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Two coefficient conjectures for nonvanishing Hardy functions, II

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Abstract. Recently the author proved that the Hummel–Scheinberg–Zalcman conjecture of 1977 on coefficients of nonvanishing H^p functions is true for all $p = 2m,\ m \in \mathbb{N}$, i.e., for the Hilbertian Hardy spaces H^{2m}. As a consequence, this also implies the proof of the Krzyz conjecture for bounded nonvanishing functions, which originated this direction.

In the present paper, we solve the problem for all spaces H^p with $p \geq 2$.

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1. Introductory remarks ad main result

This paper is devoted to construction of special quasiconformal deformations of nonvanishing Hardy functions with prescribed distortion properties and their application to proof of the Hummel–Scheinberg–Zalcman conjecture.

Recall that this conjecture posed in [7] generalizes and strengthens the Krzyz conjecture for nonvanishing H^∞ functions to the Hardy spaces H^p of holomorphic functions $f(z) = \sum\limits_0^\infty c_n z^n$ on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ with norm

$$\|f\|_p = \sup_{r<1}\Big(\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta\Big)^{1/p}.$$

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It states that Taylor's coefficients of nonvanishing functions $f(z) \in H^p$, $p > 1$, with $\|f\|_p \leq 1$ are sharply estimated by

$$|c_n| \leq (2/e)^{1-1/p}, \tag{1}$$

and this bound is realized only by the functions $\epsilon_2 \kappa_{n,p}(\epsilon_1 z)$, where $|\epsilon_1| = |\epsilon_2| = 1$ and

$$\kappa_{n,p}(z) = \Big[\frac{(1+z^n)^2}{2}\Big]^{1/p} \Big[\exp \frac{z^n - 1}{z^n + 1}\Big]^{1-1/p}. \tag{2}$$

This eminent conjecture has been investigated by many authors. A long time, the only known results here are that the conjecture is true for $n = 1$ (Brown) and $n = 2$ (Suffridge) as well as some results for special subclasses of H^p, see [5, 6, 14]. Brown also showed that (1) is true for arbitrary $n \geq 2$, provided $c_m = 0$ for all m, $1 \leq m < (n+1)/2$, and Suffridge also estimated sharply the coefficient c_1 for $0 < p \leq 1$.

Recently the author proved this conjecture for Hilbertian Hardy spaces H^{2m} with $m \in \mathbb{N}$ in [12], applying a new approach to the coefficient problems for holomorphic functions on the disk. This approach was presented in [10, 11] and involves some fundamental results of Teichmüller space theory, especially the Bers isomorphism theorem for Teichmüller spaces of punctured Riemann surfaces.

In the limit as $p = 2m \to \infty$, one obtains as a consequence of (1) and (2) that the coefficients of nonvanishing H^∞ functions with norm $\|f\|_\infty \leq 1$ are sharply estimated by $|c_n| \leq 2/e$, which proves the initial Krzyz conjecture.

In fact, the last estimate holds for some broader class containing also unbounded nonvanishning functions, see [12].

The aim of the present paper, which continues [12], is to prove the Hummel–Scheinberg–Zalcman conjecture for all spaces H^p with $p \geq 2$. The main result states:

Theorem 1. *The estimate* (1) *is valid for all spaces* H^p *with* $p \geq 2$*; that is, the coefficients of any nonvanishing function* $f \in H^p$, $p \geq 2$, *with* $\|f\|_p \leq 1$ *satisfy* $|c_n| \leq (2/e)^{1-1/p}$ *for any* $n > 1$*; the equality in* (1) *is realized only on the function* $f(z) = \kappa_{n,p}(z)$ *and its compositions with pre and post rotations about the origin.*

The proof of this general theorem is based on the same ideas as its special case $p = 2m$, $m \in \mathbb{N}$, in [12]. A new essential step is to establish that in the case of nonvanishing H^p functions the needed quasiconformal deformations exist for all $p > 1$.

Another essential step in the proof of Theorem 1 relies on the fact that for any $n \geq 2$ the coefficients of the function $\kappa_{1,p}$ satisfy

$$|c_n(\kappa_{1,p})| = |c_n^0| < |c_1^0|,$$

which follows, for example, from Parseval's equality. But this remains unknown for $f \in H^p$ with $1 < p < 2$.

The last section of the paper contains some remarks concerning the possible generalizations of this problem.

2. Quasicoformal deformatios of H^p functions

We start with establishing the existence of some special quasiconformal deformations of nonvanishning Hardy and Bergman functions.

The general result established in [9] for the generic H^p functions with $p = 2m,\ m \in \mathbb{N}$, states the follows.

Consider the functions $f(z) \in H^{2m} \cap L_\infty(\mathbb{D})$, with

$$\sup_{\mathbb{D}} |f(z)| = M > \|f\|_{2m}.$$

Let E be a ring domain bounded by a closed curve $L \subset \mathbb{D}$ containing inside the origin and by the unit circle $S^1 = \partial\mathbb{D}$. Let, in addition,

$$\mathbf{d}^0 = (0, 1, 0, \dots, 0) =: (d_k^0) \in \mathbb{R}^{n+1},$$

and $|\mathbf{x}|$ denote the Euclidean norm in $\mathbb{R}^l$.

Proposition 1. [9] *For any holomorphic function* $f(z) = \sum_{k=j}^{\infty} c_k^0 z^k \in L_{2m}(E) \cap L_\infty(E)$ *(with* $c_j^0 \neq 0,\ 0 \leq j < n$ *and* $m \in \mathbb{N}$*), which is not a polynomial of degree* $n_1 \leq n$*, there exists a positive number* ε_0 *such that for every point*

$$\mathbf{d}' = (d'_{j+1}, \dots,\ , d'_n) \in \mathbb{C}^{n-j}$$

and every $a \in \mathbb{R}$ *satisfying the inequalities*

$$|\mathbf{d}'| \leq \varepsilon, \quad |a| \leq \varepsilon, \quad \varepsilon < \varepsilon_0,$$

there exists a quasiconformal automorphism h *of the complex plane* $\widehat{\mathbb{C}}$*, which is conformal in the disk*

$$D_0 = \{w : \ |w - c_0^0| < \sup_{\mathbb{D}} |f_0(z)| + |c_0^0| + 1\}$$

(hence also outside of E*) and satisfies the conditions:*

(i) $h^{(k)}(c_0^0) = k!d_k = k!(d_k^0 + d_k'),\ k = j+1, \dots, n$ *(i.e.,* $d_1 = 1 + d_1'$ *and* $d_k = d_k'$ *for* $k \geq 2$*);*

(ii) $\|h \circ f\|_{2m}^{2m} = \|f\|_{2m}^{2m} + a$.

For any function $f \in H^p$, we have

$$\|f\|_p = \sup_{r<1}\Big(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta\Big)^{1/p} = \Big(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta\Big)^{1/p}, \qquad (3)$$

since the mean function

$$\mathcal{M}f(r)^p = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta$$

is a circularly symmetric subharmonic function on $\mathbb{D}$, monotone increasing with $r \to 1$. Any such function is logarithmically convex with respect to $\log r$ and has at least one-side derivative on $[0,1]$.

Using the appropriately thin rings E adjacent to the unit circle, one derives that for any bounded in $\mathbb{D}$ function $f \in H^{2m}$ there exists a $O(\varepsilon)$-quasiconformal automorphism h of $\widehat{\mathbb{C}}$ satisfying the conditions (i) and distorting the H^{2m}-norm of f by

$$\|h \circ f\|_{2m} = \|f\|_{2m} + O(\varepsilon) \qquad (4)$$

(where the bound of the remainder term depends on m).

The proof of Proposition 1 in [9] shows that all its assumptions are essential; the arguments do not extend to arbitrary $p \geq 2$ and unbounded holomorphic L^p functions.

One of the important steps in the proof of Theorem 1 is the following weakened extension of Lemma 1 to **nonvanishing** functions from the spaces H^p with $p > 1$.

Fix a natural $n > 1$ and consider the collections $\mathbf{d} = (d_0, d_1, \dots, d_n) \in \mathbb{C}^{n+1}$.

Proposition 2. *For every bounded nonvanishing function* $f(z) = c_0 + c_1 z + c_2 z^2 + \dots \in H^p,\ p > 1$, *with* $\|f\|_p < \infty$, *which is not a polynomial of degree at most* n, *there exists* $\varepsilon_0 > 0$ *such that for any point* $\mathbf{d} \in \mathbb{C}^{n+1}$ *with* $|\mathbf{d}| \leq \varepsilon < \varepsilon_0$, *there is a bounded nonvanishing function* $f^*(z) = c_0^* + c_1^* z + c_2^* z^2 + \dots \in H^p$ *satisfying* $c_j^* = c_j + d_j$ *for all* $j = 0, 1, \dots, n$ *and*

$$\|f^*\|_p = \|f\|_p + O(\varepsilon).$$

Note that quasiconformal deformations of such type preserve the L_p-norm of holomorphic functions, but generically increase their L_∞-norm (an illustrating example is given in [11]).

Proof. For any nonvanishing function $f(z) = \sum\limits_0^\infty c_n z^n \in H^p, p > 1$, the function

$$f_{p/2}(z) := f(z)^{p/2} = e^{(p/2)\log f(z)},$$

with a fixed branch of the logarithmic function, also is single valued, holomorphic and zero free in the unit disk $\mathbb{D}$. We take everywhere the principal branch. Explicitly,

$$f_{p/2}(z) = c_0^{p/2}\Big(1 + \frac{p}{2}\,\frac{c_1}{c_0}\,z + \dots\Big) = c_0(f_{p/2}) + c_1(f_{p/2})z + \dots\ ; \qquad (5)$$

this function belongs to the space H^2. In the case of nonvaninshing f, the correspondence $f(z) \leftrightarrow f_{p/2}(z)$ creates a biholomorphism (one-to-one and open map) between the neighborhoods of the origin in $\mathbb{C}^{n+1}$ filled by collections $\mathbf{d}(f) = (c_0, \dots, c_n)$ and $\mathbf{d}(f_{/2})p = (c_0(f_{p/2}), \dots, c_n(f_{p/2}))$.

Since the function (5) is holomorphic, one can apply to it all arguments used in [9] in the proof of Proposition 1 for $p = 2m$ (this proof essentially requires that $f(z)^m$ is single valued). In view of the importance of Proposition 2, we outline the main steps of its proof; the details omitted are given in [9].

Fix $R \ge \sup_{\mathbb{D}} |f(z)| + |c_0| + 1$ and take the annulus

$$G_R = \{w :\ R < |w - c_0^0| < R + 1\}.$$

We define for $\rho \in L_p(B),\ p \ge 2$, the operators

$$T\rho = -\frac{1}{\pi}\iint\limits_{G_R} \frac{\rho(\zeta)d\xi d\eta}{\zeta - w}, \quad \Pi\rho = \partial_w T\rho = -\frac{1}{\pi}\iint\limits_{G_R} \frac{\rho(\zeta)d\xi d\eta}{(\zeta - w)^2}$$

(the second integral exists as a principal Cauchy value). We seek the required quasiconformal automorphism $h = h^\mu$ of the form

$$h(w) = w - \frac{1}{\pi}\iint\limits_{G_R} \frac{\rho(\zeta)d\xi d\eta}{\zeta - w} = w + T\rho(w), \qquad (6)$$

with the Beltrami coefficient $\mu = \mu_h$ equal to zero outside of G_R, with $\|\mu\|_\infty < \kappa < 1$. Substituting (6) into the Beltrami equation $\partial_{\overline{w}} h = \mu\partial_w h$, we get

$$\rho = \mu + \mu\Pi\mu + \mu\Pi(\mu\Pi\mu) + \dots .$$

This series is convergent in $L_p(E)$ for some $p > 2$, and on the basis of well-known properties of operators T and Π, we have for any disk $\mathbb{D}_{R'} = \{w \in \mathbb{C} : \ |w| < R'\}, \quad 0 < R' < \infty$, that

$$\|\rho\|_{L_p(\mathbb{D}_{R'})}, \quad \|\Pi\rho\|_{L_p(\mathbb{D}_{R'})} \le M_1(\kappa, R', p)\|\mu\|_{L_\infty(\mathbb{C})};$$
$$\|h\|_{C(\mathbb{D}_{R'})} \le M_1(\kappa, R', p)\|\mu\|_\infty.$$

Therefore,

$$h(w) = w + T\mu(w) + \omega(w),$$

with $\|\omega\|_{C(\mathbb{D}_{R'})} \le M_2(\kappa, R')\|\mu\|^2_\infty$. Using the pairing

$$< \nu, \varphi >= -\frac{1}{\pi} \iint\limits_{G_R} \nu(\zeta)\varphi(\zeta) d\xi d\eta, \quad \nu \in L_\infty(G_R), \quad \varphi \in L_1(G_R),$$

one can rewrite the above representation in the form

$$h(w) = w + \sum_0^\infty < \mu, \varphi_k > (w - c_0^0)^k + \omega(w), \quad \varphi_k(\zeta) = \frac{1}{(\zeta - c_0^0)^{k+1}}. \quad (7)$$

This equality and the condition $c_j^* = c_j + d_j$ for all $j = 0, 1, \dots, n$, provide the first group of equalities to determine the desired Beltrami coefficient μ:

$$k! d_k =< \mu, \varphi_k > + \omega^{(k)}(c_0^0) =< \mu, \varphi_k > + O(\|\mu\|^2_\infty), \quad k = j + 1, \dots, n. \quad (8)$$

On the other hand, (7) and the requirement of preserving L_p norms give

$$\|h \circ f\|_p^p = \|f + T\rho \circ f\|_p^p = \int\limits_{G_R} |f(z) + T\mu \circ f(z)|^p dE_z + O(\|\mu\|^2_\infty)$$
$$= \int\limits_{G_R} [|f(z)|^2 + 2\,\mathrm{Re}(\overline{f(z)} T\mu \circ f(z)) + |T\mu \circ f(z)|^2]^{p/2} dxdy + O(\|\mu\|^2_\infty)$$
$$= \|f\|_p^p + \frac{p}{2\pi}\,\mathrm{Re}\Big[\iint\limits_{G_R} \mu(\zeta) d\xi d\eta \int\limits_{G_R} \frac{|f(z)|^{p-2}\ \overline{f(z)}}{\zeta - f(z)} dxdy\Big] + O_p(\|\mu\|^2_\infty)$$

(here $z = x + iy$). Now set

$$\phi(\zeta) = -\frac{p}{2} \int\limits_{G_R} \frac{|f(z)|^{p-2}\overline{f(z)}}{f(z) - \zeta} dxdy; \quad (9)$$

then the previous equality can be rewritten in the form

$$\|h \circ f\|_p^p - \|f\|_p^p = \mathrm{Re} < \mu, \phi > + O_p(\|\mu\|^2_\infty). \quad (10)$$

The function (9) is holomorphic in the disk $\mathbb{D}_R^* = \{w \in \widehat{\mathbb{C}} : |w - c_0^0| > R\}$ and belongs to the space B_R^p formed in $B_p(B)$ by functions holomorphic in D_R^*; moreover, $\phi(\zeta) \not\equiv 0$. The latter follows from the fact that for large $|\zeta|$ we have $\phi(\zeta) = \sum_1^\infty b_k \zeta^{-k}$ with $b_2 = \frac{p}{2} \|f\|_p^p > 0$.

We shall need the following important lemma whose proof straightforwardly follows the corresponding lemma in [9].

Lemma 1. *Under the assumptions of the theorem, the function ϕ is distinct from a linear combination of the fractions $\varphi_0, ..., \varphi_l$,* with $l \le n$.

According to Lemma 1, the series expansion of ϕ in D_R^* must contain the powers $(\zeta - c_0^0)^{-k-1}$ with $k > n$, and therefore, the remainder

$$\psi(\zeta) = \phi(\zeta) - \sum_0^n b_k(\zeta - c_0^0)^{-k-1} = \Big(\sum_0^{j-1} + \sum_s^\infty\Big) b_k(\zeta - c_0^0)^{-k-1}, \ \ s \ge n+1,$$

is distinct from zero in D_R^*. Let us note also that (3) implies

$$h \circ f(z) = c_0^* + \sum_j^\infty c_k^* z^k,$$

with $c_0^* = c_0^0 + d_0'$ and $c_j^* = c_j^0 d_1'$ $(j \ge 1)$. Thus,

$$|c_j^*|^2 - |c_j^0|^2 = \begin{cases} 2\,\mathrm{Re}(\overline{c}_0^0 d_0') = 2\,\mathrm{Re}(\overline{c}_0^0 < \mu, \varphi_0 >) + O(\|\mu\|^2), & j = 0, \\ 2|c_j^0|^2\,\mathrm{Re}\, d_1' = 2|c_j^0|^2\,\mathrm{Re} < \mu, \varphi_0 > + O(\|\mu\|^2), & j \ge 1, \end{cases}$$

and, therefore, $b_j \neq 0$.

Let us now seek the desired Beltrami coefficient μ in the form

$$\mu = \xi_j \overline{\varphi}_j + \sum_{j+1}^n \xi_k \overline{\varphi}_k + \tau \overline{\psi}, \quad \mu | \mathbb{C} \setminus B = 0, \tag{11}$$

with unknown constants $\xi_j, \xi_{j+1}, \dots, \xi_n, \tau$ to be determined from equalities (8) and (10).

Substituting the expression (11) into (8) and (10) and taking into account the mutual orthogonality of φ_k on B, one obtains the nonlinear equations

$$k! d_k = \xi_k r_k^2 + O(\|\mu\|^2), \quad k = j+1, \dots, n,$$

$$\|h \circ f_0\|_{2m}^{2m} - \|f_0\|_{2m}^{2m} = \mathrm{Re} < \xi_j \bar{\varphi}_j + \sum_{j+1}^n \xi_k \bar{\varphi}_k + \tau \bar{\psi}, \phi > + O(\|\mu\|^2) \tag{12}$$

for determining ξ_k and τ. The only remaining equation is a relation for $\operatorname{Re}\xi_j, \operatorname{Im}\xi_j, \operatorname{Re}\tau, \operatorname{Im}\tau$. To distinguish a unique solution, we add three real equations to (12). Let us require that ξ_j satisfy the equality

$$< \xi_j\overline{\varphi}_j + \sum_{j+1}^{n} \xi_k\bar{\varphi}_k, \sum_{0}^{n} b_k\varphi_k >= 0; \tag{13}$$

it is reduced to

$$\xi_j b_j r_j^2 = -\sum_{j+1}^{n} \xi_k b_k r_k^2 \quad (b_j \neq 0). \tag{14}$$

Then we obtain for τ the equation

$$\|h \circ f_0\|_p^p - \|f\|_p^p = \operatorname{Re} < \tau\overline{\psi}, \phi > +O(\|\mu\|_]\infty^2),$$

which, letting τ be real, takes the form

$$\|h \circ f\|_p^p - \|f\|_p^p = \tau\varkappa + O(\|\mu\|_\infty^2) \tag{15}$$

with $\varkappa = \sum\limits_k r_k^2$. The summation is taken here over all $k \neq j+1, \dots, n$, for which $b_k \neq 0$.

Separating the real and imaginary parts in equalities (12), (14) and adding (15), we obtain $2(n-j)+3$ real equalities, which define a nonlinear C^1 smooth (in fact, Re-analytic) map

$$\mathbf{y} = W(\mathbf{x}) = W'(\mathbf{0})\mathbf{x} + O(|\mathbf{x}|^2),$$

of the points $\mathbf{x} = (\operatorname{Re}\xi_j, \operatorname{Im}\xi_j, \operatorname{Re}\xi_{j+1}, \operatorname{Im}\xi_{j+1}, \dots, \operatorname{Re}\xi_n, \operatorname{Im}\xi_n, \tau)$ in a small neighborhood U_0 of the origin in $\operatorname{Re}^{2(n-j)+3}$, taking the values

$$\mathbf{y} = (\operatorname{Re} d_j, \operatorname{Im} d_j, \operatorname{Re} d_{j+1}, \operatorname{Im} d_{j+1}, \dots, \operatorname{Re} d_n, \operatorname{Im} d_n, \|h \circ f\|_p^p - \|f_0\|_p^p)$$

also near the origin of $\operatorname{Re}^{2(n-j)+3}$. Its linearization $\mathbf{y} = W'(\mathbf{0})\mathbf{x}$ defines a linear map $\operatorname{Re}^{2(n-j)+3} \to \operatorname{Re}^{2(n-j)+3}$ whose Jacobian only differs from $r_j^2 r_{j+1}^2 \dots r_n^2 \varkappa \neq 0$ by a constant factor. Therefore, $\mathbf{x} \mapsto W'(\mathbf{0})\mathbf{x}$ is a linear isomorphism of the space $\operatorname{Re}^{2(n-j)+3}$ onto itself, and one can apply to W the inverse mapping theorem. The latter implies the assertion of Proposition 2.

So, for any collection $\mathbf{d}_\varepsilon(f) = \mathbf{d}(f) + O(\varepsilon)$ there exists an $O(\varepsilon)$-quasiconformal homeomorphism h^μ of $\widehat{\mathbb{C}}$ conformal on $f(\mathbb{D})$ such that $\mathbf{d}_\varepsilon(f) = (c_0(h^\mu \circ f), \dots, c_j(h^\mu \circ f)$, and

$$\|h^\mu \circ f\|_{L_p} = \|f\|_{L_p}^p. \tag{16}$$

Note also that the H^p-norm is distorted similar to (4) via

$$\|h^\mu \circ f\|_{H^2}^2 = \|f_{p/2}\|_{H^2}^2 = \|f\|_{H^p}^p + O(\varepsilon). \tag{17}$$

The relations (16) and (17) show the difference under the actions of deformations given by Propositions 1 and 2 on the Hardy and Bergman spaces.

3. Proof of Theorem 1

Step 1: Underlying lemmas. As was mentioned in the introduction, the proof of the theorem for functions from H^p with $p \geq 2$ follows the lines of [12] with applying in the needed places Lemma 2.

Denote the unit ball of H^p by $B_1(H^p)$ and its subset of nonvanishing functions by $B_1^0(H^p)$. It will be convenient to regard the free coefficients $c_0(f)$ also as elements of $B_1^0(H^p)$, which are constant on the disk $\mathbb{D}$. Let

$$\widehat{B}_1^0(H^p) = B_1^0(H^p) \cup \{f_0\},$$

where $f_0(z) \equiv 0$.

We shall essentially use Brown's result quoted above and present it as

Lemma 2. [5] *For any* $f(z) = c_0 + c_1 z + c_2 z^2 + \dots \in B_1^0(H^p)$, *we have*

$$|c_1| \leq (2/e)^{1-1/p},$$

with equality only for the rotations of function $\kappa_1(z)$ *given by* (2).

The following important lemma concerns one of the basic intrinsic features of nonvanishing holomorphic functions (the openness)

Lemma 3. [12] *Every point* $f \in B_1^0(H^p)$ *has a neighborhood* $U(f, \epsilon)$ *in* H^p, *which entirely belongs to* $B_1^0(H^p)$, *i.e., contains only nonvanishing* H^p *functions on the disk* $\mathbb{D}$. *Take the maximal balls* $U(f, \epsilon)$ *with such property. Then their union*

$$\mathcal{U}^p = \bigcup_{f \in B_1^0(H^p)} U(f, \epsilon)$$

is an open path-wise connective set, hence a domain, in the space $\widehat{B}_1^0(H^p)$.

Let $\mathcal{P}_n$ be the linear space of polynomials of degree less than or equal to n, and $\mathcal{P} = \bigcup_n \mathcal{P}_n$.

Lemma 4. *The intersection $\mathcal{U}^p \bigcap \mathcal{P}$ is dense in $\mathcal{U}^p$, which means that any f from the distinguished domain $\mathcal{U}^p$ is approximated in H^p by nonvanishing polynomials.*

We shall use in the proof of Theorem 1 somewhat different (up to a biholomorphic homeomorphism) model of the universal Teichmüller space $\mathbf{T}$, which involves quasiconformally extendable univalent functions in the disk satisfying some non-standard prescribed normalization conditions. Their existence of such maps is ensured by the following lemma related to solutions of the Beltrami equation $\partial_{\overline{z}} w = \mu(z)\partial_z w$ on $\mathbb{C}$ with coefficients μ supported in the disk $\mathbb{D}^*$, i.e., from the ball

$$\mathrm{Belt}(\mathbb{D}^*)_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu|\mathbb{D} = 0, \ \|\mu\| < 1\}.$$

Lemma 5. [12] *For any Beltrami coefficient $\mu \in \mathrm{Belt}(\mathbb{D}^*)_1$ and any $\theta_0 \in [0, 2\pi]$, there exists a point $z_0 = e^{i\alpha}$ located on $\mathbb{S}^1$ so that $|e^{i\theta_0} - e^{i\alpha}| < 1$ and such that for any θ satisfying $|e^{i\theta} - e^{i\alpha}| < 1$ the equation $\partial_{\overline{z}} w = \mu(z)\partial_z w$ has a unique homeomorphic solution $w = w^\mu(z)$, which is holomorphic on the unit disk $\mathbb{D}$ and satisfies*

$$w(0) = 0, \quad w'(0) = e^{i\theta}, \quad w(z_0) = z_0. \tag{18}$$

This solution is holomorphic on the unit disk $\mathbb{D}$, and hence, $w^\mu(z_*) = \infty$ at some point z_* with $|z_*| \geq 1$.

Step 2: Holomorphic embedding of nonvanishing H^p functions into Teichmüller spaces and lifting the functional $J_n(f) = c_n$. Denote by $\mathbf{B} = \mathbf{B}(\mathbb{D})$ the space of hyperbolically bounded holomorphic functions $\varphi(z)$ (regarded as holomorphic quadratic differentials $\varphi(z)dz^2$ so that $\varphi \circ h(z)h'(z)^2 = \varphi(z)$ for any conformal coordinate map h) on the unit disk, with norm

$$\|\varphi\|_{\mathbf{B}} = \sup_{\mathbb{D}} (1 - |z|^2)^2 |f(z)|.$$

Every $\varphi \in \mathbf{B}$ is the **Schwarzian derivative**

$$S_w(z) = \left(\frac{w''(z)}{w'(z)}\right)' - \frac{1}{2}\left(\frac{w''(z)}{w'(z)}\right)^2, \quad z \in \mathbb{D},$$

of a locally univalent function $w(z)$ in the disk $\mathbb{D}$ determined (up to a Moebius map of the sphere $\widehat{\mathbb{C}}$) from the nonlinear differential equation

$$w'''/w' - 3(w''/w')^2/2 = \varphi,$$

or equivalently, as the ratio $w = \eta_2/\eta_1$ of two linearly independent solutions of the linear equation $2\eta'' + \varphi\eta = 0$ in $\mathbb{D}$.

The space $\mathbf{B}$ is dual to the space $A_1(\mathbb{D})$ of integrable holomorphic functions on $\mathbb{D}$ with L_1 norm.

The Schwarzians S_w of functions w univalent in the whole disk $\mathbb{D}$ and having quasiconformal extensions to $\widehat{\mathbb{C}}$ fill a path-wise bounded domain in $\mathbf{B}$; this domain is the most applicable model of the **universal Teichmüller space** $\mathbf{T}$ = Teich($\mathbb{D}$) (with appropriate normalization of maps w).

Let $A_p(\mathbb{D}), p \geq 1$, be the Bergman spaces of holomorphic functions in $\mathbb{D}$ with norm

$$\|f\|_{A_p} = \Big(\frac{1}{\pi} \iint_{\mathbb{D}} |f(z)| dxdy\Big)^{1/p} \quad (z = x + iy).$$

For each $f \in H^p$, we have $\|f\|_{A_p}^p \leq \frac{1}{2}\|f\|_{H^p}^p$, which yields, since $A_p(\mathbb{D}) \subset A_1(\mathbb{D}) \subset \mathbf{B}$ and $\|f\|_{\mathbf{B}} \leq \|f\|_{A_1(\mathbb{D})}$ for $\varphi \in A_1(\mathbb{D})$, that all functions $f \in H^p$ belong to the space $\mathbf{B}$. Therefore, these functions can be regarded as the Schwarzian derivatives of locally univalent functions in $\mathbb{D}$.

In particular, the functions f from the ball

$$B_\rho(H^p) = \{f \in H^p : \ \|f\| < \rho\}$$

with radius $\rho = 1/2^{1/p}$ satisfy $\|f\|_{\mathbf{B}} < 2$, and hence are the Schwarzians of univalent functions in the whole disk $\mathbb{D}$ admitting quasiconformal extension to the complementary disk

$$\mathbb{D}^* = \{z \in \widehat{\mathbb{C}} : \quad |z| > 1\}.$$

Therefore, such f are the points of the universal Teichmüller space $\mathbf{T}$. This implies a *holomorphic embedding* ι of the ball $B_\rho(H^p)$ and of its open subset

$$\frac{1}{2^p}\mathcal{U}^p = \{\frac{1}{2^p} f : \ f \in \mathcal{U}^p\}$$

into the space $\mathbf{T}$.

Now consider the family $\widehat{S}(1)$ of quasiconformally extendable to $\widehat{\mathbb{C}}$ holomorphic univalent functions

$$w(z) = a_1 z + a_2 z^2 + \dots, \quad z \in \mathbb{D},$$

with $|a_1| = 1$ and $w(z_0) = z_0$ for some point $z_0 \in \mathbb{S}^1$ (depending on w), completed in the topology of locally uniform convergence on $\mathbb{C}$. This collection is a disjunct union

$$\widehat{S}(1) = \bigcup_{-\pi \leq \theta < \pi} S_\theta,$$

where S_θ consists of quasiconformally extendable univalent functions on $\mathbb{D}$ with expansions

$$w(z) = e^{i\theta} z + a_2 z^2 + \dots$$

having a fixed point $z_0 \in \mathbb{S}^1$ (also completed in the indicated weak topology). These collections preserve conjugation with rotations $z \mapsto e^{i\alpha} z$, i.e., contain for each w the rotated functions $w_{\alpha,\alpha}(z) = e^{-i\alpha} w(e^{i\alpha} z)$. There is often enough to deal with the class S_0 related to $z_0 = 1$.

The assertion of Lemma 5 is also valid for the limit functions of sequences $\{w_n\}$ of functions $w_n \in \widehat{S}(1)$ with quasiconformal extension, but in the general case the equality $w(z_0) = z_0$ must be understand in terms of the Carathéodory prime ends. As was indicated above, any function from $\widehat{S}(1)$ with θ chosen following Lemma 5 is holomorphic on the disk $\mathbb{D}$ (has there no pole).

This family $\widehat{S}(1)$ is closely related to the canonical class S of univalent functions $w(z)$ on $\mathbb{D}$ normalized by $w(0) = 0,\ w'(0) = 1$. Every $w(z) \in S$ has its representatives $w_{\tau,\theta}$ in $\widehat{S}(1)$ obtained by pre and post compositions of w with rotations $z \mapsto e^{i\tau} z$ about the origin, related by

$$w_{\tau,\theta}(z) = e^{-i\theta} w(e^{i\tau} z) \quad \text{with} \quad \tau = \arg z_0, \tag{19}$$

where z_0 is a point of the circle $\mathbb{S}^1$ whose image $w(z_0) = e^{i\theta}$ is a common point of the unit circle and the boundary of domain $w(\mathbb{D})$.

This is trivial for the identity map $w(z) \equiv z$ (then one can take $\theta = \tau = 0$). For any another $w(z)$ the existence of such a point z_0 follows from the Schwarz lemma, which yields, together with the assumption $w'(0) = 1$, that the image $w(\mathbb{D})$ cannot lie entirely in $\mathbb{D}$; hence, its boundary $\partial w(\mathbb{D})$ has common points with the circle $\mathbb{S}^1$.

This connection also implies that the functions conformal in the closed disk $\overline{\mathbb{D}}$ are dense in each class S_θ. Note also that the classes S_θ and $\widehat{S}(1)$ are compact in the topology of locally uniform convergence on $\mathbb{D}$.

The Schwarzian derivatives of w and $w_{\tau,\theta}$ are related by

$$S_{w_{\tau,\theta}}(z) = S_w(e^{i\tau} z) e^{2i\tau},$$

which yields that *for any fixed θ the Schwarzians S_w of $w \in S_\theta$ fill the same bounded domain in the space* $\mathbf{B}$*, which models* $\mathbf{T}$.

In other words, the relation (19) allows us to model the universal Teichmüller space $\mathbf{T}$ for any fixed θ by the Schwarzians $S_w = \varphi$ of functions $w(z) = e^{i\theta} z + a_2 z^2 + \dots$ from the sets S_θ.

In this case, going to the limit $\lim\limits_{t \to 0} \|S_w(te^{i\alpha} z)\|_{\mathbf{B}} \to 0$ along a curve $\{S_w(te^{i\alpha} z) :\ 0 \le t \le 1\}$ with fixed nonzero θ and α one attains in the space $\mathbf{T}$ its base point $\varphi = \mathbf{0}$, and the corresponding function in S_θ is the elliptic fractional linear transformation

$$w = \frac{e^{i\theta} z}{(1 - e^{-i\theta}) z_0^{-1} z + 1}$$

with fixed points 0 and $z_0 = e^{i\alpha}$. For $\alpha = \theta = 0$, this is the identity map.

Note also that the relation (19) is compatible with existence and uniqueness of appropriate conformal and quasiconformal maps, holomorphy of their Taylor coefficients, the Teichmüller space theory, etc. Actually we deal with the classical model of Teichmüller spaces via domain in the Banach spaces of Schwarzian dervatives S_w in $\mathbb{D}$ (or in the disk $\mathbb{D}^*$) of univalent holomorphic functions normalized either by fixing three boundary points on the unit circle S^1 or via $w(0) = 0, \ w'(0) = 1, \ w(\infty) = \infty$ (often the disk is replaced by the half-plane).

An equivalent model of $\mathbf{T}$ is obtained by applying the inverted functions $W(z) = 1/w(1/z)$ for $w \in S_\theta$, which form the corresponding classes Σ_θ of nonvanishing univalent functions on the disk $\mathbb{D}^*$ with expansions

$$W(z) = e^{-i\theta} z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots, \quad W(1/\alpha) = 1/\alpha,$$

and $\widehat{\Sigma}(1) = \bigcup_\theta \Sigma_\theta$.

Simple computations yield that the coefficients a_n of $f \in S_\theta$ and the corresponding coefficients b_j of $W(z) = 1/f(1/z) \in \Sigma_\theta$ are related by

$$b_0 + e^{2i\theta} a_2 = 0, \quad b_n + \sum_{j=1}^{n} \epsilon_{n,j} b_{n-j} a_{j+1} + \epsilon_{n+2,0} a_{n+2} = 0, \quad n = 1, 2, \dots ,$$

where $\epsilon_{n,j}$ are the entire powers of $e^{i\theta}$ (θ is fixed). This successively implies the representations of a_n by b_j via

$$a_n = (-1)^{n-1} \epsilon_{n-1,0} b_0^{n-1} - (-1)^{n-1} (n-2) \epsilon_{1,n-3} b_1 b_0^{n-3}$$
$$+ \text{lower terms with respect to } b_0. \tag{20}$$

By abuse of notation, we shall denote the holomorphic embedding of H^p into the space $\mathbf{T}$ modelled by Schwarzians in $\mathbb{D}^*$ by the same letter ι. The image ιH^p is a non-complete linear subspace in $\mathbf{B}$, and the image of the distinguished domain $\frac{1}{2^p}\mathcal{U}^p$ is a complex submanifold in $\mathbf{T}$.

Note that the coefficients α_n of Schwarzians

$$S_w(z) = \sum_0^\infty \alpha_n z^n$$

are represented as polynomials of $n+2$ initial coefficients of $w \in S_\theta$ and, in view of (20), as polynomials of $n+1$ initial coefficients of the corresponding $W \in \Sigma_\theta$ (provided that θ and α are given and fixed and the number $e^{i\theta}$ is considered to be a constant).

We denote these polynomials by $J_n(w)$ and $\widetilde{J}_n(W)$, respectively, and will deal with these polynomial functionals only on the union of admissible classes S_θ or Σ_θ.

Step 3: Lifting to covering space $\mathbf{T}_1$ and estimating the restricted plurisubharmonic functional. Our next step is to lift both polynomial functionals $J_n(w)$ and $\widetilde{J}_n(W)$ onto the Teichmüller space $\mathbf{T}_1$ of the punctured disk $\mathbb{D}_* = \mathbb{D} \setminus \{0\}$, which covers $\mathbf{T}$.

Recall that the points of $\mathbf{T}_1$ are the classes $[\mu]_{\mathbf{T}_1}$ of **$\mathbf{T}_1$-equivalent** Beltrami coefficients $\mu \in \mathrm{Belt}(\mathbb{D})_1$ so that the corresponding quasiconformal automorphisms w^μ of the unit disk coincide on both boundary components (unit circle $\mathbb{S}^1$ and the puncture $z=0$) and are homotopic on $\mathbb{D} \setminus \{0\}$. This space also is a complex Banach manifold.

Due to the Bers isomorphism theorem [4], the space $\mathbf{T}_1$ is biholomorphically isomorphic to the **Bers fiber space**

$$\mathcal{F}(\mathbf{T}) = \{(\phi_{\mathbf{T}}(\mu), z) \in \mathbf{T} \times \mathbb{C} : \ \mu \in \mathrm{Belt}(\mathbb{D})_1, \ z \in w^\mu(\mathbb{D})\}$$

over the universal space $\mathbf{T}$ with holomorphic projection $\pi(\psi, z) = \psi$. This fiber space is a bounded hyperbolic domain in $\mathbf{B} \times \mathbb{C}$ and represents the collection of domains $D_\mu = w^\mu(\mathbb{D})$ as a holomorphic family over the space $\mathbf{T}$.

The indicated isomorphism between $\mathbf{T}_1$ and $\mathcal{F}(\mathbf{T})$ is induced by the inclusion map $j: \ \mathbb{D}_* \hookrightarrow \mathbb{D}$ forgetting the puncture at the origin via

$$\mu \mapsto (S_{w^{\mu_1}}, w^{\mu_1}(0)) \quad \text{with} \ \ \mu_1 = j_*\mu := (\mu \circ j_0)\overline{j_0'}/j_0', \tag{21}$$

where j_0 is the lift of j to $\mathbb{D}$.

Now, letting

$$\widehat{J}_n(\mu) = \widetilde{J}_n(W^\mu), \tag{22}$$

we lift these functionals from the sets S_θ and Σ_θ onto the ball $\mathrm{Belt}(\mathbb{D})_1$. Then, under the indicated $\mathbf{T}_1$-equivalence, i.e., by the quotient map

$$\phi_{\mathbf{T}_1} : \ \mathrm{Belt}(\mathbb{D})_1 \to \mathbf{T}_1, \quad \mu \to [\mu]_{\mathbf{T}_1},$$

the functional $\widetilde{J}_n(W^\mu)$ is pushed down to a bounded holomorphic functional $\mathcal{J}_n$ on the space $\mathbf{T}_1$ with the same range domain.

Equivalently, one can apply the quotient map $\mathrm{Belt}(\mathbb{D})_1 \to \mathbf{T}$ (i.e., $\mathbf{T}$-equivalence) and compose the descended functional on $\mathbf{T}$ with the natural holomorphic map $\iota_1 : \mathbf{T}_1 \to \mathbf{T}$ generated by the inclusion $\mathbb{D}_* \hookrightarrow \mathbb{D}$ forgetting the puncture. Note that since the coefficients $b_0, b_1, \dots$ of $W^\mu \in \Sigma_\theta$ are uniquely determined by its Schwarzian S_{W^μ}, the values of $\mathcal{J}_n$ in the points $X_1, X_2 \in \mathbf{T}_1$ with $\iota_1(X_1) = \iota_1(X_2)$ are equal.

Using the Bers isomorphism theorem, we regard the points of the space $\mathbf{T}_1$ as the pairs $X_{W^\mu} = (S_{W^\mu}, W^\mu(0))$, where $\mu \in \mathrm{Belt}(\mathbb{D})_1$ obey $\mathbf{T}_1$-equivalence (hence, also $\mathbf{T}$-equivalence). Denote (for simplicity of notations) the composition of $\mathcal{J}_n$ with biholomorphism $\mathbf{T}_1 \cong \mathcal{F}(\mathbf{T})$ again by $\mathcal{J}_n$. In view of (20) and (21), it is presented on the fiber space $\mathcal{F}(\mathbf{T})$ by

$$\mathcal{J}(X_{W^\mu}) = \mathcal{J}(S_{W^\mu},\ t), \quad t = W^\mu(0). \tag{23}$$

This yields a logarithmically plurisubharmonic functional $|\mathcal{J}_n(S_{W^\mu}, t)|$ on $\mathcal{F}(\mathbf{T})$.

We have to estimate a smaller plurisubharmonic functional arising after restriction of $\mathcal{J}(S_{W^\mu},\ t)$ to $S_W \in \iota\Big(\frac{1}{2^p}\mathcal{U}^p\Big)$ and to $W^\mu(0)$ filling some subdomain D_θ.

Since our functionals are polynomials, they are defined for all $S_W \in \mathbf{T}$ and t from some domain in $\mathbf{B} \times \mathbb{C}$ over D_θ. We define on D_θ the function

$$u_\theta(t) = \sup_{S_{W^\mu}} |\mathcal{J}_n(S_{W^\mu}, t)|,$$

where the supremum is taken over all $S_{W^\mu} \in \mathbf{T}$ admissible for a given $t = W^\mu(0) \in D_\theta$.

The following basic lemma is a generalization of the corresponding result in [12]. It provides that this function inherits subharmonicity of $\mathcal{J}_n$.

Lemma 6. *The function $u_\theta(t)$ is subharmonic on its domain D_θ filled by the admissible values of $W^\mu(0)$.*

The *proof* of this lemma is complicated. Similar to [12], it involves the approximation of elements from $\frac{1}{2^p}\mathcal{U}^p$ by polynomials given by Lemma 5 which provides the finite dimensional submanifolds weakly approximating $\iota((\frac{1}{2^p}\mathcal{U}^p))$ in the underlying space $\mathbf{T}$ (and simultaneously in the space $\mathbf{T}_1$) in the topology of locally uniform convergence on $\mathbb{C}$.

Since the set $\iota((\frac{1}{2^p}\mathcal{U}^p))$ is a complex submanifold in $\mathbf{T}$, the restriction of the function $|\mathcal{J}(S_{W^\mu}, t)|$ to this submanifold and to the corresponding values of $t = W^\mu(0)$ also is plurisubharmonic. The arguments from [12] are straightforwardly extended to this restriction, giving in a similar way the corresponding maximal subharmonic function

$$u_\theta(t) = \sup_{S_{W^\mu}} |\mathcal{J}_n(S_{W^\mu}, t)|;$$

the supremum here is taken over $S_{W^\mu} \in \iota((\frac{1}{2^p}\mathcal{U}^p))$ as indicated above.

One also has to extend the previous construction to the increasing unions of the quotient spaces

$$\mathcal{T}_s = \bigcup_{j=1}^{s} \widehat{\Sigma}_{\theta_j}^0 / \sim = \bigcup_{j=1}^{s} \{(S_{W_{\theta_j}}, W_\theta^\mu(0))\} \simeq \mathbf{T}_1 \cup \cdots \cup \mathbf{T}_1, \tag{24}$$

where θ_j run over a dense subset $\Theta \subset [-\pi, \pi]$, the equivalence relation $\sim$ means $\mathbf{T}_1$-equivalence on a dense subset $\widehat{\Sigma}^0(1)$ in the union $\widehat{\Sigma}(1)$ formed by univalent functions $W_{\theta_j}(z) = e^{-i\theta_j} z + b_0 + b_1 z^{-2} + \dots$ on $\mathbb{D}^*$ with quasiconformal extension to $\widehat{\mathbb{C}}$ satisfying $W_{\theta_j}(1) = 1$, and

$$\mathbf{W}_\theta^\mu(0) := (W_{\theta_1}^{\mu_1}(0), \dots, W_{\theta_s}^{\mu_s}(0)).$$

The Beltrami coefficients $\mu_j \in \mathrm{Belt}(\mathbb{D})_1$ are chosen here independently. The corresponding collection $\beta = (\beta_1, \dots, \beta_s)$ of the Bers isomorphisms

$$\beta_j : \ \{(S_{W_{\theta_j}}, W_{\theta_j}^{\mu_j}(0))\} \to \mathcal{F}(\mathbf{T})$$

determines a holomorphic surjection of the space $\mathcal{T}_s$ onto $\mathcal{F}(\mathbf{T})$.

Taking in each union (24) the corresponding collection $\iota_s\Big(\frac{1}{2^p}\mathcal{U}^p\Big)$, one obtains in a similar fashion the increasing sequence of maximal subharmonic functions

$$u_s(t) = \sup_\Theta u_{\theta_s}(t) = \sup \Big\{|\mathcal{J}_n(S_{W^\mu}, t)| : \ S_{W^\mu} \in \bigcup_s \iota_s\Big(\frac{1}{2^p}\mathcal{U}^p\Big)\Big\},$$

whose limit

$$u(t) = \lim_{s\to\infty} u_s(t) \tag{25}$$

is determined and subharmonic on a disk

$$D_\rho = \bigcup_\Theta D_{\rho,\theta_s}, \tag{26}$$

because the union of spaces (15) admits the circular symmetry.

Step 4: Determination of the range domain of $W^\mu(0)$. Our goal now is to find the domain of admissible values of $W^\mu(0)$, i.e. the radius of the disk (26). This requires a covering estimate of Koebe's type given by the following lemma.

Let G be a domain in a complex Banach space $X = \{\mathbf{x}\}$ and χ be a holomorphic map from G into the universal Teichmüller space $\mathbf{T}$

modeled as a bounded subdomain of $\mathbf{B}$. Consider in the unit disk the corresponding Schwarzian differential equations

$$S_w(z) = \chi(\mathbf{x}) \tag{27}$$

and pick their univalent solutions $w(z)$ satisfying $w(0) = w'(0) - 1 = 0$ (hence $w(z) = z + \sum_2^\infty a_n z^n$). Set

$$|a_2^0| = \sup\{|a_2| : \ S_w \in \chi(G)\}, \tag{28}$$

and let

$$w_0(z) = z + a_2^0 z^2 + \dots$$

be one of the maximizing functions for a_2.

Lemma 7. [11] *(a) For every indicated solution $w(z) = z + a_2 + \dots$ of the differential equation* (27)*, the image domain $w(\mathbb{D})$ covers entirely the disk $\{|w| < 1/(2|a_2^0|)\}$.*

The radius value $1/(2|a_2^0|)$ is sharp for this collection of functions, and the circle $\{|w| = 1/(2|a_2^0|)$ contains points not belonging to $w(\mathbb{D})$ if and only if $|a_2| = |a_2^0|$ (i.e., when w is one of the maximizing functions).

(b) The inverted functions

$$W(\zeta) = 1/w(1/\zeta) = \zeta - a_2^0 + b_1\zeta^{-1} + b_2\zeta^{-2} + \dots$$

map the disk $\mathbb{D}^*$ onto a domain whose boundary is entirely contained in the disk $\{|W + a_2^0| \le |a_2^0|\}$.

Now we show that in the case of nonvanishing H^p functions this radius $1/(2|a_2^0|)$ is naturally connected with the extremal function $\kappa_{1,p}(z)$ maximizing the coefficient $|c_1|$.

Consider the collection $\mathcal{N}_p$ $(p > 1)$ of all nonvanishing H^p functions located in the ball $\{\|\varphi\| < 2\}$ in $\mathbf{B}$ and denote the minimal radius of the balls in H^p containing these functions by $r(p)$; that is

$$r(p) = \sup\{\|f\|_p : \ \|f\|_{\mathbf{B}} \le 2, \ f(z) \ne 0 \text{ in } \mathbb{D}\}.$$

For any such f, the solutions $w(z)$ of the equation $S_w = f$ are univalent holomorphic functions on the disk $\mathbb{D}$. The set $\frac{1}{2^{1/p}}\mathcal{U}^p$ applied earlier is a proper subset of $\mathcal{N}_p$.

Lemma 8. *For any space H^p, $p > 1$, and its subset $\mathcal{N}_p$, we have the equality*

$$S_{w_0}(z) = r(p)\kappa_{1,p}(z) \tag{29}$$

which means that the Schwarzian of the extremal univalent function $w_0(z)$ maximizing the second coefficient a_2 on the set $\mathcal{N}_p$ equals the extremal function for c_1 (hence, the maximizing function for (28) *also is unique).*

Proof. In view of Lemma 2, it is enough to establish that

$$S'_{w_0}(0) = c_1^0 \neq 0 \tag{30}$$

(in other words, that the zero set of the functional $J_1(f) = c_1$ is separated from the set of rotations (19) of the function w_0). This yields that the maximal function (25) for the functional $|J_1(f)| = |c_1|$ is defined on the whole disk $\mathbb{D}_{1/(2|a_2^0|)}$, attaining its maximum on the boundary circle.

We pass to intersections

$$B_{1,M}^0(H^p) = B_1^0(H^p) \cap \{f \in L_\infty(\mathbb{D}) : \ \|f\|_\infty < M\},$$

with $M < \infty$, getting the corresponding subharmonic functions

$$u_M(t) = \sup\{|\mathcal{J}(S_{W^\mu}, t)| : \ S_{W^\mu} \in \iota\Big(\frac{1}{2^p}\mathcal{U}^p \cap \mathcal{P}\Big) \cap B_{1,M}^0(H^p)\}$$

with $\lim\limits_{M\to\infty} u_M(t) = u(t)$ and the points $f_M = S_{w_{0,M}}$, maximizing $|a_2|$ on these sets. The collection $\{f_M\}$ is weakly compact in H^p.

Applying Lemma 7, one obtains similar to [12] that both maximal values $|a_2^0|$ and $|c_1^0|$ are obtained on the same function

$$f_0(z) = \lim_{M\to\infty} f_M(z) = S_{w_0},$$

and the uniqueness in Lemma 2 yields that this function must coincide with $r(p)\kappa_{1,p}$. This completes the proof of Lemma 8.

Step 5: Finishing the proof. Now we can prove the assertion of the theorem. The assumption $p \geq 2$ insures that the boundary function $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ of any $f \in H^p$ admits Parseval's equality

$$1 \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta = \sum_0^\infty |c_n|^2. \tag{31}$$

In particular, for $f(z) = \kappa_{1,p}(z) = \sum\limits_0^\infty c_n^0 z^n$ we have from (2)

$$|c_1^0|^2 = (2/e)^{2(1-1/p)} = 0.5041...^{1-1/p} > 0.5041... \tag{32}$$

for all $p > 1$. Hence, by (31),

$$\sum_2^\infty |c_n^0|^2 < 0.5 < |c_1^0|^2. \tag{33}$$

Now take $n = 2$ and, letting $f_2(z) = f(z^2)$, consider on the set $B_1^0(H^p)$ the functional

$$I_2(f) = \max\,(|J_2(f)|, |J_2(f_2)|).$$

Since the correspondence $f(z) \mapsto f_2(z)$ is linear, the functional $J_2(f_2)$ is holomorphic with respect to f in H^{2m} norm and naturally extends to a holomorphic functional on the spaces $\mathbf{T}$. Hence, the functional I_2 is plurisubharmonic on $\mathbf{T}$. It lifted to the covering space $\mathbf{T}_1$ together with J_2.

Similar to above, this lifting generates via (25) a nonconstant radial subharmonic function $u_2(t)$ on the disk $\{|t| < 1/2|a_2^0|\},\ t = W^\mu(0)$. This function is logarithmically convex, hence monotone increasing, and thus attains its maximal value at $|t| = 1/(2|a_2^0|)$.

Taking into account the connection between the extremal value $|a_2^0|$ and the function $\kappa_{1,p}$ established by Lemma 8, one concludes that the maximal value of $I_2(f)$ on $B_1^0(H^p)$ is attained on the pair (f, f_2) with

$$f(z) = \kappa_{1,p}(z), \quad f_2(z) = \kappa_{1,p}(z^2).$$

Since the set of admissible maps $w(z) \in \widehat{S}(1)$ with $S_w = f$ for $J_2(f_2)$ is the same as for $J_2(f)$, one derives from above

$$\max_{B_1^0(H^p)} I_2(f) = \max\,\{|c_1^0|, |c_2^0|\},$$

which by (33) is equal to $|c_1^0| = (2/e)^{1-1/p}$. This yields the desired estimate (1) for $n = 2$; the extremal maximizing function is determined up to the pre and post rotations about the origin.

Now consider subsequently for $n = 3, 4, \dots$ the functionals

$$I_n(f) = \max\,\{|J_n(f)|, |J_n(f_2)|, \dots, |J_n(f_n)|\} \tag{34}$$

with $f_n(z) = f(z^n)$. Similar to I_2, this does not expand the set of admissible maps $w(z) \in \widehat{S}(1)$ with $S_w = f$, and therefore, $|I_n(f)|$ has the same maximum, as $|J_n(f_n)|$.

Each functional I_n generates similar to above the corresponding circularly symmetric subharmonic function $u_n(t)$ on the disk $\{|t| < 1/2|a_2^0|\},\ t = W^\mu(0)$, which provides in the same way the bound

$$\max_{B_1^0(H^p)} I_n(f) = \max\,\{|c_1^0|, |c_n^0|\} = (2/e)^{1-1/p},$$

with a similar description of the extremal functions. This completes the proof of Theorem 1.

4. Additional Remarks

4.1. Holomorphy in parameters

As was mentioned in the proof of Theorem 1, the holomorphic dependence of normalized quasiconformal maps on complex parameters is an underlying fact for the Teichmüller space theory and for many other applications. It was first established and applied by Ahlfors and Bers in [1] for maps with three fixed points on $\widehat{\mathbb{C}}$.

Another somewhat equivalent proof of holomorphy involves the variational technique for quasiconformal maps. For the maps w from $S_{\theta,\alpha}$, this holomorphy is a consequence of the following lemma from [8, Ch. 5] combined with appropriate Möbius maps.

Lemma 9. *Let $w(z)$ be a quasiconformal map of the plane $\widehat{\mathbb{C}}$ with Beltrami coefficient $\mu(z)$ which satisfies $\|\mu\|_\infty < \varepsilon_0 < 1$ and vanishes in the disk $\{|z| < r\}$. Suppose that $w(0) = 0$, $w'(0) = 1$, and $w(1) = 1$. Then, for sufficiently small ε_0 and for $|z| \le R < r_0(\varepsilon_0, r)$ we have the variational formula*

$$w(z) = z - \frac{z^2(z-1)}{\pi} \iint\limits_{|\zeta|>r} \frac{\mu(\zeta) d\xi d\eta}{\zeta^2(\zeta-1)(\zeta-z)} + \Omega_\mu(z),$$

where $\zeta = \xi + i\eta$; $\max_{|z|\le R} |\Omega_\mu(z) \le C(\varepsilon_0, r, R)\|\mu\|_\infty^2$; $r_0(\varepsilon_0, r)$ is a well defined function of ε_0 and r such that $\lim_{\varepsilon_0\to 0} r_0(\varepsilon_0, r) = \infty$, and the constant $C(\varepsilon_0, r, R)$ depends only on ε_0, r and R.

4.2. Remarks on the case $1 < p < 2$

All arguments in the proof of Theorem 1, excluding the Parseval equality, applied in the last step, work for any $p > 1$. In fact, this equality was applied only to the function $\kappa_{1,p}$ maximizing $|c_1|$ and was used for estimation $|c_n|$ by comparison of the initial non-free coefficient of functions $\kappa_{1,p}(z^m)$, $1 \le m \le n$.

The explicit representation (2) of $\kappa_{1,p}$ shows that this function also is bounded on the unit disk for p satisfying $1 < p < 2$; hence it belongs to H^2. However, for all such p, we have

$$\|\kappa_{1,p}\|_{H^2} > \|\kappa_{1,p}\|_{H^p}.$$

Thus the needed relation (31) giving (32), (33) fails, and the functionals J_n and I_n cannot be compared on this way.

4.3. On extremal functions in Bergman spaces

One of the interesting extensions of the Hummel–Scheinberg–Zalcman problem (also still unsolved) is to estimate the Taylor coefficients of nonvanishing holomorphic maps $f(z) = c_0 + c_1 z + \dots$ of the unit disk $\mathbb{D}$ into other complex Banach spaces X. Denote by $B(X)$ the unit ball of X.

We illustrate here on the case of Bergman's space A_2 that the features of extremal functions can be essentially different from above. Recall that the norm of A_2 is $\|f\| = (\frac{1}{\pi} \iint_{\mathbb{D}} |f(z)|^2 dxdy)^{1/2}$.

The collection $B_0(A_2)$ of nonvanishing holomorphic functions $f(z)$ mapping the disk $\mathbb{D}$ into the closed ball $\overline{B(A_2)}$ (i.e., with $f(z) \neq 0$ on $\mathbb{D}$ and $\|f\| \le 1$) is compact in the weak topology of the locally uniform convergence in $\mathbb{D}$. So any holomorphic coefficient functional

$$J(f) = J(c_{m_1}, \dots, c_{m_s}) \quad \text{with} \ \ 1 \le m_1 < m_2 < \dots < m_s = N < \infty \tag{35}$$

has an extremal f_0 on which $|J(f)|$ attains its maximum on $B_0(A_2)$.

While the extremal functions of many problems in Hardy spaces are bounded, Proposition 2 implies, for example, that *any function $f_0 \in B_0(A_2)$ maximizing the functional (35) must be unbounded on the disk $\mathbb{D}$* (compare with the extremal problems for nonvanishing Bergman functions investigated e.g. in [2, 3]).

This difference is caused by the fact mentioned after the proof of Proposition 2: quasiconformal deformations created by Propositions 1 and 2 preserve the norm in A^p, while the norm of the Hardy spaces can be increased.

Indeed, it follows from Proposition 2 that any extremal f_0 of $J(f)$ on $B_0(A_2)$ must be unbounded on $\mathbb{D}$, unless f_0 is a zero-free polynomial

$$p_N(z) = c_0 + c_1 z + \dots + c_N z^N \tag{36}$$

(with $c_0 \neq 0$); otherwise, one can vary the coefficients c_k and obtain by this lemma an admissible function $f_* \in B_0(A_2)$ with $|J(f_*)| > |J(f_0)|$.

It remains to establish that the polynomials (36) with $\|p_N\|_{A_2} \le 1$ cannot be extremal for $J(f)$. We pick a sufficiently small $\varepsilon > 0$ and consider the polynomial

$$p_\varepsilon(z) = -\varepsilon c_0 + \varepsilon z^{N+1},$$

for which

$$\max_{\mathbb{S}^1} |p_\varepsilon(z)| < \max_{\mathbb{S}^1} |p_N(z)|.$$

Then the Rouché theorem yields that the polynomial

$$P_{N+1,\varepsilon}(z) = p_N(z) + p_\varepsilon(z) = (1 - \varepsilon)c_0 + c_1 z + \dots + c_N z^N + \varepsilon z^{N+1}$$

also must be, together with p_N, zero-free on $\mathbb{D}$. Its norm is estimated by

$$\|p_{N+1,\varepsilon}\|^2_{A_2} = (1-\varepsilon)^2|c_0|^2 + \frac{|c_1|^2}{2} + \cdots + \frac{|c_N|^2}{N+1} + \frac{\varepsilon^2}{N+2}$$

$$= \|p_N\|^2_{A^2} - 2\varepsilon + O(\varepsilon^2) < \|p_N\|^2_{A^2} = 1,$$

which implies that $P_{N+1,\varepsilon}$ is an admissible function, with

$$|J(p_{N+1,\varepsilon})| = |J(p_N)| = \max\{|J(f)| : \ f \in B_0(A^2)\}. \tag{37}$$

But this contradicts to Proposition 2, because this proposition allows one to construct the variations of $p_{N+1,\varepsilon}$, which preserve its A^2-norm and increase $|J(p_{N+1,\varepsilon})|$, disturbing (37). This completes the proof of our claim.

Note also that the Parseval equality for functions $f(z) = a_0 + c_1 z + c_2 z^2 + \cdots \in A^2$ states

$$\frac{1}{\pi}\iint\limits_{\mathbb{D}} |f(z)|^2 dxdy = \sum_0^\infty |a_n|^2 = \sum_0^\infty \frac{|c_n|^2}{n+1},$$

where a_n are the Fourier coefficients of f. Hence, $a_n = o(1/\sqrt{n+1})$, and $|c_n| < 1$ for all $n \geq n_0(f)$. For these n, we have $|c_n| < |c_1|$, but in (34) all c_j with $j = 1, \ldots, n$ have been used.

4.4. Another extension of the Hummel–Scheinberg–Zalcman problem

Consider the set $B_1^0(\mathbf{B})$ of nonvanishing functions $\varphi \in \mathbf{B}$ with $\|\varphi\|_{\mathbf{B}} < 1$. As was mentioned above, any such φ is the Schwarzian derivative

$$S_w(z) = \sum_0^\infty \alpha_n z^n \quad (|z| < 1)$$

of a univalent function $w(z)$ on the disk $\mathbb{D}$. It is a holomorphic map from $\mathbb{D}$ to $\mathbf{B}$. For the even coefficients of these functions, we have a lower bound

$$\max_{\varphi \in B_1^0(\mathbf{B})} |\alpha_{2n}(\varphi)| \geq \frac{1}{6}|\alpha_{2n}(\kappa_\theta)| = n+1, \tag{38}$$

where κ_θ is the Koebe function

$$\kappa_\theta(z) = \frac{z}{(1-e^{i\theta}z)^2} = z + \sum_2^\infty n e^{in\theta} z^n$$

mapping the unit disk onto the complement of the ray $\{w = -te^{i\theta} : 1/4 \le t \le \infty\}$. Its Schwarzian derivative

$$S_{\kappa_\theta}(z) = -\frac{6}{(1 - e^{2i\theta} z^2)^2} = -6 \sum_0^\infty (n+1) e^{2in\theta} z^{2n},$$

with $\|\kappa_\theta\|_{\mathbf{B}} = 6$. This derivative is an even and zero free function in $\mathbb{D}$.

The estimate (38) shows the difference of this case from H^*. Together with the Moebius invariance of S_w, this estimate provides the lower bound for distortion at arbitrary point of $\mathbb{D}$.

The arguments applied in the proof of Lemma 3 in [12] are extended to all nonvanishing functions $\varphi \in \mathbf{B}$ and give that this collection forms a Banach submanifold. So it admits the corresponding version of Lemma 7. It does not provide an explicit expression of the extremal function for $c_1(\varphi)$ (in contrast to Lemma 2). Since $c_1(\kappa_\theta) = 0$, this function is different from const κ_θ; the same is valid at least for all odd coefficients $c_{2n-1}(\varphi)$.

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