ω is a solution of (1.8) in D with the source $\Sigma := \sigma \cdot q \circ \omega$ in the same class. Then by Lemma 1 and Remark 2 $\omega = h \circ f^\mu$, where h is a generalized analytic function with the source G of class $L_{p_*}(D_*)$ after replacement of σ by Σ in (3.1). Note that $h \in W^{1,p_*}_{\text{loc}}(D_*)$, see e.g. Theorems 1.16 and 1.37 in [59]. The proof of the vice versa conclusion of Lemma 2 is similar and it is again based on its reduction to Lemma 1. □

4. On logarithmic potential and capacity

Given a bounded Borel set E in the plane $\mathbb{C}$, a **mass distribution** on E is a nonnegative completely additive function ν of a set defined on its Borel subsets with $\nu(E) = 1$. The function

$$U^{\nu}(z) := \int\limits_{E} \log \left| \frac{1}{z-\zeta} \right| \, d\nu(\zeta) \tag{4.1}$$

is called a **logarithmic potential** of the mass distribution ν at a point $z \in \mathbb{C}$. A **logarithmic capacity** $C(E)$ of the Borel set E is the quantity

$$C(E) = e^{-V} \,, \qquad V \;=\; \inf_{\nu} \; V_{\nu}(E) \;, \qquad V_{\nu}(E) \;=\; \sup_{z} \; U^{\nu}(z) \;. \tag{4.2}$$

It is also well-known the following geometric characterization of the logarithmic capacity, see e.g. the point 110 in [46]:

$$C(E) \;=\; \tau(E) \;:=\; \lim_{n\to\infty} \; V_n^{\frac{2}{n(n-1)}} \tag{4.3}$$

where V_n denotes the supremum of the product

$$V(z_1, \ldots, z_n) \;=\; \prod_{k<l}^{l=1,\ldots,n} \; |z_k - z_l| \tag{4.4}$$

taken over all collections of points $z_1, \ldots, z_n$ in the set E. Following Fékete, see [15], the quantity $\tau(E)$ is called the **transfinite diameter** of the set E.

Remark 3. Thus, we see that if $C(E) = 0$, then $C(f(E)) = 0$ for an arbitrary mapping f that is continuous by Hölder and, in particular, for quasiconformal mappings on compact sets, see e.g. Theorem II.4.3 in [37].

In order to introduce sets that are measurable with respect to logarithmic capacity, we define, following [10], **inner** C_* and **outer** C^* **capacities**:

$$C_*(E) \;:\; = \; \sup_{F\subseteq E} \; C(E), \qquad\qquad C^*(E) \;:\; = \; \inf_{E\subseteq O} \; C(O) \tag{4.5}$$

where supremum is taken over all compact sets $F \subset \mathbb{C}$ and infimum is taken over all open sets $O \subset \mathbb{C}$. A **set** $E \subset \mathbb{C}$ is called **measurable with respect to the logarithmic capacity** if $C^*(E) = C_*(E)$, and the common value of $C_*(E)$ and $C^*(E)$ is still denoted by $C(E)$.

A **function** $\varphi : E \to \mathbb{C}$ defined on a bounded set $E \subset \mathbb{C}$ is called **measurable with respect to logarithmic capacity** if, for all open sets $O \subseteq \mathbb{C}$, the sets $\{z \in E : \varphi(z) \in O\}$ are measurable with respect to logarithmic capacity. It is clear from the definition that the set E is itself measurable with respect to logarithmic capacity.

Note also that sets of logarithmic capacity zero coincide with sets of the so-called **absolute harmonic measure** zero introduced by Nevanlinna, see Chapter V in [46]. Hence a set E is of (Hausdorff) length zero if $C(E) = 0$, see Theorem V.6.2 in [46]. However, there exist sets of length zero having a positive logarithmic capacity, see e.g. Theorem IV.5 in [10].

Remark 4. It is known that Borel sets and, in particular, compact and open sets are measurable with respect to logarithmic capacity, see e.g. Lemma I.1 and Theorem III.7 in [10]. Moreover, as it follows from the definition, any set $E \subset \mathbb{C}$ of finite logarithmic capacity can be represented as a union of a sigma-compactum (union of countable collection of compact sets) and a set of logarithmic capacity zero. Thus, the measurability of functions with respect to logarithmic capacity is invariant under Hölder continuous change of variables.

It is also known that the Borel sets and, in particular, compact sets are measurable with respect to all Hausdorff's measures and, in particular, with respect to measure of length, see e.g. theorem II(7.4) in [56]. Consequently, any set $E \subset \mathbb{C}$ of finite logarithmic capacity is measurable with respect to measure of length. Thus, on such a set any function $\varphi : E \to \mathbb{C}$ being measurable with respect to logarithmic capacity is also measurable with respect to measure of length on E. However, there exist functions that are measurable with respect to measure of length but not measurable with respect to logarithmic capacity, see e.g. Theorem IV.5 in [10].

Dealing with measurable boundary functions $\varphi(\zeta)$ with respect to the logarithmic capacity, we will use the **abbreviation q.e. (quasi-everywhere)** on a set $E \subset \mathbb{C}$, if a property holds for all $\zeta \in E$ except its subset of zero logarithmic capacity, see [36].

5. Hilbert problem with respect to angular limits

In this section, we prove the existence of nonclassical solutions of the Hilbert boundary value problem with arbitrary boundary data that are measurable with respect to logarithmic capacity for semi-linear Beltrami equations (1.9). The result is formulated in terms of the angular limit that is a traditional tool of the geometric function theory, see e.g. monographs [13, 34, 41, 48] and [49].

Recall that a straight line L is **tangent** to a curve Γ in $\mathbb{C}$ at a point $z_0 \in \Gamma$ if

$$\limsup_{z \to z_0, z \in \Gamma} \frac{\operatorname{dist}(z, L)}{|z - z_0|} = 0 . \tag{5.1}$$

Let D be a Jordan domain in $\mathbb{C}$ with a tangent at a point $\zeta \in \partial D$. A path in D terminating at ζ is called **nontangential** if its part in a neighborhood of ζ lies inside of a triangle in D with one of its vertexes at ζ. The limit along all nontangential paths at ζ is called **angular** at the point.

Following [28], we say that a Jordan curve Γ in $\mathbb{C}$ is **almost smooth** if Γ has a tangent **q.e. (quasi everywhere)** with respect to logarithmic capacity, see e.g. [36] for the term. In particular, Γ is almost smooth if Γ has a tangent at all its points except its countable collection. The nature of such a Jordan curve Γ can be complicated enough because this countable collection can be everywhere dense in Γ, see e.g. [12].

Recall that the **quasihyperbolic distance** between points z and z_0 in a domain $D \subset \mathbb{C}$ is the quantity

$$k_D(z, z_0) \ := \ \inf_{\gamma} \int_{\gamma} ds/d(\zeta, \partial D) \ ,$$

where $d(\zeta, \partial D)$ denotes the Euclidean distance from the point $\zeta \in D$ to ∂D and the infimum is taken over all rectifiable curves γ joining the points z and z_0 in D, see [19].

Further, it is said that a domain D satisfies the **quasihyperbolic boundary condition** if there exist constants a and b and a point $z_0 \in D$ such that

$$k_D(z, z_0) \ \leq \ a \ + \ b \ \ln \frac{d(z_0, \partial D)}{d(z, \partial D)} \qquad \forall \ z \in D \ . \tag{5.2}$$

The latter notion was introduced in [18] but, before it, was first implicitly applied in [6]. By the discussion in [28], every smooth (or Lipschitz) domain satisfies the quasihyperbolic boundary condition but such boundaries can be even nowhere locally rectifiable.

Note that it is well–known the so–called $(A)-$condition by Ladyzhenskaya–Ural'tseva, which is standard in the theory of boundary value problems for PDE, see e.g. [35]. Recall that a domain D in $\mathbb{R}^n$, $n \geq 2$, is called satisfying **(A)-condition** if

$$\text{mes}\ D \cap B(\zeta, \rho) \ \leq \ \Theta_0 \, \text{mes}\ B(\zeta, \rho) \qquad \forall \ \zeta \in \partial D \ , \ \rho \leq \rho_0 \tag{5.3}$$

for some Θ_0 and $\rho_0 \in (0,1)$, where $B(\zeta, \rho)$ denotes the ball with the center $\zeta \in \mathbb{R}^n$ and the radius ρ, see 1.1.3 in [35].

A domain D in $\mathbb{R}^n$, $n \geq 2$, is said to be satisfying the **outer cone condition** if there is a cone that makes possible to be touched by its top to every point of ∂D from the completion of D after its suitable rotations and shifts. It is clear that the latter condition implies (A)–condition.

Probably one of the simplest examples of an almost smooth domain D with the quasihyperbolic boundary condition and without (A)–condition is the union of 3 open disks with the radius 1 centered at the points 0 and $1 \pm i$. It is clear that this domain has zero interior angle at its boundary point 1.

Given a Jordan domain D in $\mathbb{C}$, we call $\lambda : \partial D \to \mathbb{C}$ a **function of bounded variation**, write $\lambda \in \mathcal{BV}(\partial D)$, if

$$V_\lambda(\partial D) \ : \ = \ \sup \sum_{j=1}^{k} |\lambda(\zeta_{j+1}) - \lambda(\zeta_j)| \ < \ \infty \tag{5.4}$$

where the supremum is taken over all finite collections of points $\zeta_j \in \partial D$, $j = 1, \ldots, k$, with the cyclic order meaning that ζ_j lies between ζ_{j+1} and ζ_{j-1} for every $j = 1, \ldots, k$. Here we assume that $\zeta_{k+1} = \zeta_1 = \zeta_0$. The quantity $V_\lambda(\partial D)$ is called the **variation of the function** λ.

Now, we call $\lambda : \partial D \to \mathbb{C}$ a function of **countable bounded variation**, write $\lambda \in \mathcal{CBV}(\partial D)$, if there is a countable collection of mutually disjoint arcs γ_n of ∂D, $n = 1, 2, \ldots$ on each of which the restriction of λ is of bounded variation and the set $\partial D \setminus \cup\gamma_n$ has logarithmic capacity zero. In particular, the latter holds true if the set $\partial D \setminus \cup\gamma_n$ is countable. It is clear that such functions can be singular enough.

Theorem 2. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, be in $\mathcal{CBV}(\partial D)$ and let $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose also that $q : \mathbb{C} \to \mathbb{C}$ is a continuous function with condition (1.10), $\mu : D \to \mathbb{C}$ is of class $L_\infty(D)$ with $k := \|\mu\|_\infty < 1$, μ is Hölder continuous in an open neighborhood of ∂D inside of D, $\sigma : D \to \mathbb{C}$ has compact support in D, $\sigma \in L_p(D)$ and condition (2.1) *holds for some $p > 2$.*

Then equation (1.9) *has a solution $\omega : D \to \mathbb{C}$ of class $C^\alpha_{\rm loc} \cap W^{1,s}_{\rm loc}(D)$, where $\alpha = 1 - 2/s$ and $s \in (2, p)$, that is smooth in the neighborhood of ∂D with the angular limits*

$$\lim_{z \to \zeta, z \in D} \ {\rm Re} \ \left\{ \overline{\lambda(\zeta)} \cdot \omega(z) \right\} \ = \ \varphi(\zeta) \qquad q.e. \ on \ \ \partial D \ . \tag{5.5}$$

Remark 5. By the construction in the proof below, each such solution has the representation $\omega = h \circ f|_D$, where $f = f^\mu : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping from Theorem B with a suitable extension of μ onto $\mathbb{C}$ and h is a continuous solution of class $W^{1,p_*}_{\rm loc}(D_*)$ for equation (1.5) with the multiplier $g \in L_{p_*}(D_*)$, $p_* := p^2/2(p-1) \in (2, p)$, as in

(3.1), $D_* := f(D)$, which is a generalized analytic function with a source in class $L_{p_*}(D_*)$ and has the angular limits

$$\lim_{w\to\xi, w\in D_*} \mathrm{Re}\,\left\{\overline{\Lambda(\xi)}\cdot h(w)\right\} \;=\; \Phi(\xi) \qquad \text{q.e. on } \;\partial D_*\;, \tag{5.6}$$

$\Lambda := \lambda\circ f^{-1}|_{\partial D_*}$, $\Phi := \varphi\circ f^{-1}|_{\partial D_*}$. Also, $s = p_*^2/2(p_*-1) \in (2,p_*) \subset (2,p)$.

Proof. First of all, let us choose a suitable extension of μ onto $\mathbb{C}$ outside of D. By hypotheses of Theorem 1 μ belongs to a class C^α, $\alpha\in(0,1)$, for an open neighborhood U of ∂D inside of D. By Lemma 1 in [29] μ is extended to a Hölder continuous function $\mu : U\cup\mathbb{C}\setminus D\to\mathbb{C}$ of the class C^α. Then, for every $k_*\in(k,1)$, there is an open neighborhood V of ∂D in $\mathbb{C}$, where $\|\mu\|_\infty\le k_*$ and μ in $C^\alpha(V)$. Let us choose $k_*\in(k,1)$ so close to k that $k_*C_p<1$ and set $\mu\equiv 0$ outside of $D\cup V$.

By Theorem B, there is a $\mu-$conformal mapping $f = f^\mu : \mathbb{C}\to\mathbb{C}$ a.e. satisfying the Beltrami equation (1.7) with the given extended complex coefficient μ in $\mathbb{C}$. Note that the mapping f has the Hölder continuous first partial derivatives in V with the same order of the Hölder continuity as μ, see e.g. [32] and also [33]. Moreover, its Jacobian

$$J(z)\neq 0 \qquad\qquad \forall\; z\in V\;, \tag{5.7}$$

see e.g. Theorem V.7.1 in [37]. Hence f^{-1} is also smooth in $V_* := f(V)$, see e.g. formulas I.C(3) in [1].

Now, the domain $D_* := f(D)$ satisfies the boundary quasihyperbolic condition because D is so, see e.g. Lemma 3.20 in [18]. Moreover, ∂D_* has q.e. tangents, furthermore, the points of ∂D and ∂D^* with tangents correspond each to other in one-to-one manner because the mappings f and f^{-1} are smooth there. It is evident that the function $\Lambda := \lambda\circ f^{-1}|_{\partial D_*}$ belongs to the class $\mathcal{CBV}(\partial D_*)$.

Let us also show that the function $\Phi := \varphi\circ f^{-1}|_{\partial D_*}$ is measurable with respect to logarithmic capacity. Indeed, for each open set $\Omega\subseteq\mathbb{C}$, $\Phi^{-1}(\Omega) = f\circ\varphi^{-1}(\Omega)$, where the set $\varphi^{-1}(\Omega)$ is measurable with respect to logarithmic capacity. Thus, it suffices to see that $f(S)$ is measurable with respect to logarithmic capacity whenever S is measurable with respect to logarithmic capacity.

Note for this goal that the quasiconformal mapping f is Hölder continuous on the compact set ∂D and, thus, $C(f(S)) = 0$ whenever $C(S) = 0$, see Remark 3. Moreover, it is known that Borel sets and, in particular, compact and open sets are measurable with respect to logarithmic capacity, see Remark 4. In addition, by definition a $C-$measurable set is a union of a sigma-compactum (union of a countable collection of compact sets) and a set S with $C(S) = 0$, see again Section 4, especially formulas (4.5). Thus, to conclude that f translates $C-$measurable sets into $C-$measurable sets, it remains to note that the homeomorphism f translates compact sets into compact sets.

Next, by Remark 2 the function $g : D_* \to \mathbb{C}$ in (3.1) belongs to class $L_{p_*}(D_*)$, where $p_* = p^2/2(p-1) \in (2,p)$. Thus, by Theorem 2 in [27] there is a continuous solution h of equation (1.5) that is a generalized analytic function with a source of class $L_{p_*}(D_*)$ and that has the angular limits (5.5). Note that $h \in W^{1,p_*}_{\rm loc}(D_*)$, see e.g. Theorems 1.16 and 1.37 in [59]. Finally, by Lemma 2 the function $\omega := h \circ f^{\mu}$ is a solution of (1.9) in class $C^{\alpha}_{\rm loc} \cap W^{1,s}_{\rm loc}(D)$, where $\alpha = 1 - 2/s$ and $s := p_*^2/2(p_*-1) \in (2,p_*) \subset (2,p)$. □

In particular case $\lambda \equiv 1$, we obtain the corresponding consequence of Theorem 2 on the Dirichlet problem for the semi–linear Beltrami equations (1.9).

6. On Poincare problem for semi-linear equations

In this section we study the solvability of the Poincare boundary-value problem for semi-linear Poisson type equations of the form (1.11) in anisotropic and inhomogeneous media.

It is well–known, see Theorem 16.1.6 in [3], that nonhomogeneous Beltrami equations (1.8) in a domain D of the complex plane $\mathbb{C}$ are closely connected with the divergence type equations of the form

$$\mathrm{div}\,[A(z)\,\nabla\,u(z)] \;=\; g(z)\;, \tag{6.1}$$

where $A(z)$ is the matrix function:

$$A = \begin{pmatrix} \frac{|1-\mu|^2}{1-|\mu|^2} & \frac{-2\mathrm{Im}\,\mu}{1-|\mu|^2} \\ \frac{-2\mathrm{Im}\,\mu}{1-|\mu|^2} & \frac{|1+\mu|^2}{1-|\mu|^2} \end{pmatrix}. \tag{6.2}$$

As we see, the matrix function $A(z)$ in (6.2) is symmetric and its entries $a_{ij} = a_{ij}(z)$ are dominated by the quantity

$$K_{\mu}(z) \;:=\; \frac{1\;+\;|\mu(z)|}{1\;-\;|\mu(z)|}\;,$$

and, thus, they are bounded if the Beltrami equation is not degenerate.

Vice verse, uniformly elliptic equations (6.1) with symmetric $A(z)$ whose entries are measurable and $\det A(z) \equiv 1$ just correspond to non-degenerate Beltrami equations with coefficient

$$\mu \;=\; \frac{1}{\det\,(I+A)}\;(a_{22}-a_{11}\;-\;2ia_{21})\;=\;\frac{a_{22}-a_{11}\;-\;2ia_{21}}{1\;+\;\mathrm{Tr}\,A\;+\;\det A}\;, \quad (6.3)$$

where $M^{2\times 2}(D)$ denotes the collection of all such matrix functions $A(z)$ in D. Here I is the unit 2×2 matrix and $\mathrm{Tr}\,A$ is the trace of A, i.e., $a_{11}+a_{22}$.

Note that (6.1) are the main equation of hydromechanics (mechanics of incompressible fluids) in anisotropic and inhomogeneous media.

Given such a matrix function A and a $\mu-$conformal mapping $f^{\mu}: D\to\mathbb{C}$, we have already seen in Lemma 1 of [20], by direct computation, that if a function T and the entries of A are sufficiently smooth, then

$$\mathrm{div}\,[A(z)\nabla\,(T(f^{\mu}(z)))]\;=\;J(z)\triangle\,T(f^{\mu}(z))\;. \quad (6.4)$$

In the case $T\in W^{1,2}_{\mathrm{loc}}$, we understand the identity (6.4) in the distributional sense, see Proposition 3.1 in [21], i.e., for all $\psi\in C^1_0(D)$,

$$\int\limits_D \langle A\nabla(T\circ f^{\mu}),\nabla\psi\rangle\;dm_z\;=\;\int\limits_D J(z)\langle M^{-1}((\nabla T)\circ f^{\mu}),\nabla\psi\rangle\;dm_z\;, \quad (6.5)$$

where M is the Jacobian matrix of the mapping f^{μ} and J is its Jacobian.

Theorem 3. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\nu:\partial D\to\mathbb{C}$, $|\nu(\zeta)|\equiv 1$, be of $\mathcal{CBV}(\partial D)$ and $\varphi:\partial D\to\mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose also that $A\in M^{2\times 2}(D)$ has entries in a class $C^{\alpha}(D)$, $\alpha\in(0,1)$, $\Sigma:D\to\mathbb{R}$ is a function of class $L_p(D)$, $p>2$, with a compact support in D and $Q:\mathbb{R}\to\mathbb{R}$ is a continuous function with condition (1.12).

Then there is a weak solution $u:D\to\mathbb{R}$ of class $C^{1,\gamma}_{\mathrm{loc}}\cap W^{2,p}_{\mathrm{loc}}$, $\gamma=\min(\alpha,\beta)$, $\beta=1-2/p$, of the equation (1.11) *that has the angular limits of its derivatives in the directions $\nu=\nu(\zeta)$, $\zeta\in\partial D$,*

$$\lim_{z\to\zeta,z\in D}\;\frac{\partial u}{\partial\nu}\;(z)\;=\;\varphi(\zeta)\qquad q.e.\ on\ \ \partial D\;. \quad (6.6)$$

Here u is called a **weak solution** of equation (1.11) if

$$\int\limits_D \{\langle A(z)\nabla u(z),\nabla\psi\rangle\;+\;\Sigma(z)\,Q(u(z))\,\psi(z)\}\;dm_z\;=\;0\qquad\forall\;\psi\in C^1_0(D)\;. \quad (6.7)$$

Remark 6. By the construction in the proof below, such a solution u has the representation $u = U \circ f|_D$, where $f = f^\mu : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping from Theorem B with a suitable extension of μ in (6.3) to $\mathbb{C}$, and U is a solution of class $C^{1,\beta} \cap W^{2,p}_{\rm loc}$, $\beta = (p-2)/p$, for the quasilinear Poisson equation (1.6) with the multiplier (where J is the Jacobian of f) :

$$G \ := \ \frac{\Sigma}{J} \circ f^{-1} \ , \quad G \in L_p(D_*) \ , \ D_* := f(D) \ , \tag{6.8}$$

that is a generalized harmonic function with a source of the same class $L_p(D_*)$, which has the angular limits

$$\lim_{w \to \xi, w \in D_*} \frac{\partial U}{\partial \mathcal{N}} \, (w) \ = \ \Phi(\xi) \qquad \text{q.e. on } \ \partial D_* \ , \tag{6.9}$$

where

$$\mathcal{N}(\xi) \ := \ \left\{ \frac{\partial f}{\partial \nu} \cdot \left| \frac{\partial f}{\partial \nu} \right|^{-1} \right\} \circ f^{-1}(\xi) \ , \quad \xi \in \partial D_* \ , \tag{6.10}$$

and

$$\Phi(\xi) \ := \ \left\{ \varphi \cdot \left| \frac{\partial f}{\partial \nu} \right|^{-1} \right\} \circ f^{-1}(\xi) \ , \quad \xi \in \partial D_* \ . \tag{6.11}$$

Proof. By the hypotheses of the theorem μ given by (6.3) belongs to a class $C^\alpha(D)$, $\alpha \in (0,1)$, and by Lemma 1 in [29] μ is extended to a Hölder continuous function $\mu : \mathbb{C} \to \mathbb{C}$ of the class C^α. Then, for every $k_* \in (k,1)$, there is an open neighborhood V of $\overline{D}$, where $\|\mu\|_\infty \le k_*$ and μ is of class $C^\alpha(V)$. We may assume that V is bounded and set $\mu \equiv 0$ in $\mathbb{C} \setminus V$.

Let $f = f^\mu : \mathbb{C} \to \mathbb{C}$ be the $\mu-$conformal mapping from Theorem B with the given extended complex coefficient μ in $\mathbb{C}$. Note that the mapping f has the Hölder continuous first partial derivatives in V with the same order of the Hölder continuity as μ, see e.g. [32] and also [33]. Moreover, its Jacobian

$$J(z) \ = \ |f_z|^2 \, - \, |f_{\bar z}|^2 \ > \ 0 \qquad \forall \ z \in V \ , \tag{6.12}$$

see e.g. Theorem V.7.1 in [37]. Hence f^{-1} is also smooth in $V_* := f(V)$, see e.g. formulas I.C(3) in [1].

Now, the domain $D_* := f(D)$ satisfies the boundary quasihyperbolic condition because D is so, see e.g. Lemma 3.20 in [18]. Moreover, ∂D_* has q.e. tangents, furthermore, the points of ∂D and ∂D^* with tangents correspond each to other in a one-to-one manner because the mappings

f and f^{-1} are smooth. In addition, the function $\mathcal{N}$ in (6.10) belongs to the class $\mathcal{CBV}(\partial D_*)$ because

$$\frac{\partial f}{\partial \nu} \ = \ f_z \cdot \nu \ + \ f_{\bar{z}} \cdot \overline{\nu}\ , \qquad \left|\frac{\partial f}{\partial \nu}\right| \ \geq \ |f_z| \ - \ |f_{\bar{z}}| \ > \ 0\ ,$$

and Φ in (6.11) is measurable with respect to logarithmic capacity by repeating arguments in the proof to Theorem 2.

Next, the source $G : D_* \to \mathbb{R}$ in (6.8) belongs to class $L_p(D_*)$ because by (5.7) the function $J^{-1} \circ f^{-1}$ is continuous and, consequently, bounded on the compact set $\overline{D_*}$, see also point (vi) of Theorem 5 in [2] on the replacement of variables in integrals. Thus, by Theorem 4 in [27] there is a solution of class $C^{1,\beta}_{\rm loc}(D_*) \cap W^{2,p}_{\rm loc}(D_*)$, $\beta = (p-2)/p$, of the quasilinear Poisson equation (1.6) with the multiplier (6.8) that is a generalized harmonic function with a source of the same class $L_p(D_*)$ and which has the angular limits (6.9) q.e. on ∂D_*.

Note that the function $u := U \circ f$ belongs to class $W^{2,p}_{\rm loc}(D)$ because f is a quasi-isometry in D of class C^1, see e.g. 1.1.7 in [43]. Finally, by Proposition 3.1 in [21] the function u gives the desired solution of the equation (6.1) because by Lemma 10 and the point (i) of Theorem 5 in [2]

$$\frac{\partial u}{\partial \nu} \ = \ u_z \cdot \nu + u_{\bar{z}} \cdot \overline{\nu} \ = \ \nu \cdot (U_w \circ f \cdot f_z + U_{\bar{w}} \circ f \cdot \overline{f_{\bar{z}}}) + \overline{\nu} \cdot (U_w \circ f \cdot f_{\bar{z}} + U_{\bar{w}} \circ f \cdot \overline{f_z})$$

$$= \ U_w \circ f \cdot (\nu \cdot f_z + \overline{\nu} \cdot f_{\bar{z}}) + U_{\overline{w}} \circ f \cdot (\nu \cdot \overline{f_{\bar{z}}} + \overline{\nu} \cdot \overline{f_z}) \ = \ U_w \circ f \cdot \frac{\partial f}{\partial \nu} + U_{\bar{w}} \circ f \cdot \overline{\frac{\partial f}{\partial \nu}}$$

$$= \ \left(\mathcal{N} \cdot U_w \ + \ \overline{\mathcal{N}} \cdot U_{\bar{w}} \ \right) \circ f \cdot \left|\frac{\partial f}{\partial \nu}\right| \ = \ \frac{\partial U}{\partial \mathcal{N}} \circ f \cdot \left|\frac{\partial f}{\partial \nu}\right|\ ,$$

where the direction $\mathcal{N}$ is given by (6.10). This solution u belongs to the class $C^{1,\gamma}_{\rm loc}(D)$, $\gamma = \min(\alpha, \beta)$, because by the above calculations with $\nu = 1$ and i

$$u_x \ = \ U_w \circ f \cdot f_x + U_{\bar{w}} \circ f \cdot \overline{f_x}\ , \quad u_y \ = \ U_w \circ f \cdot f_y - U_{\bar{w}} \circ f \cdot \overline{f_y}\ , \quad z = x + iy\ .$$

Remark 7. We are able to say more in the case of Re $n(\zeta)\overline{\nu(\zeta)} > 0$, where $n(\zeta)$ is the inner normal to ∂D at the point ζ. Indeed, the latter magnitude is a scalar product of $n = n(\zeta)$ and $\nu = \nu(\zeta)$ interpreted as vectors in $\mathbb{R}^2$ and it has the geometric sense of projection of the vector ν into n. In view of (6.6), since the limit $\varphi(\zeta)$ is finite, there is a finite limit $u(\zeta)$ of $u(z)$ as $z \to \zeta$ in D along the straight line passing through the point ζ and being parallel to the vector ν because along this line

$$u(z) \ = \ u(z_0) \ - \ \int\limits_0^1 \ \frac{\partial u}{\partial \nu} \ (z_0 + \tau(z - z_0))\ d\tau\ . \qquad (6.13)$$

Thus, at each point with condition (6.6), there is the directional derivative

$$\frac{\partial u}{\partial \nu}(\zeta) \ := \ \lim_{t\to 0} \ \frac{u(\zeta + t\cdot\nu) - u(\zeta)}{t} \ = \ \varphi(\zeta)\ . \tag{6.14}$$

In particular case of the Neumann problem, Re $n(\zeta)\overline{\nu(\zeta)} \equiv 1 > 0$, where $n = n(\zeta)$ denotes the unit interior normal to ∂D at the point ζ, and we have by Theorem 3 and Remark 7 the following significant result.

Corollary 1. *Let D be a Jordan domain in $\mathbb{C}$ with the quasihyperbolic boundary condition, the unit inner normal $n(\zeta)$, $\zeta \in \partial D$, belong to the class $\mathcal{CBV}(\partial D)$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose also that $A \in M^{2\times 2}(D)$ has entries in a class $C^{\alpha}(D)$, $\alpha \in (0,1)$, $\Sigma : D \to \mathbb{R}$ is a function of class $L_p(D)$, $p > 2$, with a compact support in D and $Q : \mathbb{R} \to \mathbb{R}$ is a continuous function with condition (1.12).

Then one can find a weak solution $u : D \to \mathbb{R}$ of class $C^{1,\gamma}_{\rm loc} \cap W^{2,p}_{\rm loc}$ with $\gamma = \min(\alpha, \beta)$, $\beta = 1 - 2/p$, of equation (1.11) such that q.e. on ∂D there exist:

1) the finite limit along the normal $n(\zeta)$

$$u(\zeta) \ := \ \lim_{z\to\zeta} \ u(z)\ ,$$

2) the normal derivative

$$\frac{\partial u}{\partial n}(\zeta) \ := \ \lim_{t\to 0} \ \frac{u(\zeta + t\cdot n(\zeta)) - u(\zeta)}{t} \ = \ \varphi(\zeta)\ ,$$

3) the angular limit

$$\lim_{z\to\zeta} \ \frac{\partial u}{\partial n}(z) \ = \ \frac{\partial u}{\partial n}(\zeta)\ .$$

Remark 8. In addition, such a solution u has the representation $u = U \circ f|_D$, where $f = f^{\mu} : \mathbb{C} \to \mathbb{C}$ is the $\mu-$conformal mapping from Theorem B with a suitable extension of μ in (6.3) onto $\mathbb{C}$ outside of D, described in the proof of Theorem 3, and U is a weak solution of the class $C^{1,\beta}_{\rm loc} \cap W^{2,p}_{\rm loc}$, $\beta = 1 - 2/p$, of the quasilinear Poisson equation (1.6) with the multiplier G in (6.8) that is a generalized harmonic function with a source of the same class $L_p(D_*)$, which has the angular limits

$$\lim_{w\to\xi, w\in D_*} \ \frac{\partial U}{\partial n_*}(w) \ = \ \varphi_*(\xi) \qquad \text{q.e. on} \ \ \partial D_*\ , \tag{6.15}$$

with

$$n_*(\xi) \ := \ \left\{ \frac{\partial f}{\partial n} \cdot \left| \frac{\partial f}{\partial n} \right|^{-1} \right\} \circ f^{-1}(\xi) \ , \quad \xi \in \partial D_* \ , \tag{6.16}$$

and

$$\varphi_*(\xi) \ := \ \left\{ \varphi \cdot \left| \frac{\partial f}{\partial n} \right|^{-1} \right\} \circ f^{-1}(\xi) \ , \quad \xi \in \partial D_* \ . \tag{6.17}$$

7. The Poincare problem in physical applications

Theorem 3 on the Poincare boundary-value problem with arbitrary measurable boundary data over the logarithmic capacity in Jordan domains can be applied to mathematical models of physical and chemical absorption with diffusion, plasma states, stationary burning etc.

The first group of such applications is relevant to reaction-diffusion problems. Problems of this type are discussed in [11], p. 4, and, in detail, in [4]. A nonlinear system is obtained for the density U and the temperature T of the reactant. Upon eliminating T the system can be reduced to equations of the form

$$\triangle U \ = \ \sigma \cdot Q(U) \tag{7.1}$$

with $\sigma > 0$ and, for isothermal reactions, $Q(U) = U^{\lambda}$ where $\lambda > 0$ that is called the order of the reaction. It turns out that the density of the reactant U may be zero in a subdomain called a **dead core**. A particularization of results in Chapter 1 of [11] shows that a dead core may exist just if and only if $\beta \in (0,1)$, see also the corresponding examples in [21].

In the case of anisotropic and inhomogeneous media, we come to the semi-linear Poisson type equations (1.11). In this connection, the following statement may be of independent interest.

Corollary 2. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, be of $\mathcal{CBV}(\partial D)$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose also that $A \in M^{2\times 2}(D)$ has entries in a class $C^{\alpha}(D)$, $\alpha \in (0,1)$, $\sigma : D \to \mathbb{R}$ is a function of class $L_p(D)$, $p > 2$, with a compact support in D and $Q : \mathbb{R} \to \mathbb{R}$ is a continuous function with condition (1.12).

Then there is a weak solution $u : D \to \mathbb{R}$ of class $C^{1,\gamma}_{\rm loc} \cap W^{2,p}_{\rm loc}$, $\gamma = \min(\alpha, \beta)$, $\beta = 1 - 2/p$, of the semi-linear Poisson type equation

$$\mathrm{div}\,[A(z)\,\nabla\, u(z)] \ = \ \sigma(z) \cdot u^{\lambda}(z) \ , \quad 0 \ < \ \lambda \ < \ 1 \ , \qquad a.e.\ in\ D \tag{7.2}$$

satisfying the Poincare boundary condition on directional derivatives

$$\lim_{z\to\zeta}\ \frac{\partial u}{\partial \nu}\ (z)\ =\ \varphi(\zeta) \qquad \text{q.e. on } \partial D \tag{7.3}$$

in the sense of the angular limits.

Note also that certain mathematical models of a thermal evolution of a heated plasma lead to nonlinear equations of the type (7.1). Indeed, it is known that some of them have the form $\triangle\psi(u) = f(u)$ with $\psi'(0) = \infty$ and $\psi'(u) > 0$ if $u \neq 0$ as, for instance, $\psi(u) = |u|^{q-1}u$ under $0 < q < 1$, see e.g. [11]. With the replacement of the function $U = \psi(u) = |u|^q \cdot \operatorname{sign} u$, we have that $u = |U|^Q \cdot \operatorname{sign} U$, $Q = 1/q$, and, with the choice $f(u) = |u|^{q^2} \cdot \operatorname{sign} u$, we come to the equation $\triangle U = |U|^q \cdot \operatorname{sign} U = \psi(U)$. For anisotropic and inhomogeneous media, we obtain the corresponding equation (7.4) below:

Corollary 3. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, be of $\mathcal{CBV}(\partial D)$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose also that $A \in M^{2\times 2}(D)$ has entries in a class $C^{\alpha}(D)$, $\alpha \in (0,1)$, $\sigma : D \to \mathbb{R}$ is a function of class $L_p(D)$, $p > 2$, with a compact support in D and $Q : \mathbb{R} \to \mathbb{R}$ is a continuous function with condition (1.12).

Then there is a weak solution $u : D \to \mathbb{R}$ of class $C^{1,\gamma}_{\rm loc} \cap W^{2,p}_{\rm loc}$, $\gamma = \min(\alpha,\beta)$, $\beta = 1 - 2/p$, of the semi-linear Poisson type equation

$$\operatorname{div}\left[A(z)\,\nabla\, u(z)\right]\ =\ \sigma(z)\cdot|u(z)|^{\lambda-1}u(\xi)\ ,\quad 0\ <\ \lambda\ <\ 1\ , \qquad \text{a.e. in } D \tag{7.4}$$

satisfying the Poincare boundary condition on directional derivatives (7.3) *q.e.*

Finally, we recall that in the combustion theory, see e.g. [5, 47] and the references therein, the following model equation

$$\frac{\partial u(z,t)}{\partial t}\ =\ \frac{1}{\delta}\cdot\triangle u\ +\ e^{u}\ ,\quad \delta\ >\ 0\ ,\ t\geq 0,\ z\in D, \tag{7.5}$$

takes a special part. Here $u \geq 0$ is the temperature of the medium. We restrict ourselves here by the stationary case, although our approach makes it possible to study the parabolic equation (7.5), see [21]. The corresponding equation of the type (1.11), see (7.6) below, appears in anisotropic and inhomogeneous media with the function $Q(u) = e^{-|u|}$ that is uniformly bounded at all.

Corollary 4. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, be of $\mathcal{CBV}(\partial D)$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose also that $A \in M^{2\times 2}(D)$ has entries in a class $C^{\alpha}(D)$, $\alpha \in (0,1)$, $\sigma : D \to \mathbb{R}$ is a function of class $L_p(D)$, $p > 2$, with a compact support in D and $Q : \mathbb{R} \to \mathbb{R}$ is a continuous function with condition (1.12).

Then there is a weak solution $u : D \to \mathbb{R}$ of class $C^{1,\gamma}_{\rm loc} \cap W^{2,p}_{\rm loc}$, $\gamma = \min(\alpha, \beta)$, $\beta = 1 - 2/p$, of the semi-linear Poisson type equation

$$\mathrm{div}\,[A(z)\,\nabla\,u(z)] \;=\; \sigma(z)\cdot e^{-|u(z)|} \qquad a.e.\ in\ D \tag{7.6}$$

satisfying the Poincare boundary condition on directional derivatives (7.3) *q.e.*

Remark 9. Such solutions u in Corollaries 2–4 have the representation $u = U \circ f|_D$, where $f = f^{\mu} : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping in Theorem B with a suitable extension of μ in (6.3) to $\mathbb{C}$ described in the proof of Theorem 3, and U is a solution of class $C^{1,\beta} \cap W^{2,p}_{\rm loc}$, $\beta = (p-2)/p$, for the quasilinear Poisson equation (1.6) with the functions $Q(t) = t^{\lambda}$, $|t|^{\lambda-1}t$, $\lambda \in (0,1)$ and $e^{-|t|}$, correspondingly, and with the multiplier (here J is the Jacobian of f) :

$$G \;:=\; \frac{\sigma}{J} \circ f^{-1}\ , \quad G \in L_p(D_*)\ ,\ D_* := f(D)\ , \tag{7.7}$$

that is a generalized harmonic function with a source g of the same class $L_p(D_*)$, which satisfy the Poincare boundary condition on directional derivatives (6.9) in the sense of the angular limits q.e. on ∂D_*.

8. Neumann problem in physical applications

In turn, Corollary 1 can be applied to the study of the physical phenomena discussed by us in the last section. In this connection, the particular cases of the function $Q(t) = t^{\lambda}$, $|t|^{\lambda-1}t$, $\lambda \in (0,1)$, and $e^{-|t|}$ will be again useful.

Corollary 5. *Let D be a Jordan domain in $\mathbb{C}$ with the quasihyperbolic boundary condition, the unit inner normal $n(\zeta)$, $\zeta \in \partial D$, belong to the class $\mathcal{CBV}(\partial D)$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose also that $A \in M^{2\times 2}(D)$ has entries in a class $C^{\alpha}(D)$, $\alpha \in (0,1)$, $\sigma : D \to \mathbb{R}$ is a function of class $L_p(D)$, $p > 2$, with a compact support in D.

Then one can find a weak solution $u : D \to \mathbb{R}$ *of class* $C^{1,\gamma}_{\mathrm{loc}} \cap W^{2,p}_{\mathrm{loc}}$ *with* $\gamma = \min(\alpha, \beta)$ *and* $\beta = 1 - 2/p$ *of the semi-linear Poisson type equation* (7.2) *such that q.e. on* ∂D *there exist:*

1) the finite limit along the normal $n(\zeta)$

$$u(\zeta) \ := \ \lim_{z \to \zeta} \ u(z) \ ,$$

2) the normal derivative

$$\frac{\partial u}{\partial n}\,(\zeta) \ := \ \lim_{t \to 0} \ \frac{u(\zeta + t \cdot n(\zeta)) - u(\zeta)}{t} \ = \ \varphi(\zeta) \ ,$$

3) the angular limit

$$\lim_{z \to \zeta} \ \frac{\partial u}{\partial n}\,(z) \ = \ \frac{\partial u}{\partial n}\,(\zeta) \ .$$

Corollary 6. *Under hypotheses of Corollary 5, there is a weak solution u* $u : D \to \mathbb{R}$ *of class* $C^{1,\gamma}_{\mathrm{loc}} \cap W^{2,p}_{\mathrm{loc}}$ *with* $\gamma = \min(\alpha, \beta)$ *and* $\beta = 1 - 2/p$ *of the semi-linear Poisson type equation* (7.4) *such that q.e. on* ∂D *all the conclusion 1)-3) of Corollary 5 hold, i.e., u is a generalized solution of the Neumann problem for* (7.4) *in the given sense.*

Corollary 7. *Under hypotheses of Corollary 5, there is a weak solution u* $u : D \to \mathbb{R}$ *of class* $C^{1,\gamma}_{\mathrm{loc}} \cap W^{2,p}_{\mathrm{loc}}$ *with* $\gamma = \min(\alpha, \beta)$ *and* $\beta = 1 - 2/p$ *of the semi-linear Poisson type equation* (7.6) *such that q.e. on* ∂D *all the conclusion 1)–3) of Corollary 5 hold, i.e., u is a generalized solution of the Neumann problem for* (7.6) *in the given sense.*

Remark 10. Such solutions u in Corollaries 5–7 have the representation $u = U \circ f|_D$, where $f = f^{\mu} : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping in Theorem B with a suitable extension of μ in (6.3) to $\mathbb{C}$ described in the proof of Theorem 3, and U is a solution of class $C^{1,\beta} \cap W^{2,p}_{\mathrm{loc}}$, $\beta = (p-2)/p$, for the quasilinear Poisson equation (1.6) with the functions $Q(t) = t^{\lambda}$, $|t|^{\lambda - 1} t$, $\lambda \in (0, 1)$ and $e^{-|t|}$, correspondingly, and the multiplier G in (7.7), that is a generalized harmonic function with a source g of the same class $L_p(D_*)$, which satisfy the Neumann boundary condition (6.15) in the sense of the angular limits q.e. on ∂D_*.

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