Український математичний вісник
Том 19 **(2022)**, № 4, 541 – 583
https://doi.org/10.37069/1810-3200-2022-19-4-6

On the behavior of Orlicz–Sobolev mappings with branching on the unit sphere

Miodrag Mateljevic, Evgeny Sevost'yanov

(*Presented by V. Ryazanov*)

Abstract. We study mappings of the Orlicz–Sobolev classes with a branching defined in the unit ball of the Euclidean space. We have obtained estimates of the distortion of the distance under these mappings at the points of the unit sphere. Under some conditions we also have obtained the Hölder continuity of the mappings mentioned above. If we suppose that considered mappings are solutions to certain Laplacian-gradient inequalities, we get Lipschitz property. In section 7–9 we review some results and prove a new result, Theorem 7.1 and outline proof of Theorem 9.2.

2010 MSC. 30C65, 31A15, 31B25.

Key words and phrases. Quasiconformal mappings, Hölder and Lipschitz continuity, harmonic mappings.

1. Introduction

In our previous publication [19], we discussed the question of Hölder continuity of homeomorphisms in Orlicz–Sobolev classes of the unit ball. In particular, it was proved that the corresponding mappings are locally Hölder continuous if their inner dilatations has bounded integral means over the infinitesimal spheres. In this article, we will enhance the results obtained in several directions at once. First of all, we will consider the case when the dilatations of mappings has an arbitrary order $n-1 < p \leqslant n$. In addition, the principal attention is paid to mappings with branching, the study of which differs significantly from the already mentioned case of homeomorphisms. Finally, in this article we do not limit ourselves to studying only the continuity of mappings in the sense of Lipschitz and/or Hölder, as it was before in [19]. The principal object of study is the order of growth of mappings at a fixed point, which can turn out

Received 21.10.2022

to be Hölder continuous, logarithmic Hölder continuous, or even more generally, described by some general expression defined by the dilatations of mappings.

Throughout this manuscript, D denotes a domain in $\mathbb{R}^n$, $n \geqslant 2$. Let $x_0 \in \mathbb{R}^n$, $x_0 \neq \infty$,

$$B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}\,, \quad \mathbb{B}^n := B(0,1)\,,$$

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}\,, \quad S_i = S(x_0, r_i)\,, \quad i = 1, 2\,, \tag{1.1}$$

$$\mathbb{S}^{n-1} := S(0,1)\,,$$

$$A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}\,. \tag{1.2}$$

We assume that the reader is familiar with the definitions of Sobolev classes $W^{1,1}_{\rm loc}$ and some of their basic properties, see, for example, [30, 2.I]. Here only recall if $f : D \to \mathbb{R}^m$ has ACL (absolutely continuous on lines) property on D we write that $f \in ACL(D)$.

We write $f \in W^{1,\varphi}_{\rm loc}(D)$ for a locally integrable vector-function $f = (f_1, \ldots, f_m)$ of n real variables $x_1, \ldots, x_n$ if $f_i \in W^{1,1}_{\rm loc}$ and

$$\int\limits_{D^*} \varphi\left(|\nabla f(x)|\right)\, dm(x) < \infty \tag{1.3}$$

for every subdomain D^* with a compact closure, where

$$|\nabla f(x)| = \sqrt{\sum_{i,j} \left(\frac{\partial f_i}{\partial x_j}\right)^2}\,.$$

If additionally $f \in W^{1,1}(D)$ and

$$\int\limits_{D} \varphi\left(|\nabla f(x)|\right)\, dm(x) < \infty\,, \tag{1.4}$$

we write $f \in W^{1,\varphi}(D)$. For a mapping $f : D \to \mathbb{R}^n$ having partial derivatives almost everywhere in D, we set

$$J(x, f) := \det f'(x), \quad l\left(f'(x)\right) = \min_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|f'(x)h|}{|h|} \tag{1.5}$$

for the Jacobian and smallest distortion respectively. Fix $\alpha \geqslant 1$. We define the *inner* dilatation of the mapping f at a point x of the order α by the relation

$$K_{I,\alpha}(x, f) = \begin{cases} \frac{|J(x,f)|}{l(f'(x))^{\alpha}}, & J(x,f) \neq 0, \\ 1, & f'(x) = 0, \\ \infty, & \text{otherwise} \end{cases}\,. \tag{1.6}$$

Given a mapping $f : D \to \mathbb{R}^n$, a set $E \subset D$ and $y \in \mathbb{R}^n$, we define the *multiplicity function* $N(y, f, E)$ as a number of preimages of the point y in a set E, i.e.

$$N(y, f, E) \,=\, \text{card}\,\{x \in E : f(x) = y\}\ ,$$

$$N(f, E) \,=\, \sup_{y\in\mathbb{R}^n} N(y, f, E)\,. \tag{1.7}$$

Note that, the concept of a multiplicity function may also be extended to sets belonging to the closure of a given domain. Indeed, given a set $G \subset \overline{D}$ we set

$$N(y, f, G) = \text{card}\,\{x \in G : \exists\ x_k \in D,\ x_k \to x : f(x_k) \to y,\ k \to \infty\}\ .$$

In this case, the function $N(f, G)$ may be defined similarly to (1.7).

Let h be a chordal metric in $\overline{\mathbb{R}^n}$,

$$h(x,\infty) = \frac{1}{\sqrt{1+|x|^2}},\quad h(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}\,,\qquad x \ne \infty \ne y\,,$$

and let $h(E) := \sup\limits_{x,y\in E} h(x, y)$ be a chordal diameter of a set $E \subset \overline{\mathbb{R}^n}$ (see, e.g., [41, Definition 12.1]). Let X and Y be metric spaces. A mapping $f : X \to Y$ is *discrete* if $f^{-1}(y)$ is discrete for all $y \in Y$ and f is *open* if f maps open sets onto open sets. A mapping $f : X \to Y$ is called *closed* if $f(A)$ is closed in $f(X)$ whenever A is closed in X. As usual, put

$$\|f'(x)\| \,=\, \max_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f'(x)h|}{|h|}\,. \tag{1.8}$$

Recall that a mapping f between domains D and D' in $\mathbb{R}^n$, $n \geqslant 2$, is of *finite distortion* if $f \in W^{1,1}_{\rm loc}$ and $\|f'(x)\|^n \leqslant K(x)J(x, f)$ for a.e. $x \in D$ and some finite function $K(x) < \infty$.

We say that a function $\varphi : D \to \mathbb{R}$ has a *finite mean oscillation* at a point $x_0 \in D$, write $\varphi \in FMO(x_0)$, if

$$\limsup_{\varepsilon\to 0} \frac{1}{\Omega_n\varepsilon^n} \int\limits_{B(x_0,\,\varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon|\ dm(x) < \infty\,,$$

where $\overline{\varphi}_\varepsilon = \frac{1}{\Omega_n\varepsilon^n} \int\limits_{B(x_0,\,\varepsilon)} \varphi(x)\, dm(x)$. We also say that a function $\varphi : D \to \mathbb{R}$ has a finite mean oscillation at $A \subset \overline{D}$, write $\varphi \in FMO(A)$, if φ has a finite mean oscillation at any point $x_0 \in A$.

Given $n \geqslant 3$, a Lebesgue measurable function $Q : \mathbb{R}^n \to [0,\infty]$, $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus \mathbb{B}^n$, a nondecreasing function $\varphi : (0,\infty) \to [0,\infty)$ and numbers $R > 1$, $m > 0$, $N \in \mathbb{N}$, $n-1 < \alpha \leqslant n$ denote by $F^{\varphi}_{Q,R,m,N,\alpha}(\mathbb{B}^n)$ the family of all open discrete and closed mappings f with a finite distortion of $\mathbb{B}^n$ onto $\mathbb{B}^n$ of the class $W^{1,\varphi}(\mathbb{B}^n)$ such that $N(f,\overline{\mathbb{B}^n}) = N(f,\mathbb{B}^n) \leqslant N$, $K_{I,\alpha}(x,f) \leqslant Q(x)$ a.a. in $\mathbb{B}^n$ and $f(\Psi(y)) \geqslant m$ for $\{1 < |y| < R\}$, where $\Psi(y) := \frac{y}{|y|^2}$. The following results hold.

Theorem 1.1. *Let $n \geqslant 3$ and $\alpha < n-1 \leqslant n$, and let $Q \in FMO(x_0)$ for any $x_0 \in \mathbb{S}^{n-1}$. Suppose that, a function $\varphi : (0,\infty) \to [0,\infty)$ satisfies Calderon's condition*

$$\int\limits_1^{\infty} \left(\frac{t}{\varphi(t)}\right)^{\frac{1}{n-2}} dt < \infty \tag{1.9}$$

and, in addition, there exist constants $C > 0$ and $T > 0$ such that

$$\varphi(2t) \leqslant C \cdot \varphi(t) \ \forall\ t \geqslant T\,. \tag{1.10}$$

Then any mapping $f \in F^{\varphi}_{Q,R,m,N,\alpha}(\mathbb{B}^n)$ has a continuous extension $\overline{f} : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$ such that $\overline{f}(\overline{\mathbb{B}^n}) = \overline{\mathbb{B}^n}$. In addition,

I. *If $\alpha = n$, then, for any $x_0 \in \mathbb{S}^{n-1}$, there are constants $\varepsilon_0 = \varepsilon_0(x_0) > 0$ and $0 < {\varepsilon_0}' = {\varepsilon_0}'(x_0) < \varepsilon_0$ such that the relation*

$$h(\overline{f}(x), \overline{f}(x_0)) \leqslant C_n \cdot \left(\frac{\log \frac{1}{\varepsilon_0}}{\log \frac{1}{|x-x_0|}}\right)^{\beta_n} \tag{1.11}$$

holds for any $x \in B(x_0, {\varepsilon_0}') \cap \overline{\mathbb{B}^n}$ and any $\overline{f} \in F^{\varphi}_{Q,R,m,N,n}(\overline{\mathbb{B}^n})$, where $F^{\varphi}_{Q,R,m,N,n}(\overline{\mathbb{B}^n})$ denotes the family of all extended mappings $\overline{f} : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$, $C_n > 0$ depends only on n, R and m, and $\beta_n > 0$ depends only n, R, m, and N.

II. *If $n-1 < \alpha \leqslant n$, then, for any $x_0 \in \mathbb{S}^{n-1}$, there are constants $\varepsilon_0 = \varepsilon_0(x_0) > 0$ and $0 < {\varepsilon_0}' = {\varepsilon_0}'(x_0) < \varepsilon_0$ such that the relation*

$$|\overline{f}(x) - \overline{f}(x_0)| \leqslant C_n \cdot \left(\log \frac{\log \frac{1}{|x-x_0|}}{\log \frac{1}{\varepsilon_0}}\right)^{\frac{(1-\alpha)(n-1)}{\alpha}} \tag{1.12}$$

holds for every $x \in B(x_0, {\varepsilon_0}') \cap \overline{\mathbb{B}^n}$ and any $\overline{f} \in F^{\varphi}_{Q,R,m,N,\alpha}(\overline{\mathbb{B}^n})$, where $F^{\varphi}_{Q,R,m,N,\alpha}(\overline{\mathbb{B}^n})$ denotes the family of all extended mappings $\overline{f} : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$, $C_n > 0$ is a constant depending only on n, α, R, m, N.

Given a Lebesgue measurable function $Q:\mathbb{R}^n\rightarrow[0,\infty]$, we set

$$q^*_{x_0}(r):=\frac{1}{\omega_{n-1}r^{n-1}}\int\limits_{|x-x_0|=r}Q^*(x)\,d\mathcal{H}^{n-1}\,, \tag{1.13}$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, and Q^* is defined by the equality

$$Q^*(x)=\begin{cases}Q(x), & x\in\mathbb{B}^n\,,\\ Q\left(\frac{x}{|x|^2}\right)\,, & x\in\mathbb{R}^n\setminus\mathbb{B}^n\end{cases}. \tag{1.14}$$

Theorem 1.2. *Let $n\geqslant 3$ and let $\alpha<n-1\leqslant n$. Assume that, for any $x_0\in\mathbb{S}^{n-1}$ there exists $\varepsilon_0=\varepsilon_0(x_0)>0$ such that for sufficiently small $\varepsilon>0$ the relations*

$$\int\limits_{\varepsilon}^{\varepsilon_0}\frac{dt}{t^{\frac{n-1}{\alpha-1}}q_{x_0}^{*\frac{1}{\alpha-1}}(t)}<\infty \tag{1.15}$$

and

$$\int\limits_{0}^{\varepsilon_0}\frac{dt}{t^{\frac{n-1}{\alpha-1}}q_{x_0}^{*\frac{1}{\alpha-1}}(t)}=\infty \tag{1.16}$$

hold. Suppose also that a function $\varphi:(0,\infty)\rightarrow[0,\infty)$ satisfies Calderon's condition (1.9) and, in addition, there exist constants $C>0$ and $T>0$ such that the relation (1.10) holds. Then any mapping $f\in F^{\varphi}_{Q,R,m,N,\alpha}(\mathbb{B}^n)$ has a continuous extension $\overline{f}:\overline{\mathbb{B}^n}\rightarrow\overline{\mathbb{B}^n}$ such that $\overline{f}(\overline{\mathbb{B}^n})=\overline{\mathbb{B}^n}$. In addition,

I. *If $\alpha=n$, then, for any $x_0\in\mathbb{S}^{n-1}$, there is $0<{\varepsilon_0}'={\varepsilon_0}'(x_0)<\varepsilon_0$ such that the relation*

$$h(\overline{f}(x),\overline{f}(x_0))\leqslant C_n\cdot\exp\left\{-\beta_n\int\limits_{|x-x_0|}^{\varepsilon_0}\frac{dt}{tq_{x_0}^{*\frac{1}{n-1}}(t)}\right\} \tag{1.17}$$

holds for any $x\in B(x_0,{\varepsilon_0}')\cap\overline{\mathbb{B}^n}$ and any $\overline{f}\in F^{\varphi}_{Q,R,m,N,n}(\overline{\mathbb{B}^n})$, where $F^{\varphi}_{Q,R,m,N,n}(\overline{\mathbb{B}^n})$ denotes the family of all extended mappings $\overline{f}:\overline{\mathbb{B}^n}\rightarrow\overline{\mathbb{B}^n}$, $C_n>0$ depends only on n, R and m, and $\beta_n>0$ depends only n, R, m, and N.

II. *If $n-1<\alpha\leqslant n$, then, for any $x_0\in\mathbb{S}^{n-1}$, there is $0<{\varepsilon_0}'={\varepsilon_0}'(x_0)<\varepsilon_0$ such that the relation*

$$|\overline{f}(x)-\overline{f}(x_0)|\leqslant C_n'\cdot\left(\int\limits_{|x-x_0|}^{\varepsilon_0}\frac{dt}{t^{\frac{n-1}{\alpha-1}}q_{x_0}^{*\frac{1}{\alpha-1}}(t)}\right)^{\frac{(1-\alpha)(n-1)}{\alpha}} \tag{1.18}$$

holds for every $x \in B(x_0, \varepsilon_0{}') \cap \overline{\mathbb{B}^n}$ *and any* $\overline{f} \in F^{\varphi}_{Q,R,m,N,\alpha}(\overline{\mathbb{B}^n})$, *where* $F^{\varphi}_{Q,R,m,N,\alpha}(\overline{\mathbb{B}^n})$ *denotes the family of all extended mappings* $\overline{f} : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$, $C_n^{\prime} > 0$ *is a constant depending only on* n, α, R, m, N.

The following simple corollary follows directly from Theorem 1.2.

Corollary 1.1. *Let us assume that under the conditions of Theorem 1.2* $\alpha = n$ *and, instead of assumptions* (1.15)–(1.16), *a stronger condition is satisfied:* $q^*_{x_0}(r) \leqslant q^* = const$ *for any* $r \in (0, \varepsilon_0)$. *Then, for any* $x_0 \in \mathbb{S}^{n-1}$, *the relation*

$$h(\overline{f}(x), \overline{f}(x_0)) \leqslant C_n \cdot \frac{1}{\varepsilon_0^{\frac{\beta_n}{q^{*1/(n-1)}}}} |x - x_0|^{\frac{\beta_n}{q^{*1/(n-1)}}}$$

holds for any $x \in B(x_0, \varepsilon_0{}') \cap \overline{\mathbb{B}^n}$ *and any* $\overline{f} \in F^{\varphi}_{Q,R,m,N,n}(\overline{\mathbb{B}^n})$, *where* $F^{\varphi}_{Q,R,m,N,n}(\overline{\mathbb{B}^n})$ *denotes the family of all extended mappings* $\overline{f} : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$, *and constants* β_n *and* C_n *are described in Theorem 1.2. Moreover, the inequality*

$$|\overline{f}(x) - \overline{f}(x_0)| \leqslant \widetilde{C_n} \cdot \frac{1}{\varepsilon_0^{\frac{\beta_n}{q^{*1/(n-1)}}}} |x - x_0|^{\frac{\beta_n}{q^{*1/(n-1)}}}$$

holds for some another constant $\widetilde{C_n} > 0$ *depending only on* n, R *and* m.

2. Preliminaries

Given $\alpha \geqslant 1$, we say that the boundary ∂D of a domain D is *strongly accessible at a point* $x_0 \in \partial D$ *with respect to* α*-modulus* if for each neighborhood U of x_0 there exist a compact set $E \subset D$, a neighborhood $V \subset U$ of x_0 and $\delta > 0$ such that

$$M_\alpha(\Gamma(E, F, D)) \geqslant \delta \tag{2.1}$$

for each continuum F in D that intersects ∂U and ∂V. When $\alpha = n$, we will usually drop the prefix in "α-modulus" when speaking about (2.1). Recall that a pair $E = (A,\, C)$, where A is an open set in $\mathbb{R}^n$, and C is a compact subset of A, is called *condenser* in $\mathbb{R}^n$. The quantity

$$\operatorname{cap}_\alpha E \quad = \quad \operatorname{cap}_\alpha (A,\, C) = \inf_{u \in W_0(E)} \int\limits_A |\nabla u|^\alpha \, dm(x)\,, \tag{2.2}$$

where $W_0(E) = W_0\,(A,\, C)$ is a family of all nonnegative absolutely continuous on lines (ACL) functions $u : A \to \mathbb{R}$ with compact support in

A and such that $u(x)\geqslant 1$ on C, is called *α-capacity* of the condenser E. We set $\operatorname{cap} E := \operatorname{cap}_n E$.

Let $f : D\rightarrow \mathbb{R}^n$, f is open, $n\geqslant 2$, $x_0\in D$, $0<r_1<r_2<d_0=\operatorname{dist}(x_0, \partial D)$, $E=(A, C)$ where $A=B(x_0, r_2)$, $C=\overline{B(x_0, r_1)}$. By the continuity and openness of f, the pair $f(E)=(f(A), f(C))$ is also a condenser.

The proof of the main assertions of the article is connected with the use of modulus techniques, in particular, mappings that distort the modulus of families of paths according to the Poletsky inequality type. In this regard, consider the following definition.

Given $\alpha\geqslant 1$. An open mapping $f:D\rightarrow \overline{\mathbb{R}^n}$ is called a *ring Q-mapping at the point $x_0\in \overline{D}\setminus\{\infty\}$ with respect to α-modulus in the sense of condenser*, if the condition

$$\operatorname{cap}_{\alpha}(f(E))\leqslant \int\limits_{A(x_0,r_1,r_2)} Q(x)\cdot \eta^{\alpha}(|x-x_0|)\, dm(x) \tag{2.3}$$

holds for all $0<r_1<r_2< r_0$ and some $0<r_0=r_0(x_0)\leqslant d_0$ and all Lebesgue measurable functions $\eta:(r_1,r_2)\rightarrow [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2}\eta(r)\, dr\geqslant 1\,. \tag{2.4}$$

Here $A=A(x_0,r_1,r_2)$ is defined in (1.2) and S_i are defined in (1.1). Similarly, a mapping f is called a *ring Q-mapping with respect to α-modulus in D in the sense of condenser,* if condition (2.3) is satisfied at every point $x_0\in D$. We need the following statement, see e.g. [17, Lemma 7.4, Ch. 7] for $\alpha=n$ and [34, Lemma 2.2] for $\alpha\ne n$.

Proposition 2.1. *Let $x_0\in \mathbb{R}^n$, let Q be a Lebesgue measurable function $Q:\mathbb{R}^n\rightarrow [0,\infty]$, $Q\in L_{loc}^1(\mathbb{R}^n)$. Set $\eta_0(r)=\frac{1}{Ir^{\frac{n-1}{\alpha-1}}q_{x_0}^{\frac{1}{\alpha-1}}(r)}$, where $J:=J(x_0,r_1,r_2)=\int\limits_{r_1}^{r_2}\frac{dr}{r^{\frac{n-1}{\alpha-1}}q_{x_0}^{\frac{1}{\alpha-1}}(r)}$ and $q_{x_0}(r)$ is defined in the relation (1.13). Then*

$$\frac{\omega_{n-1}}{J^{\alpha-1}}=\int\limits_{A} Q(x)\cdot \eta_0^{\alpha}(|x-x_0|)\, dm(x)\leqslant \int\limits_{A} Q(x)\cdot \eta^{\alpha}(|x-x_0|)\, dm(x) \tag{2.5}$$

for any Lebesgue measurable function $\eta:(r_1,r_2)\rightarrow [0,\infty]$ such that $\int\limits_{r_1}^{r_2}\eta(r)\,dr=1$, where A is defined in (1.2).

Remark 2.1. Note that, if (2.5) holds for any function η with a condition

$$\int\limits_{r_1}^{r_2} \eta(r)\,dr = 1\,, \tag{2.6}$$

then the same relationship holds for any function η with the condition

$$\int\limits_{r_1}^{r_2} \eta(r)\,dr \geqslant 1\,. \tag{2.7}$$

Indeed, let η be a nonnegative Lebesgue function that satisfies the condition (2.7). If $I := \int\limits_{r_1}^{r_2} \eta(t)\,dt < \infty$, then we put $\eta_0 := \eta/I$. Obviously, the function η_0 satisfies condition (2.6). Then the relation (2.5) gives that

$$\frac{\omega_{n-1}}{J^{\alpha-1}} \leqslant \frac{1}{I^{\alpha}} \int\limits_{A} Q_*(x)\cdot\eta^{\alpha}(|x-x_0|)\,dm(x) \leqslant \int\limits_{A} Q_*(x)\cdot\eta^{\alpha}(|x-x_0|)\,dm(x)$$

because $I \geqslant 1$. Let now $I=\infty$. Then, by [33, Theorem I.7.4], a function η is a limit of a nondecreasing nonnegative sequence of simple functions η_m, $m=1,2,\ldots$. Set $I_m := \int\limits_{r_1}^{r_2} \eta_m(t)\,dt < \infty$ and $w_m(t) := \eta_m(t)/I_m$. Then, it follows from (2.7) that

$$\frac{\omega_{n-1}}{J^{\alpha-1}} \leqslant \frac{1}{I_m^{\alpha}} \int\limits_{A} Q_*(x)\cdot\eta_m^{\alpha}(|x-x_0|)\,dm(x) \leqslant$$

$$\leqslant \int\limits_{A} Q_*(x)\cdot\eta_m^{\alpha}(|x-x_0|)\,dm(x)\,, \tag{2.8}$$

because $I_m \to I = \infty$ as $m\to\infty$ (see [33, Lemma I.11.6]), and, consequently, $I_m \geqslant 1$ for sufficiently large $m\in\mathbb{N}$. Observe that, a functional sequence $f_m(x) = Q_*(x)\cdot\eta_m^{\alpha}(|x-x_0|)$, $m=1,2\ldots$, is nonnegative, monotone increasing and converges to a function $f(x) := Q_*(x)\cdot\eta^{\alpha}(|x-x_0|)$ almost everywhere. By the Lebesgue theorem on the monotone convergence (see [33, Theorem I.12.6]), it is possible to go to the limit on the right side of the inequality (2.8), which gives us the desired inequality (2.5).

Let (X,μ) be a metric space with measure μ. For each real number $n \geqslant 1$, we define *the Loewner function* $\phi_n : (0,\infty) \to [0,\infty)$ on X by the relation

$$\phi_n(t) = \inf\{M_n(\Gamma(E,F,X)) : \Delta(E,F) \leqslant t\}\,,$$

where the infimum is taken over all disjoint nondegenerate continua E and F in X and

$$\Delta(E,F) := \frac{\mathrm{dist}\,(E,F)}{\min\{\mathrm{diam}\,E, \mathrm{diam}\,F\}}\,.$$

A pathwise connected metric measure space (X,μ) is said to be a *Loewner space* of exponent n, or an n-Loewner space, if the Loewner function $\phi_n(t)$ is positive for all $t>0$ (see [17, section 2.5], cf. [10, Ch. 8]). Following [10, section 7.22], given a real-valued function u in a metric space X, a Borel function $\rho\colon X\rightarrow[0,\infty]$ is said to be an *upper gradient* of a function $u:X\rightarrow\mathbb{R}$ if $|u(x)-u(y)|\leqslant\int\limits_{\gamma}\rho\,|dx|$ for each rectifiable curve γ joining x and y in X. Let (X,μ) be a metric measure space and let $1\leqslant\alpha<\infty$. We say that X admits *a* $(1;\alpha)$-*Poincare inequality* if there is a constant $C\geqslant 1$ such that

$$\frac{1}{\mu(B)}\int\limits_{B}|u-u_B|\,d\mu(x)\leqslant C\cdot(\mathrm{diam}\,B)\left(\frac{1}{\mu(B)}\int\limits_{B}\rho^{\alpha}\,d\mu(x)\right)^{1/\alpha}$$

for all balls B in X, for all bounded continuous functions u on B, and for all upper gradients ρ of u. Metric measure spaces where the inequalities

$$\frac{1}{C}R^{n}\leqslant\mu(B(x_0,R))\leqslant CR^{n}$$

hold for a constant $C\geqslant 1$, every $x_0\in X$ and all $R<\mathrm{diam}\,X$, are called *Ahlfors n-regular.* Let us to prove the following statement.

Lemma 2.1. *Given $n\geqslant 2$ and $n-1<\alpha\leqslant n$, the set $\partial\mathbb{B}^n$ is strongly accessible with respect to α-modulus.*

Proof. Observe that, $\mathbb{B}^n$ is a Loewner space (see [10, Example 8.24(a)]) and, therefore, is Ahlfors regular, see [10, Proposition 8.19]. Moreover, by [11, Theorem 10.5], the Poincaré $(1;\alpha)$-inequality is fulfilled in $\mathbb{B}^n$ for any $\alpha\geqslant 1$. By [1, Proposition 4.7], for $n-1<\alpha\leqslant n$ we have that

$$M_{\alpha}(\Gamma(E,F,\mathbb{B}^n))\geqslant\frac{1}{C}\min\{\mathrm{diam}\,E,\mathrm{diam}\,\mathrm{F}\},\tag{2.9}$$

where $C>0$ is a constant. Let $x_0\in\partial\mathbb{B}^n$, and let U be an arbitrary neighborhood of x_0. We choose $\varepsilon_1>0$ in such a way that, putting $V:=B(x_0,\varepsilon_1)$, we have $\overline{V}\subset U$. Let $\partial U\ne\varnothing$, then $\varepsilon_2:=d(\partial U,\partial V)>0$. Note that, $\mathrm{diam}(F_1)\geqslant\varepsilon_2$ and $\mathrm{diam}(F_2)\geqslant\varepsilon_2$ for any continua F_1 and F_2 in $\mathbb{B}^n$ satisfying $F_1\cap\partial U\ne\varnothing\ne F_1\cap\partial V$ and $F_2\cap\partial U\ne\varnothing\ne F_2\cap\partial V$. Therefore, by (2.9), we obtain that $M_{\alpha}(\Gamma(F_1,F_2,\mathbb{B}^n))\geqslant\varepsilon_2$, as required. $\Box$

3. On the local behavior of ring homeomorphisms in the sense of condensers

Note that, studies of the local behavior of mappings close to (2.3) have been repeatedly carried out in our papers (see, for example, [35] and [8]). However, in order to study it, we need to establish some facts specifically for mappings of the form (2.3). An analog of the following lemma in a slightly different form was proved in [16, Lemma 2.9], see [36, Lemma 3].

Lemma 3.1. *Let $E=(A,C)$ be a condenser such that $A\subset B(0,r)$, $r>0$, and a set C is connected. Then the estimate*

$$\operatorname{cap} E\geqslant \frac{\omega_{n-1}}{\left\{\log\frac{2\lambda_n^2}{h(C)h\left(\overline{\mathbb{R}^n}\setminus B(0,r)\right)}\right\}^{n-1}}$$

holds, where $\lambda_n\in[4,2e^{n-1})$ is some constant depending only on n.

The following statement holds (see, e.g., [35, Lemma 3.3]).

Lemma 3.2. *Let $f:D\to\mathbb{R}^n$, $n\geqslant 2$, be an open mapping satisfying (2.3) at $x_0\in D$ for $\alpha=n$ such that $D'=f(D)\subset B(0,r)$ for some $r>0$. Assume that, there are $p\leqslant n$, $\varepsilon_0\in(0,\operatorname{dist}(x_0,\partial D))$, $\varepsilon_0'\in(0,\varepsilon_0)$ and a Lebesgue measurable function ψ, $\psi:(\varepsilon,\varepsilon_0)\to[0,\infty]$, $\varepsilon\in(0,\varepsilon_0')$, such that the relation*

$$\int\limits_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x)\cdot\psi^n(|x-x_0|)\,dm(x)\leqslant$$

$$\leqslant K\cdot I^p(\varepsilon,\varepsilon_0)\qquad\forall\,\varepsilon\in(0,\varepsilon_0')\,, \tag{3.1}$$

holds, where

$$0<I(\varepsilon,\varepsilon_0):=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt<\infty\qquad\forall\,\varepsilon\in(0,\varepsilon_0')\,. \tag{3.2}$$

Then

$$h(f(x),f(x_0))\leqslant\frac{\alpha_n}{\delta}\exp\{-\beta_n I^{\gamma_{n,p}}(|x-x_0|,\varepsilon_0)\} \tag{3.3}$$

for any $x\in B(x_0,{\varepsilon_0}')$, where $\delta:=h\left(\overline{\mathbb{R}^n}\setminus B(0,r)\right)$, besides that, λ_n, α_n and β_n are some constants depending only on n, and $\gamma_{n,p}=1-\frac{p-1}{n-1}$.

Proof. Let $E=(B(x_0,\varepsilon_0),\overline{B(x_0,\varepsilon)})$, $0<\varepsilon<\varepsilon_0'$. Let us note that,

$$\operatorname{cap} f(E)\leqslant K\cdot I^{p-n}(\varepsilon,\varepsilon_0)\,. \tag{3.4}$$

Indeed, setting $\eta_\varepsilon(t)=\psi(t)/I(\varepsilon,\varepsilon_0)$, $t\in(\varepsilon,\,\varepsilon_0)$, we obtain that $\int\limits_{\varepsilon}^{\varepsilon_0}\eta_\varepsilon(t)\,dt$ $=1$. Now, substituting $\eta_\varepsilon(t)$ into the relation (2.3) and using the condition (3.1), we obtain (3.4), as required.

Since $f(A)\subset B(0,r)$, by Lemma 3.1 we obtain that

$$\operatorname{cap} f(E)\geqslant\frac{\omega_{n-1}}{\left\{\log\frac{2\lambda_n^2}{h(f(C))h\left(\overline{\mathbb{R}^n}\setminus B(0,r)\right)}\right\}^{n-1}}\,, \tag{3.5}$$

where $\lambda_n\in[4,2e^{n-1})$. Since $\delta=h\left(\overline{\mathbb{R}^n}\setminus B(0,r)\right)$, by (3.4) and (3.5) we obtain that

$$h\left(f(C)\right)\leqslant\frac{2\,\lambda_n^2}{\delta}\,\exp\left\{-\left(\frac{\omega_{n-1}}{K}\right)^{\frac{1}{n-1}}\left(I(\varepsilon,\varepsilon_0)\right)^{\frac{n-p}{n-1}}\right\}.$$

Setting $\alpha_n=2\lambda_n^2$, $\beta_n=\left(\frac{\omega_{n-1}}{K}\right)^{\frac{1}{n-1}}$ and $\gamma_{n,p}=1-\frac{p-1}{n-1}$, we obtain that

$$h\left(f(C)\right)\leqslant\frac{\alpha_n}{\delta}\exp\left\{-\beta_n I^{\gamma_{n,p}}(\varepsilon,\varepsilon_0)\right\}. \tag{3.6}$$

Let $x\in D$ be such that $|x-x_0|=\varepsilon$, $0<\varepsilon<\varepsilon_0'$. Then $x\in\overline{B\left(x_0,\,\varepsilon\right)}$ and $f(x)\in f\left(\overline{B\left(x_0,\,\varepsilon\right)}\right)=f(C)$, in addition, by (3.6) we obtain that the relation

$$h\left(f(x),\,f(x_0)\right)\leqslant\frac{\alpha_n}{\delta}\,\exp\left\{-\beta_n I^{\gamma_{n,p}}(|x-x_0|,\varepsilon_0)\right\} \tag{3.7}$$

holds for any $\varepsilon\in(0,\varepsilon_0')$. Due to the arbitrariness of $\varepsilon\in(0,\varepsilon_0')$, we obtain the relation (3.7) for all $x\in B(x_0,\varepsilon_0')$. □

The case $\alpha\neq n$ will be considered now separately. Recall the basic lower estimate of p-capacity of a condenser $E=(A,C)$ in $\mathbb{R}^n$:

$$\operatorname{cap}_\alpha\,E=\operatorname{cap}_\alpha\,(A,C)\geqslant\left(c_1\frac{(d(C))^\alpha}{(m(A))^{1-n+\alpha}}\right)^{\frac{1}{n-1}}\,,\qquad\alpha>n-1, \tag{3.8}$$

where c_1 depends only on n and p, and $d(C)$ denotes the Euclidean diameter of C (see, e.g., [14, Proposition 6]).

Lemma 3.3. *Let $n-1<\alpha\leqslant n$, $n\geqslant 2$, and let $f:D\to\mathbb{R}^n$, be an open mapping satisfying* (2.3) *at $x_0\in D$, $D':=f(D)\subset B(0,r)$. Suppose that there exist numbers $q\leqslant\alpha$, $\varepsilon_0\in(0,\,\operatorname{dist}(x_0,\partial D))$, $\varepsilon_0'\in(0,\varepsilon_0)$ and a* *nonnegative Lebesgue measurable function* $\psi : (\varepsilon, \varepsilon_0) \to [0,\infty]$, $\varepsilon \in (0, \varepsilon_0')$ *such that*

$$\int\limits_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x)\cdot\psi^{\alpha}(|x-x_0|)\ dm(x) \leqslant K\cdot I^q(\varepsilon,\varepsilon_0) \qquad \forall\,\varepsilon\in(0,\varepsilon_0')\,, \tag{3.9}$$

where $I(\varepsilon,\varepsilon_0)$ *is defined by* (3.2)*. Then*

$$|f(x)-f(x_0)| \leqslant Cr^{\frac{(1-n+\alpha)n}{\alpha}} K^{\frac{n-1}{\alpha}} I^{\frac{(q-\alpha)(n-1)}{\alpha}}(|x-x_0|,\varepsilon_0)$$

for any $x\in B(x_0,{\varepsilon_0}')$*, where* C *is a constant depending only on* n *and* α*.*

Proof. Let $E = (B(x_0,\varepsilon_0), \overline{B(x_0,\varepsilon)})$, $0<\varepsilon<\varepsilon_0'$. Setting $\eta_{\varepsilon}(t) = \psi(t)/I(\varepsilon,\varepsilon_0)$, $t\in(\varepsilon,\,\varepsilon_0)$, we obtain that $\int\limits_{\varepsilon}^{\varepsilon_0}\eta_{\varepsilon}(t)\,dt = 1$. Now, substituting $\eta_{\varepsilon}(t)$ into the relation (2.3), one obtains from (3.9) that

$$\mathrm{cap}_{\alpha}\, f(E) \leqslant K\cdot I^{q-\alpha}\,(\varepsilon,\varepsilon_0)\ . \tag{3.10}$$

Since $f(A)\subset B(0,r)$, the bound (3.8) yields

$$\mathrm{cap}_{\alpha}\, f(E)\geqslant$$

$$\geqslant\left(c_1\frac{(d(f(C)))^{\alpha}}{(m(f(A)))^{1-n+\alpha}}\right)^{\frac{1}{n-1}} \geqslant \left(c_1\frac{(d(f(C)))^{\alpha}}{(\Omega_n r^n))^{1-n+\alpha}}\right)^{\frac{1}{n-1}}\ . \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$d\,(f(C))\leqslant\left(\frac{1}{c_1}\right)^{1/\alpha}\Omega_n^{\frac{1-n+\alpha}{\alpha}} r^{\frac{(1-n+\alpha)n}{\alpha}} K^{\frac{n-1}{\alpha}} I^{\frac{(q-\alpha)(n-1)}{\alpha}}(\varepsilon,\varepsilon_0)\,, \tag{3.12}$$

where Ω_n is the volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. Now let $x\in D$ be such that $|x-x_0|=\varepsilon$, $0<\varepsilon<\varepsilon_0'$. Then, $x\in\overline{B\,(x_0,\,\varepsilon)}$ and $f(x)\in f\left(\overline{B\,(x_0,\,\varepsilon)}\right) = f(C)$, and from (3.12) we obtain the estimate

$$|f(x)-f(x_0)|\leqslant$$

$$\leqslant\left(\frac{1}{c_1}\right)^{1/\alpha}\Omega_n^{\frac{1-n+\alpha}{\alpha}} r^{\frac{(1-n+\alpha)n}{\alpha}} K^{\frac{n-1}{\alpha}} I^{\frac{(q-\alpha)(n-1)}{\alpha}}(|x-x_0|,\varepsilon_0)\,. \tag{3.13}$$

Since $\varepsilon\in(0,\varepsilon_0')$ is arbitrary, the relation (3.13) holds in $B(x_0,\varepsilon_0')$. $\Box$

4. On upper distortion of the modulus under Orlicz–Sobolev classes

Let ω be an open set in $\mathbb{R}^{n-1}$. A continuous mapping $\sigma\colon \omega\rightarrow\mathbb{R}^n$ is called a *surface.* Accordingly, we say that a property P holds for almost every surface, if P holds for all surfaces except a family of zero p-modulus. Let Γ be a family of surfaces S. A Borel function $\rho\colon \mathbb{R}^n\rightarrow\overline{\mathbb{R}^+}$ is said to be *admissible* for Γ (briefly: $\rho\in{\rm adm}\Gamma$) if

$$\int\limits_S\rho^{n-1}\,d\mathcal{A}\geqslant 1 \tag{4.1}$$

for every surface $S\in\Gamma$, where the integral on the left-hand side of (4.1) is defined by relation

$$\int\limits_S\rho\,d\mathcal{A}=\int\limits_{\mathbb{R}^n}\rho(y)N(S,y)\,d\mathcal{H}^{n-1}y\,, \tag{4.2}$$

and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-measured Hausdorff measure.

If $p\geqslant 1$, the p-modulus of the family Γ is defined to be the quantity

$$M_p(\Gamma)=\inf\limits_{\rho\in{\rm adm}\Gamma}\int\limits_{\mathbb{R}^n}\rho^p(x)\,dm(x).$$

Following [17], a metric ρ is said to be *extensively admissible* for Γ with respect to p-modulus, write $\rho\in{\rm ext}_p{\rm adm}\,\Gamma$, if $\rho\in{\rm adm}\,(\Gamma\backslash\Gamma_0)$ such that $M_p(\Gamma_0)=0$. The next class of mappings is a generalization of quasi-conformal mappings in the sense of Gehring's ring definition (see [5]; cf. [17, Chapter 9]). Let D and $D^{\,\prime}$ be domains in $\mathbb{R}^n$ with $n\geqslant 2$. Suppose that $x_0\in\overline{D}\setminus\{\infty\}$ and $Q\colon D\rightarrow(0,\infty)$ is a Lebesgue measurable function. A mapping $f\colon D\rightarrow D^{\,\prime}$ is called a *lower Q-mapping at a point x_0 relative to the p-modulus* if

$$M_p(f(\Sigma_{\varepsilon}))\geqslant \inf\limits_{\rho\in{\rm ext}_p\,{\rm adm}\Sigma_{\varepsilon}}\int\limits_{D\cap A(x_0,\varepsilon,r_0)}\frac{\rho^p(x)}{Q(x)}\,dm(x)\,, \tag{4.3}$$

where $A(x_0,\varepsilon,r_0)$ is defined in (1.2), $r_0\in(0,d_0)$, $d_0=\sup\limits_{x\in D}|x-x_0|$, in addition, Σ_{ε} denotes the family of all intersections of the spheres $S(x_0,r)$ with the domain D, $r\in(\varepsilon,r_0)$. The following statement holds (see e.g. [38, Lemma 2]).

Lemma 4.1. *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 3$, and let $\varphi\colon (0,\infty) \to (0,\infty)$ be a nondecreasing function satisfying* (1.9)*. If $p > n-1$, then every open discrete mapping $f\colon D \to \mathbb{R}^n$ with a finite distortion of the class $W^{1,\varphi}_{\rm loc}$ such that $N(f,D) < \infty$ is a lower Q-mapping relative to the p-modulus at every point $x_0 \in \overline{D}$ for*

$$Q(x) = N(f,D) \cdot K_{I,\alpha}^{\frac{p-n+1}{n-1}}(x,f),$$

$\alpha := \frac{p}{p-n+1}$, *where the inner dilation $K_{I,\alpha}(x,f)$ for f at x of order α is defined by* (1.6)*, and the multiplicity $N(f,D)$ is defined by the second relation in* (1.7)*.*

The following statement is proved in [39, Lemma 4.2].

Lemma 4.2. *Let $n \geqslant 2$, $p > n-1$, let D be a domain in $\mathbb{R}^n$, let $x_0 \in D$ and let $Q\colon D \to [0,\infty]$ be a function in $L^s(D)$, where $s = \frac{n-1}{p-n+1}$. Assume that D' is a domain in $\mathbb{R}^n$ with a compact closure $\overline{D'}$. If $f\colon D \to D'$ is an open discrete lower Q-mapping at x_0 with respect to p-modulus, then there is $C > 0$ such that*

$$\operatorname{cap}_\beta f(E) \leqslant \int\limits_A Q^*(x) \cdot \eta^\beta(|x-x_0|)\, dm(x)$$

for $\beta = \frac{p}{p-n+1}$, $Q^* = C \cdot Q^{\frac{n-1}{p-n+1}}$, $E = (B(x_0,r_2), \overline{B(x_0,r_1)})$, any $0 < r_1 < r_2 < \varepsilon_0 := \operatorname{dist}(x_0, \partial D)$, and any Lebesgue measurable function $\eta\colon (r_1,r_2) \to [0,\infty]$ such that the relation (2.4) *holds. Here $A = A(x_0,r_1,r_2)$ is defined in* (1.2)*.*

Observe that, $p \in \left[n, n+\frac{1}{n-2}\right)$ if and only if $\alpha := \frac{p}{p-n+1} \in (n-1,n]$. Now, combining Lemmas 4.1 and 4.2 we obtain the following.

Lemma 4.3. *Let D, D' be domains in $\mathbb{R}^n$, $n \geqslant 3$, $x_0 \in D$, let $\alpha \in (n-1,n]$, and let $\varphi\colon (0,\infty) \to (0,\infty)$ be a nondecreasing function satisfying* (1.9)*. Assume that, $f\colon D \to D'$ is an open discrete mapping with a finite distortion of the class $W^{1,\varphi}_{\rm loc}(D)$ such that $N(f,D) < \infty$ and $K_{I,\alpha} \in L^1_{\rm loc}(D)$. Then there is $C_1 > 0$ depending only on domains D and D' such that*

$$\operatorname{cap}_\alpha f(E) \leqslant \int\limits_A Q^{**}(x) \cdot \eta^\alpha(|x-x_0|)\, dm(x)$$

for $Q^{**} = C_1 \cdot N^{\alpha-1}(f,D) \cdot K_{I,\alpha}(x,f)$, $E = (B(x_0,r_2), \overline{B(x_0,r_1)})$, $0 < r_1 < r_2 < \varepsilon_0 := \operatorname{dist}(x_0, \partial D)$, and any Lebesgue measurable function $\eta\colon (r_1,r_2) \to [0,\infty]$ with (2.4)*.*

5. The main Lemma

Analogues of the following lemma have been repeatedly proved by various authors in the study of the local and boundary behavior of mappings, see, for example, [17, Lemma 6.2], [31, Lemma 4.9].

Lemma 5.1. *Let $n\geqslant 3$, $n-1<\alpha\leqslant n$, and let $\varphi:(0,\infty)\rightarrow [0,\infty)$ be a non-decreasing Lebesgue measurable function which satisfies Calderon's condition* (1.9) *and the condition* (1.10)*. Let $Q:{\mathbb B}^n\rightarrow [0,\infty]$ be integrable function in ${\mathbb B}^n$,*

$$Q^*(x)=\begin{cases}Q(x), & x\in {\mathbb B}^n\,,\\ Q\left(\frac{x}{|x|^2}\right)\,, & x\in {\mathbb R}^n\setminus {\mathbb B}^n\end{cases}.$$

Assume that f is an open discrete and closed mapping of ${\mathbb B}^n$ onto ${\mathbb B}^n$ such that $f\in W^{1,\varphi}({\mathbb B}^n)$ and, in addition, $N(f, {\mathbb B}^n)=N(f, \overline{{\mathbb B}^n})$. Let, moreover, $K_{I, \alpha}(x, f)\leqslant Q(x)$ for a.e. $x\in {\mathbb B}^n$ and, besides that, for any $x_0\in {\mathbb S}^{n-1}$ there is $0<\varepsilon_0=\varepsilon_0(x_0)$ and $0<\varepsilon_0^{\prime}<\varepsilon_0$ and some positive Lebesgue measurable function $\psi:(0, \varepsilon_0)\rightarrow (0,\infty)$ such that

$$0<I(\varepsilon, \varepsilon_0):=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt < \infty \tag{5.1}$$

for any $\varepsilon\in (0, \varepsilon_0^{\prime})$ and, in addition,

$$\int\limits_{A(x_0, \varepsilon, \varepsilon_0)} Q^*(x)\cdot\psi^{\,\alpha}(|x-x_0|)\;dm(x)\leqslant K\cdot I^p(\varepsilon, \varepsilon_0)\,, \tag{5.2}$$

for some $p<\alpha$, for some constant $K>0$ and for any $\varepsilon\in (0, \varepsilon_0^{\prime})$, where $A:=A(x_0, \varepsilon, \varepsilon_0)$ is defined in (1.2)*. Assume that $I(\varepsilon, \varepsilon_0)\rightarrow\infty$ as $\varepsilon\rightarrow 0$, while $\int\limits_{\varepsilon}^{\varepsilon_0}\frac{dt}{tq_{x_0}^{\frac{1}{n-1}}(t)}<\infty$ for sufficiently small $0<\varepsilon<\varepsilon_0$ and any $x_0\in {\mathbb S}^{n-1}$. Then:*

I. *A mapping f has a continuous extension $\overline{f}:\overline{{\mathbb B}^n}\rightarrow \overline{{\mathbb B}^n}$, while $\overline{f}(\overline{{\mathbb B}^n})=\overline{{\mathbb B}^n}$. Moreover, there is a continuous extension $F:B(0, R_0)\rightarrow \overline{{\mathbb R}^n}$ for any $R_0\geqslant 1$, which is open and discrete. Namely,*

$$F(x)=\left\{\begin{array}{rr} f(x), & |x|<1\,,\\ \Psi(f(\Psi(x))), & |x|\geqslant 1\,.\end{array}\right. \tag{5.3}$$

II. *There are $m>0$ and $R>0$ such that the relation*

$$|f(\Psi(y))|\geqslant m\,,\qquad 1\leqslant |y|\leqslant R\,, \tag{5.4}$$

holds, where $\Psi(x) := \frac{x}{|x|^2}$.

III. *Let* $0 < \varepsilon_0 < 1/R$.

III. 1) If $\alpha = n$*, then, for any* $x_0 \in \mathbb{S}^{n-1}$*, the relation*

$$h(F(x), F(x_0)) \leqslant \frac{\alpha_n}{\delta} \exp\{-\beta_n I^{\gamma_{n,p}}(|x-x_0|, \varepsilon_0)\} \tag{5.5}$$

holds for any $x \in B(x_0, {\varepsilon_0}')$*, where* α_n *is a number depending only on* n*,*

$$\beta_n = \left(\frac{\omega_{n-1}}{KC \cdot N^{\alpha-1}(F, B(0,R))}\right)^{\frac{1}{n-1}}, \quad \gamma_{n,p} = 1 - \frac{p-1}{n-1}, \tag{5.6}$$

$\delta := h\left(\overline{\mathbb{R}^n} \setminus F(B(0,R))\right)$ *and* $C > 0$ *depends on* n*,* m *and* R*.*

III. 2) if $n-1 < \alpha \leqslant n$*, then, for any* $x_0 \in \mathbb{S}^{n-1}$*, the relation*

$$|F(x) - F(x_0)| \leqslant$$

$$\leqslant Cr^{\frac{(1-n+\alpha)n}{\alpha}} \left(KC \cdot N^{\alpha-1}(F, B(0,R))\right)^{\frac{n-1}{\alpha}} I^{\frac{(p-\alpha)(n-1)}{\alpha}}(|x-x_0|, \varepsilon_0) \tag{5.7}$$

holds for every $x \in B(x_0, {\varepsilon_0}')$*, where* C *is a constant depending only on* n *and* α*, and* $r > 0$ *is any radius of the ball consisting* $F(B(0,R))$*.*

Proof. **I.** Note that, a mapping f has a continuous extension $\overline{f}$ onto $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$. Indeed, $\mathbb{B}^n$ is locally connected on $\partial\mathbb{B}^n$, in addition, by Lemma 2.1 the set $\partial\mathbb{B}^n$ is strongly accessible with respect to α-modulus. Observe that a function $\eta := \psi/I(\varepsilon, \varepsilon_0)$ satisfies the relation (2.4). Now, by Proposition 2.1 and by the relation (5.2) we obtain that

$$\frac{\omega_{n-1}}{J^{\alpha-1}} = \int\limits_A Q^*(x) \cdot \eta_0^{\alpha}(|x-x_0|)\, dm(x) \leqslant$$

$$\leqslant \frac{1}{I^{\alpha}(\varepsilon, \varepsilon_0)} \int\limits_A Q^*(x) \cdot \psi^{\alpha}(|x-x_0|)\, dm(x) \leqslant K \cdot I^{p-\alpha}(\varepsilon, \varepsilon_0)\,, \tag{5.8}$$

where $J := J(x_0, \varepsilon, \varepsilon_0) = \int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{r^{\frac{n-1}{\alpha-1}} q_{x_0}^{*\frac{1}{\alpha-1}}(r)}$, $A := A(x_0, \varepsilon, \varepsilon_0)$ is defined in (1.2) and $q^*_{x_0}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{|x-x_0|=r} Q^*(x)\, d\mathcal{H}^{n-1}$. Since, by the assumption, $I(\varepsilon, \varepsilon_0) \to \infty$ as $\varepsilon \to 0$, it follows by (5.8) that $J \to \infty$ as $\varepsilon \to 0$. Since by the assumption $\int\limits_{\varepsilon}^{\varepsilon_0} \frac{dt}{t q_{x_0}^{\frac{1}{n-1}}(t)} < \infty$ for sufficiently small $0 < \varepsilon < \varepsilon_0$ and any $x_0 \in \mathbb{S}^{n-1}$, by Theorem 1 in [38] f has a continuous extension $\overline{f} : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$, as required.

II. Let $R_0 > 0$. Using the conformal transformation $\Psi(x) = \frac{x}{|x|^2}$, we extend the mapping f continuously onto $B(0, R_0)$ by (5.3). Let us show that the mapping f is open and discrete in $\mathbb{R}^n$. Since f is open discrete and closed, $N(f, \mathbb{B}^n) = N(f, \overline{\mathbb{B}^n}) < \infty$ (see [18, Theorem 2.8]). Now, f is discrete in $B(0, R_0)$.

It is known that any discrete open map defined in $\mathbb{B}^n$ is either sense-preserving or sense reversing (see, for example, [29, Ch. I, § 4]). To be definite, let f be sense-preserving in $\mathbb{B}^n$. Now, f is sense-preserving in $B(0, R_0) \setminus \mathbb{S}^{n-1}$. Let G be a domain in $B(0, R_0)$ such that $\overline{G}$ is a compactum and let $y \in (f(G) \setminus f(\partial G)) \cap f(\mathbb{S}^{n-1})$. Given a mapping $f : D \to \overline{\mathbb{R}^n}$ and a set $E \subset \overline{D}$ we use the notation

$$C(f, E) = \bigcup_{x \in E} C(f, x)$$

and

$$C(f, x) := \{y \in \overline{\mathbb{R}^n} : \exists\, x_k \in D : x_k \to x, f(x_k) \to y, k \to \infty\}\,.$$

Since f is closed, $C(f, \mathbb{S}^{n-1}) \subset \mathbb{S}^{n-1}$ (see, e.g., [42, Theorem 3.3]). Then there is a point $y_0 \in (\mathbb{R}^n \setminus f(\partial G)) \setminus \mathbb{S}^{n-1}$ belonging to the connected component of the set $\mathbb{R}^n \setminus f(\partial G)$ that contains y. Denote, as usual, by $\mu(y, f, G)$ the topological degree of the mapping f at the point y with respect to the domain G, and by $i(x, f)$ the local topological index of the mapping f at the point x (see e.g. [29, Ch. I, § 4]). Since the topological index is constant on every connected component of the set $\mathbb{R}^n \setminus f(\partial G)$ (see [29, Proposition 4.4, Ch. I]), we obtain $\mu(y, f, G) = \mu(y_0, f, G) = \sum\limits_{x \in G \cap f^{-1}(y_0)} i(x, f) > 0$. Thus, the map f is sense-preserving in $\mathbb{R}^n$. In this case, f is open and discrete in $\mathbb{R}^n$, as required (see [40], p. 333).

III. Using the condition $f \in W^{1,\varphi}(\mathbb{B}^n)$, we show that the inclusion $F \in W^{1,\varphi}(B(0, R))$ for some $R > 1$. Observe that

$$\overline{f}(\mathbb{S}^{n-1}) = \mathbb{S}^{n-1}\,. \tag{5.9}$$

Indeed, by proving above, $\overline{f}(\mathbb{S}^{n-1}) \subset \mathbb{S}^{n-1}$. On the other hand, let $y \in \mathbb{S}^{n-1}$. Since $f(\mathbb{B}^n) = \mathbb{B}^n$, there is a sequence $y_m \in \mathbb{B}^n$, $m = 1, 2, \ldots$, such that $y_m \to y$ as $m \to \infty$ and, simultaneously, $y_m = f(x_m)$, $x_m \in \mathbb{B}^n$. We may assume that x_m converge to some $x_0 \in \overline{\mathbb{B}^n}$ as $m \to \infty$. Then $x_0 \in \mathbb{S}^{n-1}$. Now $f(x_m) \to \overline{f}(x_0) = y$ as $m \to \infty$ because f has a continuous extension to x_0. Thus $y \in \overline{f}(\mathbb{S}^{n-1})$, as required. The relation (5.9) is proved. It follows from (5.9) that $\overline{f}(\overline{\mathbb{B}^n}) = \overline{\mathbb{B}^n}$.

Observe that $f(x) \neq 0$ for any $x \in A(0, r_*, 1)$ and some sufficiently small $r_* > 0$, where A is defined in (1.2). Indeed, in the contrary case, for any $m\in\mathbb{N}$ there is $x_m\in A(0,1-1/m,1)$ such that $f(x_m)=0$. According to the Bolzano–Weierstrass theorem, we may assume that $x_m\to x_0\in\overline{\mathbb{B}^n}$ as $m\to\infty$ for some $x_0\in\overline{\mathbb{B}^n}$. Since $x_m\in A(0,1-1/m,1)$, we obtain that $1-1/m<|x_m|<1$. Now $x_0\in\mathbb{S}^{n-1}$. Since by the item **I** f has a continuous extension to $\mathbb{S}^{n-1}=\partial\mathbb{B}^n$, we obtain that $f(x_m)=0\to f(x_0)$ as $m\to\infty$. This contradicts with (5.9). The contradiction obtained above proves that $f(x)\neq 0$ for any $x\in A(0,r_*,1)$ and some sufficiently small $r_*>0$.

IV. Let us to prove that the functions $|\nabla F|$ and $\varphi(|\nabla F|)$ are integrable in $B(0,R)$ for some $R>1$. For this, we observe that, for $|x|>1$, by the differentiation rule of a superposition of mappings,

$$F'(x)=\Psi'(f(\Psi(x))\circ f'(\Psi(x))\circ\Psi'(x)\,. \tag{5.10}$$

Here we used the fact that homeomorphisms of the Orlicz–Sobolev classes under the Calderon condition are differentiable almost everywhere, see, for example, [15, Theorem 1]. Using direct calculations, we may establish the inequality

$$\|f'(x)\|\leqslant|\nabla f(x)|\leqslant n^{1/2}\cdot\|f'(x)\| \tag{5.11}$$

at all points $x\in D$ where the map f has formal partial derivatives. Observe that $\|\Psi'(x)\|=\frac{1}{|x|^2}$ (see, e.g., [37, paragraph 7]). Recall that, for any two linear mappings g and h the relation

$$\|g\circ h\|\leqslant\|g\|\cdot\|h\| \tag{5.12}$$

holds, and here, equality holds as soon as at least one of the mappings is generalized orthogonal (see, e.g., [30, I.4, relation (4.13)]).

By the item **III**, $f(\Psi(y))\neq 0$ for $1<|y|\leqslant R$ and some $R>1$. Since the map $f(\Psi(y))$ is continuous in $\{1\leqslant|y|\leqslant R\}$ and does not vanish, there is $m>0$ such that the relation (5.4) holds. In this case, from (5.4), (5.10), (5.11) and (5.12) we obtain that

$$\int\limits_{1<|x|<R}|\nabla F(x)|\,dm(x)\leqslant\int\limits_{1<|x|<R}n^{1/2}\cdot\|F'(x)\|\,dm(x)=$$

$$=n^{1/2}\cdot\int\limits_{1<|x|<R}\|\Psi'(f(\Psi(x))\|\cdot\|f'(\Psi(x))\|\cdot\|\Psi'(x)\|\,dm(x)=$$

$$=n^{1/2}\cdot\int\limits_{1<|x|<R}\frac{1}{|f(\Psi(x))|^2}\cdot\|f'(\Psi(x))\|\cdot\frac{1}{|x|^2}\,dm(x)\leqslant$$

$$\leqslant \frac{n^{1/2}}{m^2}\cdot\int\limits_{1<|x|<R}\|f'(\Psi(x))\|\,dm(x)=$$

$$=\frac{n^{1/2}}{m^2}\cdot\int\limits_{1/R<|y|<1}\frac{\|f'(y)\|}{|y|^{2n}}\,dm(y)\leqslant$$

$$\leqslant \frac{n^{1/2}R^{2n}}{m^2}\cdot\int\limits_{1/R<|y|<1}|\nabla f(y)|\,dm(y)<\infty\,. \tag{5.13}$$

V. Quite similarly, applying the same arguments to the function $\varphi(|\nabla F|)$ instead of $|\nabla F|$, and taking into account relation (1.10) together with the non-decreasing property of the function φ, we obtain that

$$\int\limits_{1<|x|<R}\varphi(|\nabla F(x)|)\,dm(x)\leqslant \widetilde{C_1}\cdot\int\limits_{1<|x|<R}\varphi(\|F'(x)\|)\,dm(x)=$$

$$=\widetilde{C_1}\cdot\int\limits_{1<|x|<R}\varphi(\|\Psi'(f(\Psi(x)))\|\cdot\|f'(\Psi(x))\|\cdot\|\Psi'(x)\|)\,dm(x)=$$

$$=\widetilde{C_1}\cdot\int\limits_{1<|x|<R}\varphi\left(\frac{1}{|f(\Psi(x))|^2}\cdot\|f'(\Psi(x))\|\cdot\frac{1}{|x|^2}\right)\,dm(x)\leqslant$$

$$\leqslant \widetilde{C_2}\cdot\int\limits_{1<|x|<R}\varphi(\|f'(\Psi(x))\|)\,dm(x)=$$

$$=\widetilde{C_2}\cdot\int\limits_{1/R<|x|<1}\frac{\varphi\left(\|f'(x)\|\right)}{|x|^{2n}}\,dm(x)\leqslant$$

$$\leqslant \widetilde{C_2}R^{2n}\cdot\int\limits_{1/R<|x|<1}\varphi(|\nabla f(x)|)\,dm(x)<\infty\,, \tag{5.14}$$

where $\widetilde{C_1}$ and $\widetilde{C_2}$ are some constants.

VI. It follows from (5.13) and (5.14) that

$$\int\limits_{B(0,R)}|\nabla F(x)|\,dm(x)<\infty\,,$$

$$\int\limits_{B(0,R)}\varphi(|\nabla F(x)|)\,dm(x)<\infty\,,\quad R>1\,. \tag{5.15}$$

Reasoning in a similar way, we may also obtain similar relations for the inner dilatation of the map F. Indeed, since $l((f\circ g)'(x))\geqslant l(f'(g(x)))\cdot l(g'(x))$ for any mappings f and g at the corresponding points x and, in addition, $J(x, f\circ g)=J(g(x), f)\cdot J(x, f)$ we obtain that

$$K_{I,\alpha}(x, F)\leqslant K_{I,\alpha}(f(\Psi(x)), \Psi)\cdot K_{I,\alpha}(\Psi(x), f)\cdot K_{I,\alpha}(x, \Psi)\,.$$

Using the the calculation of $l(\Psi'(x))$ and $J(x, \Psi)$ through the radial and tangential stretchings (see, e.g., [30]), we obtain that $l(\Psi'(x))=\frac{1}{|x|^2}$ and $|J(x, \Psi)|=\frac{1}{|x|^{2n}}$, so that

$$K_{I,\alpha}(x, F)\leqslant \frac{1}{\left|f\left(\frac{x}{|x|^2}\right)\right|^{2(n-\alpha)}}\cdot K_{I,\alpha}\left(\frac{x}{|x|^2}, f\right)\cdot \frac{1}{|x|^{2(n-\alpha)}}\,. \tag{5.16}$$

Due to the relations in (5.4) and (5.16) we obtain that

$$K_{I,\alpha}(x, F)\leqslant \frac{1}{m^{2(n-\alpha)}}\cdot K_{I,\alpha}\left(\frac{x}{|x|^2}, f\right)\cdot \frac{1}{|x|^{2(n-\alpha)}}\,. \tag{5.17}$$

Now, we obtain that

$$\int\limits_{B(0,R)} K_{I,\alpha}(x, F)\,dm(x)\leqslant$$

$$\leqslant\int\limits_{\mathbb{B}^n} K_{I,\alpha}(x, f)\,dm(x)+\frac{1}{m^{2(n-\alpha)}}\cdot\int\limits_{1<|x|<R} K_{I,\alpha}\left(\frac{x}{|x|^2}, f\right)\cdot\frac{1}{|x|^{2(n-\alpha)}}\,dm(x)\,.$$

Making a change of variables here, and taking into account that $K_{I,\alpha}(x, f)\in L^1(\mathbb{B}^n)$ by the assumption, we obtain that

$$\int\limits_{B(0,R)} K_{I,\alpha}(x, F)\,dm(x)\leqslant$$

$$\leqslant\int\limits_{\mathbb{B}^n} K_{I,\alpha}(x, f)\,dm(x)+$$

$$+\frac{1}{m^{2(n-\alpha)}}\cdot\int\limits_{1/R<|y|<1} K_{I,\alpha}\left(y, f\right)\cdot\frac{|y|^{2(n-\alpha)}}{|y|^{2n}}\,dm(y)\leqslant \tag{5.18}$$

$$\leqslant\int\limits_{\mathbb{B}^n} K_{I,\alpha}(x, f)\,dm(x)+R^{2\alpha}\cdot\int\limits_{1/R<|y|<1} K_{I,\alpha}(y, f)\,dm(y)<\infty.$$

VII. Let us check that $F \in ACL(B(0,R))$. It is known if $f \in W^{1,1}(\mathbb{B}^n)$, that the unit ball $\mathbb{B}^n$ may be divided in a standard way into no more than a countable number of parallelepipeds I_s, $s \geqslant 1$, with disjoint interiors, such that F is absolutely continuous on almost all coordinate segments in each I_s, $s \geqslant 1$. We call a segment coordinate segment if it is parallel to a coordinate axis. Let us prove:

(A) F is absolutely continuous on almost all segments in $\overline{\mathbb{B}^n}$, parallel to the coordinate axes.

It is enough to consider segments r for which F is absolutely continuous (shortly AC) on $r_s := r \cap I_s$ for every $s \geqslant 1$. Suppose that $r(t) = \{x \in \mathbb{R}^n : x = x_0 + te, t \in [a,b]\}$ is such a segment in $\overline{\mathbb{B}^n}$, where e is some coordinate unit vector, and $x_0 \in \mathbb{B}^n$.

Two cases are possible: when $z_0 := x_0 + be$ belongs to the interior of the ball, and when the same point lies on the unit sphere. Set $\alpha(t) = f(x_0 + te)$. In the first case, there are finite number of integers $s_1, s_2, ..., s_l$ such that $r = \bigcup\limits_{\nu=1}^{l} r_{s_\nu}$. Hence F is AC on r. Note also here that by ACL-characterization of the Sobolev classes (see, e.g., [26, Theorems 1.1.2 and 1.1.3]) and by the fact that for a real-valued functions defined on an interval of the real line, absolute continuity may be formulated by the validity of the fundamental theorem of calculus in terms of Lebesgue integration, (see, for example, see [33, Theorem IV.7.4]), we have $\int\limits_a^b \alpha'(t)\,dt = \alpha(b) - \alpha(a)$. Let now $z_0 \in \mathbb{S}^{n-1}$. Then, as it was proved above with respect to the inner points of the ball, for an arbitrary $a < c < b$ we have that

$$\int\limits_a^c \alpha'(t)\,dt = \alpha(c) - \alpha(a)\,. \tag{5.19}$$

Since it was also proved above, that the map $\overline{f}$ is a continuous mapping in the closed unit ball $\overline{\mathbb{B}^n}$, the passage to the limit on the right-hand side of (5.19) as $c \to b$ gives that $\alpha(b) - \alpha(a)$.

Since (5.19) holds for every subinterval of r, we first conclude that F is AC on r, and (A) follows. Now consider the family $J(B(0,R))$ of all coordinate segments in $B(0,R)$. It follows from the integrability of the gradient of the mapping F on $B(0,R)$ (see (5.15) and by virtue of Fubini's theorem (see, for example, [33, Theorem III.8.1]) that the derivative of the function α is integrable on almost all segments in $B(0,R)$ parallel to the coordinate axes. Without loss of generality, we may assume that a segment $r(t)$ has exactly this property.

Since the reflection with respect to the unit sphere is C^{∞} change of variables, and $f\in W^{1,1}(\mathbb{B}^n)$, we conclude that $F\in W^{1,1}\big(B(0,R)\setminus \mathbb{B}^n\big)$ (see item 1.1.7 in [26] and also definitions of Sobolev spaces on manifolds in literature). Similarly as above, we may verify that:

(B) F is absolutely continuous on almost all segments in $\mathbb{R}^n\setminus\mathbb{B}^n$, parallel to the coordinate axes.

Since F is continuous on $B(0,R)$, this immediately implies that F is absolutely continuous on the same segments in $B(0,R)$, as required.

VIII. Since $F\in ACL(B(0,R))$, by (5.15) $F\in W^{1,\varphi}_{\rm loc}(B(0,R))$. In addition, by (5.18) $K_I(x,F)\in L^1(B(0,R))$. Now, by Lemma 4.3 and by (5.17) F is a ring Q^{**}-mapping in $B(0,R)$ with respect to α-modulus with respect to a condenser , where $Q^{**}(x)=C_2\cdot Q(x)$ for $x\in\mathbb{B}^n$, $C_2:= C_1\cdot N^{\alpha-1}(F,B(0,R))$ and $C_1>0$ depends only on R, and $Q^{**}(x)=C_2\cdot \frac{1}{m^{2(n-\alpha)}}\cdot Q\left(\frac{x}{|x|^2}\right)$ for $x\in B(0,R)\setminus\mathbb{B}^n$, where C_2 is given above. Generally, F is a ring $C\cdot N^{\alpha-1}(F,B(0,R))\cdot Q^*$-mapping, where C depends on n, m, α and R.

IX. Let $x_0\in\mathbb{S}^{n-1}$. Now, by Lemma 3.2 there exists $0<{\varepsilon_0}'<\varepsilon_0$ such that, for $\alpha=n$

$$h(F(x),F(x_0))\leqslant\frac{\alpha_n}{\delta}\,\exp\{-\beta_n I^{\gamma_{n,p}}(|x-x_0|,\varepsilon_0)\}\tag{5.20}$$

for any $x\in B(x_0,{\varepsilon_0}')$, where $\delta:=h\left(\overline{\mathbb{R}^n}\setminus F(B(0,R))\right)$, besides that, α_n depends only on n, $\beta_n=\left(\frac{\omega_{n-1}}{KC\cdot N^{\alpha-1}(F,B(0,R))}\right)^{\frac{1}{n-1}}$ and $\gamma_{n,p}=1-\frac{p-1}{n-1}$.

If $\alpha\ne n$, by Lemma 3.3 we obtain that

$$|F(x)-F(x_0)|\leqslant\tag{5.21}$$

$$\leqslant Cr^{\frac{(1-n+\alpha)n}{\alpha}}\left(KC\cdot N^{\alpha-1}(F,B(0,R))\right)^{\frac{n-1}{\alpha}}I^{\frac{(p-\alpha)(n-1)}{\alpha}}(|x-x_0|,\varepsilon_0)$$

for every $x\in B(x_0,{\varepsilon_0}')$ and for some $0<{\varepsilon_0}'<\varepsilon_0<1/R$, where C is a constant depending only on n and α, and $r>0$ is any radius of the ball consisting $F(B(0,R))$. Lemma is proved. □

The following statement holds.

Lemma 5.2. *Assume that, for any $x_0\in\mathbb{S}^{n-1}$ there is $0<\varepsilon_0=\varepsilon_0(x_0)$ and $0<\varepsilon_0'<\varepsilon_0$ and some positive Lebesgue measurable function $\psi:(0,\varepsilon_0)\to(0,\infty)$ such that the relation*

$$0<I(\varepsilon,\varepsilon_0):=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt<\infty\tag{5.22}$$

holds for any $\varepsilon\in(0,\varepsilon_0')$ *and, in addition, the condition*

$$\int\limits_{A(x_0,\varepsilon,\varepsilon_0)} Q(x)\cdot\psi^{\,\alpha}(|x-x_0|)\ dm(x)\leqslant K\cdot I^p(\varepsilon,\varepsilon_0) \tag{5.23}$$

holds for some $p<\alpha$*, for some constant* $K>0$ *and for any* $\varepsilon\in(0,\varepsilon_0')$*, where* $A:=A(x_0,\varepsilon,\varepsilon_0)$ *is defined in* (1.2)*. Assume that* $I(\varepsilon,\varepsilon_0)\rightarrow\infty$ *as* $\varepsilon\rightarrow 0$*, and that* $\varphi:(0,\infty)\rightarrow[0,\infty)$ *satisfies Calderon's condition* (1.9) *and the condition* (1.10)*. Then:*

I. *A mapping* f *has a continuous extension* $\overline{f}:\overline{\mathbb{B}^n}\rightarrow\overline{\mathbb{B}^n}$*. Moreover, there is a continuous extension* $F:B(0,R)\rightarrow\mathbb{R}^n$ *for any* $R\geqslant 1$*, where* F *is defined by* (5.3)*.*

II. *Let* $x_0\in\mathbb{S}^{n-1}$*.*

II. 1) If $\alpha=n$*, then*

$$h(\overline{f}(x),\overline{f}(x_0))\leqslant\frac{\alpha_n}{\delta}\cdot\exp\{-\beta_n I^{\gamma_{n,p}}(|x-x_0|,\varepsilon_0)\} \tag{5.24}$$

for any $x\in B(x_0,{\varepsilon_0}')\cap\overline{\mathbb{B}^n}$ *and any mapping* $\overline{f}\in F^{\varphi}_{Q,R,m,N,n}(\overline{\mathbb{B}^n})$*, where* $F^{\varphi}_{Q,R,m,N,\alpha}(\overline{\mathbb{B}^n})$ *denotes the family of all extended mappings* $\overline{f}:\overline{\mathbb{B}^n}\rightarrow\overline{\mathbb{B}^n}$*,* α_n *depends only on* n*;* β_n *depends only* n*,* R*,* m*, and* N*;* δ *depends only* n*,* R *and* m*; and* $\gamma_{n,p}=1-\frac{p-1}{n-1}$*.*

II. 2) if $n-1<\alpha\leqslant n$*, then*

$$|\overline{f}(x)-\overline{f}(x_0)|\leqslant C_n'\cdot I^{\frac{(p-\alpha)(n-1)}{\alpha}}(|x-x_0|,\varepsilon_0) \tag{5.25}$$

for any $x\in B(x_0,{\varepsilon_0}')\cap\overline{\mathbb{B}^n}$ *and* $\overline{f}\in F^{\varphi}_{Q,R,m,N,\alpha}(\overline{\mathbb{B}^n})$*, where* $F^{\varphi}_{Q,R,m,N,\alpha}(\overline{\mathbb{B}^n})$ *denotes the family of all extended mappings* $\overline{f}:\overline{\mathbb{B}^n}\rightarrow\overline{\mathbb{B}^n}$*,* $C_n'>0$ *is a constant depending only on* n*,* α*,* R *and* m*.*

Proof. The desired conclusion follows directly from Lemma 5.1. In particular, denoting as above $F(y)=\psi(f(\psi(y))$, $y\not\in\mathbb{B}^n$, $\psi(y):=\frac{y}{|y|^2}$, we observe that $|F(y)|=|\psi(f(\psi(y)))|=\frac{1}{|f(\psi(y))|}\leqslant m$ for any $f\in F^{\varphi}_{Q,R,m,N,\alpha}(\mathbb{B}^n)$ and $1<|y|<R$. So, we may set in Lemma 5.1 $\delta:=h(\mathbb{R}^n\setminus B(0,m))$ for $\alpha=n$ and $r=B(0,m)$ for $\alpha\ne n$. $\Box$

6. Proof of the main results

The proof of the following results may be found in [8, Lemma 2.5, Proof of Theorem 3.3].

Lemma 6.1. *Let $Q:\mathbb{R}^n\to[0,\infty]$ be a Lebesgue measurable function in $D\subset\mathbb{R}^n$, $n\geqslant 2$, $x_0\in\mathbb{R}^n$ and $n-1<\alpha\leqslant n$. Assume that $Q\in FMO(x_0)$. Then the relations* (5.22)–(5.23) *hold for all sufficiently small $\varepsilon_0>0$ with $\psi(t)=\frac{1}{\left(t\log\frac{1}{t}\right)^{\frac{n}{\alpha}}}$ and $p=1$. In this case, $I(\varepsilon,\varepsilon_0)=\log\frac{\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon_0}}$.*

Lemma 6.2. *Let $Q:\mathbb{R}^n\to[0,\infty]$ be a Lebesgue measurable function in $D\subset\mathbb{R}^n$, $n\geqslant 2$, $x_0\in\mathbb{R}^n$ and $\alpha<n-1\leqslant n$. Assume that, for sufficiently small $\varepsilon_0>0$,*

$$\int\limits_{\varepsilon}^{\varepsilon_0}\frac{dt}{t^{\frac{n-1}{\alpha-1}}q_{x_0}^{\frac{1}{\alpha-1}}(t)}<\infty \tag{6.1}$$

and

$$\int\limits_{0}^{\varepsilon_0}\frac{dt}{t^{\frac{n-1}{\alpha-1}}q_{x_0}^{\frac{1}{\alpha-1}}(t)}=\infty\,. \tag{6.2}$$

Then the relations (5.22)–(5.23) *hold for all sufficiently small $\varepsilon_0>0$ with*

$$\psi(t)\quad=\quad\begin{cases}1/[t^{\frac{n-1}{\alpha-1}}q_0^{\frac{1}{\alpha-1}}(t)]\,, & t\in(\varepsilon,\varepsilon_0)\ ,\\ 0\,, & t\notin(\varepsilon,\varepsilon_0)\ ,\end{cases}$$

where $K=\omega_{n-1}$ and $p=1$. In this case, $I(\varepsilon,\varepsilon_0)=\int\limits_{\varepsilon}^{\varepsilon_0}\frac{dt}{t^{\frac{n-1}{\alpha-1}}q_{x_0}^{\frac{1}{\alpha-1}}(t)}$.

The following lemma holds.

Lemma 6.3. *Let $Q:\mathbb{R}^n\to[0,\infty]$ be a Lebesgue measurable function such that $Q(x)\equiv 0$ for $x\notin\mathbb{B}^n$, and let $\zeta_0\in\mathbb{S}^{n-1}$. Assume that, there is $0<C_1<\infty$ such that*

$$\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(\zeta_0,\varepsilon)}Q(x)\,dm(x)\leqslant C_1 \tag{6.3}$$

as $\varepsilon\to 0$. Then

$$\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(\zeta_0,\varepsilon)}Q^*(x)\,dm(x)\leqslant C_2$$

as $\varepsilon\to 0$, where $C_2:=C_1(4^n+1)$ and

$$Q^*(x)=\begin{cases}Q(x), & x\in\mathbb{B}^n\,,\\ Q\left(\frac{x}{|x|^2}\right)\,, & x\in\mathbb{R}^n\setminus\mathbb{B}^n\end{cases}\,. \tag{6.4}$$

Proof. In essence, the assertion of the lemma was established in [19, Proof of Theorem 1.1, items VII-VIII], however, for the sake of completeness, we present its proof in full. Let $\zeta_0\in\mathbb{S}^{n-1}$ and $\varepsilon_0>0$. Notice, that

$$\psi(B_+(\zeta_0,\varepsilon_0))\subset B_-(\zeta_0,\varepsilon_0)\quad\forall\,\varepsilon_0\in(0,1)\,, \tag{6.5}$$

where

$$B_+(\zeta_0,\varepsilon_0)=\{x\in\mathbb{R}^n:\exists\, e\in\mathbb{S}^{n-1}, t\in[0,\varepsilon_0): x=\zeta_0+te, |x|>1\}=$$
$$=B(\zeta_0,\varepsilon_0)\cap(\mathbb{R}^n\setminus\mathbb{B}^n)\,,$$
$$B_-(\zeta_0,\varepsilon_0)=\{x\in\mathbb{R}^n:\exists\, e\in\mathbb{S}^{n-1}, t\in[0,\varepsilon_0): x=\zeta_0+te, |x|<1\}=$$
$$=B(\zeta_0,\varepsilon_0)\cap\mathbb{B}^n\,,$$

and, as above, $\psi(x)=\frac{x}{|x|^2}$. Indeed, for a given $x=\zeta_0+te\in B_+(\zeta_0,\varepsilon_0)$, computing the square of the module of the vector by means of the scalar product $(\cdot,\cdot)$, we obtain that

$$|\psi(x)-\zeta_0|^2=\left|\frac{\zeta_0+te}{|\zeta_0+te|^2}-\zeta_0\right|^2=$$
$$=\frac{1}{|\zeta_0+te|^2}-\frac{2(1+t(\zeta_0,e))}{|\zeta_0+te|^2}+\frac{|\zeta_0+te|^2}{|\zeta_0+te|^2}=$$
$$=\frac{1-2(1+t(\zeta_0,e))+1+2t(\zeta_0,e)+t^2}{|\zeta_0+te|^2}= \tag{6.6}$$
$$=\frac{t^2}{|\zeta_0+te|^2}<t^2\,,$$

that is, $|\psi(x)-\zeta_0|<t$, as required.

Similarly, let us to show that

$$\psi(\mathbb{R}^n\setminus(B(\zeta_0,\varepsilon)\cup\mathbb{B}^n))\subset\mathbb{B}^n\setminus B(\zeta_0,\varepsilon)\quad\forall\,\varepsilon\in(0,1)\,. \tag{6.7}$$

Indeed, let $x\in\mathbb{R}^n\setminus(B(\zeta_0,\varepsilon)\cup\mathbb{B}^n)$, $x=\zeta_0+te$, $|x|>1$, $e\in\mathbb{S}^{n-1}$, $\zeta_0\in\mathbb{S}^{n-1}$, $t\geqslant\varepsilon$. Arguing similarly to (6.6), we obtain that

$$|\psi(x)-\zeta_0|^2=\frac{t^2}{|\zeta_0+te|^2}=\frac{t^2}{|\zeta_0|^2+t^2+2t(\zeta_0,e)}\,. \tag{6.8}$$

By the Cauchy–Bunyakovsky inequality, we obtain that $|\zeta_0|^2+t^2+2t(\zeta_0,e)\leqslant 1+2t+t^2=(1+t)^2$. Now, we obtain from (6.8) that

$$|\psi(x)-\zeta_0|^2\geqslant\frac{t^2}{(1+t)^2}\geqslant t^2\geqslant\varepsilon^2\,. \tag{6.9}$$

In addition, since $x \in \mathbb{R}^n \setminus (B(\zeta_0, \varepsilon) \cup \mathbb{B}^n)$, we obtain that $x \notin \mathbb{B}^n$, so that $\psi(x) \in \mathbb{B}^n$. Due to the relation (6.9), $\psi(x) \in \mathbb{B}^n \setminus B(\zeta_0, \varepsilon)$, as required.

Let $0 < \varepsilon < 1/2$. Now, by (6.5) and by formula for the change of variable in the integral (see, e.g., [4, Theorem 3.2.5]) we obtain that

$$\int\limits_{B(\zeta_0,\varepsilon)\cap(\mathbb{R}^n\setminus\overline{\mathbb{B}^n})} Q^*(y)\,dm(y) = \int\limits_{B(\zeta_0,\varepsilon)\cap(\mathbb{R}^n\setminus\overline{\mathbb{B}^n})} Q(\psi(y))\,dm(y) \leqslant$$

$$\leqslant \int\limits_{B(\zeta_0,\varepsilon)\cap\mathbb{B}^n} Q(y)\cdot\frac{1}{|y|^{2n}}\,dm(y)\,. \tag{6.10}$$

Let $y \in B(\zeta_0, \varepsilon) \cap \mathbb{B}^n$. Now $y = \zeta_0 + et$, where $e \in \mathbb{S}^{n-1}$ and $0 \leqslant t < \varepsilon < 1/2$. Hence, by the Cauchy–Bunyakovsky inequality, we have that

$$|y|^2 = |\zeta_0 + et|^2 = 1 + 2t(\zeta_0, e) + t^2 \geqslant 1 - 2t + t^2 = (1-t)^2 \geqslant 1/4\,. \tag{6.11}$$

By (6.10) and (6.11),

$$\int\limits_{B(\zeta_0,\varepsilon)\cap(\mathbb{R}^n\setminus\overline{\mathbb{B}^n})} Q^*(y)\,dm(y) \leqslant 4^n\cdot\int\limits_{B(\zeta_0,\varepsilon)\cap\mathbb{B}^n} Q(y)\,dm(y)\,. \tag{6.12}$$

It immediately follows from (6.12) that

$$\int\limits_{B(\zeta_0,\varepsilon)} Q^*(y)\,dm(y) \leqslant (4^n+1)\cdot\int\limits_{B(\zeta_0,\varepsilon)\cap\mathbb{B}^n} Q(y)\,dm(y) < \infty\,. \tag{6.13}$$

It follows by (6.13) that

$$\frac{1}{\Omega_n r^n}\int\limits_{B(\zeta_0,\varepsilon)} Q^*(y)\,dm(y) \leqslant$$

$$\leqslant \frac{4^n+1}{\Omega_n r^n}\cdot\int\limits_{B(\zeta_0,\varepsilon)\cap\mathbb{B}^n} Q(y)\,dm(y) \leqslant C_1(4^n+1)\,. \tag{6.14}$$

□

Lemma 6.4. *Let $Q : \mathbb{R}^n \to [0,\infty]$ be a Lebesgue measurable function such that $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus \mathbb{B}^n$.*

If Q $Q \in FMO(\zeta_0)$ for some $\zeta_0 \in \mathbb{S}^{n-1}$, then $Q^* \in FMO(\zeta_0)$, where Q^* is defined in (6.4).

Proof. Let $Q \in FMO(\zeta_0)$. Denote $Q_\varepsilon := \frac{1}{\Omega_n \varepsilon^n} \int\limits_{B(\zeta_0,\varepsilon)} Q(x)\, dm(x)$. Then, by Lemma 6.3

$$\frac{1}{\Omega_n \varepsilon^n} \int\limits_{B(\zeta_0,\varepsilon)} |Q^*(x) - Q_\varepsilon|\, dm(x) \leqslant \frac{4^n+1}{\Omega_n \varepsilon^n} \int\limits_{B(\zeta_0,\varepsilon)\cap \mathbb{B}^n} |Q(x) - Q_\varepsilon|\, dm(x) < \infty$$

for sufficiently small $\varepsilon > 0$. Now, $Q^* \in FMO(\zeta_0)$ (see [17, Proposition 6.1]). □

Proof of Theorem 1.1 directly follows from Lemmas 5.2, 6.1 and 6.4. □

Proof of Theorem 1.2 directly follows from Lemmas 5.2 and 6.2. □

The following theorem holds.

Theorem 6.1. *Let us assume that under the conditions of Theorem 1.2* $n-1 < \alpha < n$ *and, instead of assumptions* (1.15)–(1.16), *the following condition holds:* $Q \in L^l(\mathbb{B}^n)$, $l \geqslant \frac{n}{n-\alpha}$. *Let* $x_0 \in \mathbb{S}^{n-1}$. *Then under notions of Theorem 1.2 the relations*

$$|\overline{f}(x) - \overline{f}(x_0)| \leqslant C_n' \cdot \log^{\frac{(\frac{\alpha}{n}-\alpha)(n-1)}{\alpha}} \frac{\varepsilon_0}{|x-x_0|} \tag{6.15}$$

for $l = \frac{n}{n-\alpha}$, $x \in B(x_0, \varepsilon_0{}') \cap \overline{\mathbb{B}^n}$ *and any* $\overline{f} \in F^{\varphi}_{Q,R,m,N,\alpha}(\overline{\mathbb{B}^n})$, *and*

$$|\overline{f}(x) - \overline{f}(x_0)| \leqslant C_n' \cdot \log^{-(n-1)} \frac{\varepsilon_0}{|x-x_0|} \tag{6.16}$$

hold for $l > \frac{n}{n-\alpha}$, $x \in B(x_0, \varepsilon_0{}') \cap \overline{\mathbb{B}^n}$ *and any* $\overline{f} \in F^{\varphi}_{Q,R,m,N,\alpha}(\overline{\mathbb{B}^n})$, *where* $F^{\varphi}_{Q,R,m,N,\alpha}(\overline{\mathbb{B}^n})$ *denotes the family of all extended mappings* $\overline{f} : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$, $C_n' > 0$ *is a constant depending only on* n, α, R, m, N.

Proof. Choose $\psi(t) := \frac{1}{t}$ in Lemma 5.2 and arguing similarly to the proof of [8, Theorem 3.2], we obtain the relations (5.22)–(5.23) with $p = \frac{\alpha}{n}$ for $l = \frac{n}{n-\alpha}$ and $p = 0$ for $l > \frac{n}{n-\alpha}$. □

Finally, we have the following statement.

Theorem 6.2. *If, under conditions of Theorem 6.1 and* $\alpha = n$, *we replace the assumption* $Q \in FMO(\mathbb{S}^{n-1})$ *by the condition* (6.3), *then*

$$|\overline{f}(x) - \overline{f}(x_0)| \leqslant 2\alpha_n \varepsilon_0^{-\gamma} |x-x_0|^{\gamma}$$

as $x \to x_0$, *where* $\alpha_n > 0$ *depends only on* n, *and*

$$\gamma = \left(\frac{\omega_{n-1} \log 2}{\Omega_n (4^n+1) 2^{n+1} C C_1 N^{n-1}(F, B(0,R))} \right)^{1/(n-1)},$$

where C *depends on* n, m *and* R, *and* F *is defined by* (5.3).

Proof. Firstly, $Q\in FMO(\mathbb{S}^{n-1})$ (see Corollary 6.1 in [17]), so that all of conclusions of Theorem 6.1 hold. Due to Lemma 6.1, all of the conditions and arguing from Lemma 5.1 hold, as well. In particular, F is a ring $C\cdot N^{n-1}(f, B(0,R))\cdot Q$-mapping, where F is defined by (5.3).

By Lemma 6.3

$$\sup\limits_{\varepsilon\in(0,\varepsilon_0)}\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0,\varepsilon)} Q^*(x)\,dm(x)\leqslant (4^n+1)\cdot C_1\,.$$

By Lemma 3.1 in [32], for $C_*=(4^n+1)\cdot C_1$ and $\varphi(t)=1$,

$$\int\limits_{A(x_0,\varepsilon,\varepsilon_0)}\frac{Q^*(x)\,dm(x)}{|x-x_0|^n}\leqslant$$

$$\leqslant\frac{\Omega_n(4^n+1)2^nC_1}{\log 2}\left(\log\frac{1}{\varepsilon}\right),\quad \forall\,\varepsilon\in(0,\varepsilon_0)\,,\quad\forall\,x_0\in\partial\mathbb{B}^n\,.$$

Observe that $\frac{\log\frac{1}{\varepsilon}}{\log\left(\frac{\varepsilon_0}{\varepsilon}\right)}=1+\frac{\log\frac{1}{\varepsilon_0}}{\log\left(\frac{\varepsilon_0}{\varepsilon}\right)}<2$ for $\varepsilon\in(0,\delta_0)$, where $\delta_0=\min\left\{\frac{1}{2},\varepsilon_0^2\right\}$. Now

$$\left(\log\left(\frac{\varepsilon_0}{\varepsilon}\right)\right)^{-1}\cdot\int\limits_{A(x_0,\varepsilon,\varepsilon_0)}\frac{Q^*(x)\,dm(x)}{|x-x_0|^n}\leqslant$$

$$\leqslant\frac{\Omega_n(4^n+1)2^nC_1}{\log 2}\frac{\log\frac{1}{\varepsilon}}{\log\left(\frac{\varepsilon_0}{\varepsilon}\right)}\leqslant\frac{\Omega_n(4^n+1)2^{n+1}C_1}{\log 2}\,. \tag{6.17}$$

Applying Lemma 5.1 for $\psi(t)=1/t$ we obtain by (6.17) that

$$h(F(x),F(x_0))\leqslant\alpha_n\left(\frac{|x-x_0|}{\varepsilon_0}\right)^{\gamma}\,,$$

for every $x\in B(x_0,\varepsilon_0^{\prime})$ and any $x_0\in\mathbb{R}^n\setminus\{0\}$, where γ is defined above, and α_n is some constant depending only on n. Finally, since $h(x,y)\geqslant\frac{|x-y|}{1+r_0^2}$ for $x,y\in\overline{B(0,r_0)}$ and any $r_0>0$, and $F|_{\overline{\mathbb{B}^n}}=\overline{f}$, we obtain that

$$|\overline{f}(x)-\overline{f}(x_0)|\leqslant 2\alpha_n\varepsilon_0^{-\gamma}|x-x_0|^{\gamma}\,.$$

Theorem is proved. $\Box$

7. Boundary behaviour of partial derivatives for solutions to certain Laplacian-gradient inequalities and spatial qr maps

For the subject see papers cited here and literature cited there (in particular [22]). Here we shortly review some results from [22] and prove a new result, Theorem 7.1. We also use notation from [22] (in particular see Definition 1 there). For the convenient of reader we recall a part of this definitions.

Definition 7.1. 1. We say that a bounded domain Ω in $\mathbb{R}^n$ and its boundary belong to the class $C^{k,\alpha}$, $0 \leqslant \alpha \leqslant 1$, if for every point $x_0 \in \partial\Omega$ there exists a ball $B = B(x_0)$ and we have mapping ψ from B onto D such that (cf. [7], page 95)

(a) $\psi(B \cap \Omega) \subset \mathbb{R}^n_+$

(b) $\psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$

(c) $\psi \in C^{k,\alpha}(B)$, $\psi^{-1} \in C^{k,\alpha}(D)$.
We refer to ψ in the above definition as a local coordinate diffeomorphism flattering the boundary in a neighboorhod of x_0.

If ψ is bi-Lipschitz we say that the domain Ω is weakly Lipschitz. On some place instead of Lipschitz we write Lip.

2. Suppose that $f : D \to D'$ is differentiable at a point $x \in D$. By $f'(x)$ (or $(df)_x$) we denote the linear operator which can be identified with the matrix $[D_j f_i(x)]$ and maps the tangent space at x into the tangent space at $f(x)$.

3. We adopt the standard terminology and notation for K-quasiconformal (K-qc) mappings [41]. If, in addition, f is a C^1 homemorphism on G and there is a constant $K \in [1, \infty)$ such that $\|f'(x)\|^n \leqslant KJ(x, f)$, $x \in G$, where $J(x, f)$ denotes the Jacobian of f, and $\|f'(x)\|$ is defined by (1.8), then we say that it is a K-quasiconformal (shortly K-qc) mapping. A map is called quasiconformal (shortly qc) if it is K-quasiconformal with some K.

   For harmonic quasiconformal mappings we use short notation HQC mappings.

4. If f is a twice-differentiable real-valued function, then the Laplacian of f is the real-valued function defined by:

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f\,.$$

Explicitly, the Laplacian of f is thus the sum of all the unmixed second partial derivatives in the Cartesian coordinates x_i:

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

Let D be a domain in $\mathbb{R}^n$ and $s : D \to \mathbb{R}$. If

$$|\Delta s| \leqslant a|\nabla s|^2 + b$$

on D, then we say that s satisfies a, b - Laplacian-gradient (Poisson differential inequality) inequality on D. It turns out that it is convenient to adapt this definition to vector valued functions on the following way. Namely, if $w : D \to \mathbb{R}^n$ satisfies the above inequality with w instead of s, then we say that w satisfies a, b - Laplacian-gradient inequality on D. If $w = (w_1, ..., w_n)$ and w_k, $k = 1, 2, ..., n$, satisfy a, b - Laplacian-gradient inequality on D, we say that w satisfies a, b - Laplacian-gradient inequality with respect to the coordinate functions on D.

Throughout this paper D, D_1, D_2, G denote domains in $\mathbb{R}^n$ space. In [22] we proved the following results:

Clame 7.1. ψ is bi-Lip on B if $k \geqslant 1$ and that $|D^2_{ij}\psi|$, $1 \leqslant i, j \leqslant n$ are bounded on B if $k \geqslant 2$.

Clame 7.2. if $w = (w_1, ..., w_n) : D \to D_1$ is a C^2 function, Δw and $|\nabla w|$ are bounded on D and the partial derivatives of H of the second order, i.e., $D^2_{ij}H$ are bounded, then $\Delta(H \circ w)$ is bounded on D.

Clame 7.3. Suppose that f is a C^1 homemorphism on $G \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and there is a constant $K \in [1, \infty)$ such that $\|f'(x)\|^n \leqslant KJ(x, f)$, $x \in G$, where $J(x, f)$ denotes the Jacobian of f. Then $\|f'(x)\| \approx l(f'(x)) \approx |\nabla f_i(x)|$, $x \in G$.

Recall that we study mappings in plane and space which satisfy the Laplacian-gradient inequality.

7.1. Local C^2 flattering method

First we outline approach which we refer as Local C^2 -coordinate method flattering the boundary used in [22](shortly local C^2 flattering method).

In order to explain this method suppose that D and D_1 are domains in $\mathbb{R}^n$ and

(1) $w = (w_1, ..., w_n) : D \to D_1$, $H : D_1 \to \mathbb{R}$ be C^2 functions and set $\hat{w}_H = H \circ w$. Using the chain rule formula to compute the derivative of a composite function we first have

(2) $D_k \hat{w}_H = \sum\limits_{i=1}^{n} D_i H D_k w_i$,

and hence

$$\Delta(H \circ w) =$$

$$= \sum_{i=1}^{n} D^2_{ii} H |\nabla w_i|^2 + 2 \sum_{i<j}^{n} D^2_{ij} H \left\langle \nabla w_i, \nabla w_j \right\rangle + \sum_{i=1}^{n} D_i H \Delta w_i. \qquad (7.1)$$

Using the change of variables formula (7.1) for Laplacian we can prove a preliminary result:

Clame 7.4. if $w = (w_1, ..., w_n) : D \to D_1$ is a C^2 mapping, Δw and $|\nabla w|$ are bounded on D and the partial derivatives of H of the second order $D^2_{ij} H$, where $1 \leqslant i, j \leqslant n$, are bounded, then $\Delta(H \circ w)$ is bounded on D.

Next in order to outline our approach concerning spatial versions of Kellogg's theorem suppose in addition that

(3) the considered mapping w in (1) is harmonic (more generally Δw is bounded on D or w satisfies Laplacian-gradient inequality on D),

(4) the codomain D_1 is a C^2 domain,

(5) w is proper and it has continuous extension on $\overline{D}$.
For simplicity we suppose in addition a stronger hypothesis than (5), that

(6) w is homeomorphisms of $\overline{D}$ onto $\overline{D}_1$,

(7) D is a smooth domain.

Here in general even with hypothesis (7) and with the continuity hypothesis (6) we can not conclude a priori that the boundary functions w^*, which is the restriction of w on ∂D, has some kind of smoothness. To get locally a smooth boundary function we use the hypothesis (4) which provides local coordinates.

Namely, let $x_0 \in \partial D$ and $y_0 = w(x_0)$ and let ψ be the local coordinate around y_0 from Definition 7.1 (1.) defined on a ball B and $\tilde{w} = \psi \circ w$. If $W = f^{-1}(B \cap D_1)$, then by Claim 7.1 there is a ball $W_1 = B(x_0, r_1)$ with center x_0 such that $\overline{W_1} \cap W \subset W$ and $|\nabla \tilde{w}| \approx |\nabla w|$ on $V = W_1 \cap W$. Hence we get the following auxiliary result:

Clame 7.5. If the considered mapping w in (1) satisfies the Laplacian-gradient inequality on D and partial derivatives of H of the second order $D^2_{ij}H$, where $1 \leqslant i, j \leqslant n$, are bounded, then $\tilde{w}$ satisfies the Laplacian-gradient inequality on V.

We call a domain $U \subset \mathbb{R}^m$ an elementary closed m-dimensional domain if it is homeomorphic to closed m-dimensional ball.

Now we consider $\tilde{w}_n = H \circ w$, where $H = \psi_n$. Note first that n-th coordinate ψ_n is 0 on some part $T_1 \subset \partial D_1$ of the neighborhood of y_0 with respect to ∂D_1. We can choose T_1 to be domain in ∂D_1 homeomorphic to closed $n-1$-dimensional ball. Hence we conclude that

(1'): $\tilde{w}_n$ is 0 on some part $T \subset \partial D$ of the neighborhood of x_0 with respect to ∂D (we can choose T to be an elementary closed $n-1$-dimensional domain), and that by Claim 7.5, we have (2'): $\tilde{w}_n$ satisfies the Laplacian-gradient inequality on $V_1 = B_2 \cap D$, where $B_2 = B(x_0, r_2)$ for every $r_2 \in (0, r_1)$.

In particular we consider the case $D = \mathbb{B}^n$.

This approach leads us to study the boundary behavior of gradient of real valued functions which satisfy the Laplacian-gradient inequality with smooth boundary condition. In this setting we can apply Claim 7.7 below which states that $|\nabla \tilde{w}_n|$ is bounded on V.

On the following figure we illustrate Local C^2-coordinate method flattening the boundary. Recall, using the previous setting and notation, let $x_0 \in \partial D, y_0 = f(x_0)$ and let ψ be local coordinate defined on $B = B(y_0, r_0)$, $r_0 > 0$. Note first that there is a domain W such that $x_0 \in W$ and $f(W \cap D) \subset B$. Next we can choose a ball $W_1 = B(x_0, r_1)$ with center x_0 such that $f(W_1 \cap D) \subset B$ and $T = W \cap \partial D$ is domain in ∂D, set $V = W_1 \cap D$.

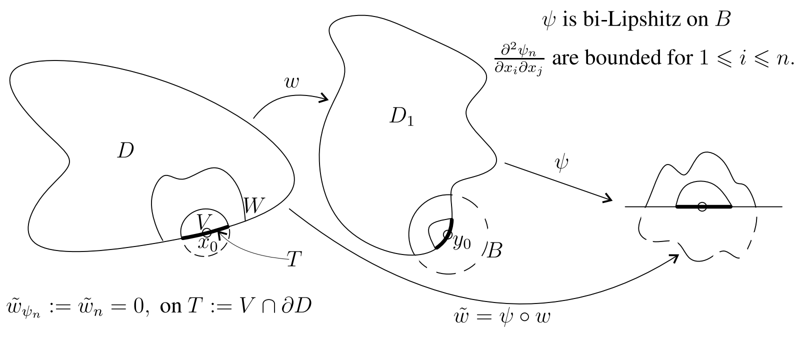

Figure 1. Flattening the boundary

The proof of next result in [22] is related to Heinz's approach. The proof of Lemma 9 and $9'$ of Heinz's paper [13] clearly applies to $n \geqslant 2$.

We can use Heinz's approach (cf. also Kalaj paper [12]) to prove Lemma 7.2 stated here as

Lemma 7.1 (Local gradient lemma Version 1)**.** *Consider the hypothesis:*

(h_1) *For a given $x_0 \in \mathbb{S}^{n-1}$ the real-valued function u is defined and continuous on $B(x_0, r_0) \cap \overline{\mathbb{B}^n}$, and C^2 on $V_0 = V_0(r_0) := B(x_0, r_0) \cap \mathbb{B}^n$.*

(h_2) *Δu is bounded on V_0.*

(h_3) *u is $C^{1,\alpha}$ on $B(x_0, r_0) \cap \mathbb{S}^{n-1}$.*

Conclusion (I): Then (h_1), (h_2) and (h_3) imply that for every $r < r_0$ partial derivatives of u are bounded on on $B_r \cap \mathbb{B}^n$, where $B_r = B(x_0, r)$.

Further in order to make an auxiliary statement that is interesting in itself let us consider the hypothesis (h_4): u satisfies a, b – Laplacian-gradient inequality on V_0.

Clame 7.6 (Local gradient lemma Version 2)**.** Under hypothesis (h_1) and (h_3) the hypothesis (h_2) and (h_4) are equivalent. In particular, the hypothesis (h_1), (h_3) and (h_4) imply that $|\nabla u|$ is bounded on $V_0(r)$, $r < r_0$.

Clame 7.7. The hypothesis (h_1), (h_3) and (h_4) imply that Δu is bounded on $V_0(r)$ for every $r < r_0$.

Thus under hypothesis (h_1) and (h_3) we have (h_2) is equivalent with (h_4). It is interesting that in this setting (h_4) is only a priori more general than (h_2).

Lemma 7.2 (Local gradient lemma Version 1)**.** *Consider the hypothesis:*

(h_1) *For a given $x_0 \in \mathbb{S}^{n-1}$ a real-valued function u is defined and continuous on $B(x_0, r_0) \cap \overline{\mathbb{B}^n}$, and C^2 on $V_0(r_0) := B(x_0, r_0) \cap \mathbb{B}^n$.*

(h_2) *Δu is bounded on V_0.*

(h_3) *u is $C^{1,\alpha}$ on $B(x_0, r_0) \cap \mathbb{S}^{n-1}$.*

Conclusion (IV)*: Then (h_1), (h_2) and (h_3) imply that for every $r < r_0$, the partial derivatives of u are bounded on $V_0(r) := B_r \cap \mathbb{B}^n$, where $B_r = B(x_0, r)$.*

Recall that Conclusion (IV) also holds if we replace the hypothesis (h_2) with a priori more general hypothesis (h_4): u satisfies a, b-Laplacian-gradient inequality on V_0. Namely, if we suppose (h_1), (h_3) and (h_4) we have by Claim 7.7 and Lemma 7.2 above, that there is a constant $M > 0$ such that $|\nabla u| \leqslant M$ on $V_0(r)$, $r < r_0$.

A natural question is to consider in which extent we can extend the above results using a more general inequality then Laplacian-gradient inequality.

For $a, d \geqslant 0$, we say that a map u between space(in particular planar) domains satisfy d-Poisson differential inequality (a, b-Laplacian-gradient inequality with gradient power d) if

$$|\Delta u| \leqslant a|\nabla u|^d + b\,.$$

Here using the above approach, we can prove:

Theorem 7.1. *Let D be a C^2 domain in $\mathbb{R}^n$ and let $f : \mathbb{B}^n \xrightarrow{onto} D$ be a C^2 proper K-qr mapping. If Δf is bounded (more generally f satisfies the Laplacian-gradient inequality with gradient power* 1) *on $\mathbb{B}^n$, then f is* Lip *on $\mathbb{B}^n$.*

Proof. Suppose first that Δf is bounded. Let $x_0 \in \mathbb{S}^{n-1}$ and $y_0 = f(x_0)$. For every point $y_0 \in \partial D$ there exists a ball $B = B(y_0, r_0)$ and a mapping ψ from B onto B^* such that ([7], p. 95) $\psi(B \cap D) \subset \mathbb{R}^n_+$ and $\psi(B \cap \partial D) \subset \partial\mathbb{R}^n_+$. By Claim 7.1 (see the introduction) we can choose B such ψ is Bi-Lip on B and that $D^2_{ij}\psi$ are bounded on B. Further we conclude, see Theorem 4.10 [42], that

(i-1) f has continuous extension on $\overline{\mathbb{B}^n}$ and since f is proper, $f(\mathbb{S}^{n-1}) \subset \partial D$.

If $\tilde{f} = \psi \circ f$ and $W = f^{-1}(B)$, then by (i-1) $\tilde{w}_n = 0$ on $W \cap \mathbb{S}^{n-1}$ and by Claim 7.2 $\Delta\tilde{w}_n$ is bounded on $W \cap \mathbb{B}^n$. Hence an application of Lemma 7.2, Local gradient lemma Version 1, shows that there is a ball W_1 with center x_0 such that $\overline{W_1} \subset W$ and that $\tilde{w}_n$ is Lip on $V = W_1 \cap \mathbb{B}^n$. Hence since ψ is Bi-Lip on B and f is K-qr, $\tilde{f}$ is K_1-qr on V. Next using that $\tilde{f}_n$ is Lip on V, and $\tilde{f}$ is K_1-qr on V and Claim 7.3 property of a qc mapping from the introduction, we conclude that $\tilde{f}$ is Lip on V and therefore since $f = \psi^{-1} \circ \tilde{f}$ it is Lip on V. Since x_0 is an arbitrary point we conclude f is Lip on $\mathbb{B}^n$. If f satisfies the Laplacian-gradient inequality on $\mathbb{B}^n$ the proof can be based on Claim 7.6, Local gradient lemma Version 2. □

7.2. Distortion of harmonic functions and harmonic quasiconformal quasi-isometry

If D and G are domains in $\mathbb{R}^n$, by $QRH(D,G)$ (respectively $QCH(D,G)$) we denote the family of quasiregular (respectively quasiconformal) harmonic maps of D onto G. If $D=G$ instead of $QCH(D,D)$ we write $QCH(D)$.

In [21] we proved:

Theorem 7.2. *If $h \in QCH(\mathbb{H}^n)$ and $h(\infty)=\infty$ then h is Euclidean bi-Lipschitz and a quasi-isometry with respect to the Poincare distance.*

Theorem 7.3. *If $h \in QRH(\mathbb{H}^n)$ and $h(\infty)=\infty$ then h is Lipschitz with respect to Euclidean and the Poincare distance.*

We now outline a proof of Theorem. Suppose that $n=3$ (the same proof works in general). Let $h=(h_1,h_2,h_3)$. Since $h_3(x)=x_3$, we have $h'_{x_3}(x)=1$ and, therefore, $|h'(x)|\leqslant c$. Since $h_3(x)=x_3$, we have

$$\frac{|h'(x)|}{h_3(x)} \leqslant \frac{c}{x_3}$$

and hence $\lambda(h(a),h(b))\leqslant c\lambda(a,b)$, where λ is a hyperbolic metric in $\mathbb{H}^3$.

For a domain $G\subset\mathbb{R}^n$ let $\rho: G\to(0,\infty)$ be a function. We say that ρ is a weight function or a metric density if for every locally rectifiable curve γ in G, the integral

$$l_\rho(\gamma)=\int_\gamma \rho(x)ds$$

exists. In this case we call $l_\rho(\gamma)$ the ρ-length of γ. A metric density defines a metric $d_\rho: G\times G\to(0,\infty)$ as follows. For $a,b\in G$, let

$$d_\rho(a,b)=\inf_\gamma l_\rho(\gamma)$$

where the infimum is taken over all locally rectifiable curves in G joining a and b. It is an easy exercise to check that d_ρ satisfies the axioms of a metric. For instance, the hyperbolic (or Poincaré) metric of $\mathbb{D}$ is defined in terms of the density $\rho(x)=c/(1-|x|^2)$ where $c>0$ is a constant. The quasihyperbolic metric $k=k_G$ of G is a particular case of the metric d_ρ when $\rho(x)=\frac{1}{d(x,\partial G)}$. By $\overline{\mathbb{H}}^n$ we denote closure in $\mathbb{R}^n$. Suppose that G is a domain in $\mathbb{R}^n$ and $\overline{G}$ is homeomorphic to $\overline{\mathbb{H}}^n$.

Theorem 7.4. *If $h\in QCH(G,\mathbb{H}^n)$ and $h(\infty)=\infty$ then h is Euclidean bi-Lipschitz and a quasi-isometry with respect to the quasihyperbolic metric $k=k_G$ of G and the Poincare distance $\mathbb{H}^n$.*

8. Pseudo-isometry and $OC^1(G)$

In this section, we review some results from [25].

More precisely, we give a sufficient condition for a qc mapping $f : G \to f(G)$ to be a pseudo-isometry w.r.t. quasihyperbolic metrics on G and $f(G)$. First we adopt the following notation.

If V is a subset of $\mathbb{R}^n$ and $u : V \to \mathbb{R}^m$, we define

$$\mathrm{osc}_V u = \sup\{|u(x) - u(y)| \, : x, y \in V\}\,.$$

Suppose that $G \subset \mathbb{R}^n$ and $B_x = B(x, d(x)/2)$. Let $OC^1(G)$ denote the class of $f \in C^1(G)$ such that

$$d(x)\|f'(x)\| \leqslant c_1\, \mathrm{osc}_{B_x} f \tag{8.1}$$

for every $x \in G$. Similarly, let $SC^1(G)$ be the class of functions $f \in C^1(G)$ such that

$$\|f'(x)\| \leqslant a r^{-1}\, \omega_f(x, r) \quad \text{for all } B(x, r) \subset G, \tag{8.2}$$

where $\omega_f(x, r) = \sup\{|f(y) - f(x)| : \, y \in B(x, r)\}$.

For a domain $G \subset \mathbb{R}^n, n \geqslant 2, x, y \in G$, let

$$r_G(x, y) = \frac{|x - y|}{\min\{d(x), d(y)\}}$$

where

$$d(x) = d(x, \partial G) \equiv \inf\{|z - x| : z \in \partial G\}\,.$$

If the domain G is understood from the context, we write r instead r_G. This quantity is used, for instance, in the study of quasiconformal and quasiregular mappings, cf. [42]. It is a basic fact that [41, Theorem 18.1] for $n \geqslant 2, K \geqslant 1, c_2 > 0$ there exists $c_1 \in (0, 1)$ such that whenever $f : G \to f(G)$ is a quasiconformal mapping with $G, f(G) \subset \mathbb{R}^n$ then $x, y \in G$ and $r_G(x, y) \leqslant c_1$ imply $r_{f(G)}(f(x), f(y)) \leqslant c_2$. We call this property the local uniform boundedness of f with respect to r_G. Note that quasiconformal mappings satisfy the local uniform boundedness property and so do quasiregular mappings under appropriate conditions; it is known that one to one mappings satisfying the local uniform boundedness property may not be quasiconformal.

We also consider a weaker form of this property and say that $f : G \to f(G)$ with $G, f(G) \subset \mathbb{R}^n$ satisfies the weak uniform boundedness property on G (with respect to r_G) if there is a constant $c > 0$ such that $r_G(x, y) \leqslant 1/2$ implies $r_{f(G)}(f(x), f(y)) \leqslant c$. Univalent harmonic mappings fail to satisfy the weak uniform boundedness property as a rule

The proof of Theorem 2.13 [25] gives the following more general result:

Theorem 8.1. *Suppose that $G \subset \mathbb{R}^n$, $f : G \to G'$, $f \in OC^1(G)$ and it satisfies the weak property of uniform boundedness with a constant c on G. Then*

(e) $f : (G, k_G) \to (G', k_{G'})$ *is Lipschitz.*

(f) *In addition, if f is K-qc, then f is pseudo-isometry w.r.t. quasi-hyperbolic metrics on G and $f(G)$.*

Proof. By the hypothesis f satisfies the weak property of uniform boundedness: $|f(t) - f(x)| \leqslant c_2\, d(f(x)$ for every $t \in B_x$, that is

$$\mathrm{osc}_{B_x} f \leqslant c_2\, d(f(x)) \tag{8.3}$$

for every $x \in G$. This inequality together with (8.1) gives $d(x)\|f'(x)\| \leqslant c_3\, d(f(x))$. Now an application of Lemma 2.10 [25] gives part (e). Since f^{-1} is qc, an application of [6, Theorem 3] on f^{-1} gives part (f). □

In order to apply the above method we introduce subclasses of $OC^1(G)$ (see, for example, below (8.4)).

Let $f : G \to G'$ be a C^2 function and $B_x = B(x, d(x)/2)$. We denote by $OC^2(G)$ the class of functions which satisfy the following condition:

$$\sup_{B_x} d^2(x)|\Delta f(x)| \leqslant c\ \mathrm{osc}_{B_x} f \tag{8.4}$$

for every $x \in G$.

If $f \in OC^2(G)$, then by Theorem 3.9 in [7], applied to $\Omega = B_x$,

$$\sup_{t\in B_x} d(t)\|f'(t)\| \leqslant C(\sup_{t\in B_x} |f(t) - f(x)| + \sup_{t\in B_x} d^2(t)|\Delta f(t)|)$$

and hence by (8.4)

$$d(x)\|f'(x)\| \leqslant c_1\, \mathrm{osc}_{B_x} f \tag{8.5}$$

for every $x \in G$ and therefore $OC^2(G) \subset OC^1(G)$.

Now the following result follows from the previous theorem.

Corollary 8.1. *Suppose that $G \subset \mathbb{R}^n$ is a proper subdomain, $f : G \to G'$ is K-qc and f satisfies the condition (8.4)* (that is $f \in OC^2(G)$). Then $f : (G, k_G) \to (G', k_{G'})$ is Lipschitz.

We will now give some examples of classes of functions to which Theorem 8.1 is applicable. Let $SC^2(G)$ denote the class of $f \in C^2(G)$ such that

$$|\Delta f(x)| \leqslant a r^{-1} \sup\{\|f'(y)\| : y \in B(x, r)\},$$

for all $B(x,r) \subset G$, where a is a positive constant. Note that the class $SC^2(G)$ contains every function for which $d(x)|\Delta f(x)| \leqslant a\|f'(x)\|$, $x \in G$. It is clear that $SC^1(G) \subset OC^1(G)$ and by the mean value theorem, $OC^2(G) \subset SC^2(G)$. Note that $SC^2(G) \subset SC^1(G)$ and that the class $SC^2(G)$ contains harmonic functions, eigenfunctions of the ordinary Laplacian if G is bounded, eigenfunctions of the hyperbolic Laplacian if $G = \mathbb{B}^n$ and therefore our results are applicable for instance to the mentioned classes.

Let P denote harmonic Poisson kernel for the unit ball $\mathbb{B}^n$. It is interesting that P maps $\Lambda_\alpha(\mathbb{S}^{n-1})$ into $\Lambda_\alpha(\mathbb{B}^n)$, $0 < \alpha < 1$, and if $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$, then in general $P[f]$ is not in $\mathrm{Lip}(\mathbb{B}^n)$. Here by Λ_α we denote the class of Hölder continuous function with power exponent α.

It is natural to consider the corresponding question for the hyperbolic Poisson kernel P_h.

Question 8.1. Whether partial derivatives of P_h are bounded on the set $\mathrm{Lip}(\mathbb{S}^{n-1})$, where P_h is hyperbolic Poisson kernel for the unit ball $\mathbb{B}^n$?

It is true; see [2,24]. More precisely, if $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$, then in general, $P_h[f]$ is in $\mathrm{Lip}(\mathbb{B}^n)$. This is not true for harmonic.

9. Further results 1

Here we outline a proof of Theorem 9.2 below. In the recent article [9] D. Kalaj and A. Gjokaj proved:

Theorem 9.1. *If*

(i) : there is a C^{1+} diffeomorphism $\phi : \overline{\mathbb{B}^n} \to \overline{D}$ and

(ii) : f is a harmonic quasiconformal mapping between the unit ball in $\mathbb{R}^n$ and D,
then f is Lipschitz continuous in $\mathbb{B}^n$.

In [22] the first author of this paper consider local version of results of this type and announced more general results. This generalizes some known results for $n = 2$ and improves some others in higher dimensional case. Here it seems a natural to ask:

Question 9.1. If D is a C^{1+} space domain homeomorphic to the unit ball $\mathbb{B}^n$ whether D satisfies (i)?

Definition 9.1. 1. D is a domain with C^{1+} boundary if there is $\alpha \in (0,1]$ such that it is $C^{1,\alpha}$ domain.

2. We say that D is good Green-ian domain if $|D_k \overline{g}_D(x,y)| \leqslant c\frac{1}{|x-y|^{n-1}}$, $k=1,2,...,n$, $x,y\in D$, where D_k denotes D_{x_k} and locally good Green-ian domain at $x_0\in\partial D$ if for every $\delta>0$ there is a C^{1+} domain $W=W_{x_0}\subset D\cap B(x_0,\delta)$ such that $x_0\in\partial W$ and ∂W is an open set in ∂D.

3. D is a locally good Green-ian domain with respect to all ∂D, if it is a locally good Green-ian domain at every $x_0\in\partial D$.

In [3] it is proved that sufficiently smooth domains are good Green-ian. Furthermore it seems that we can use Theorem 2.3 in [43] to prove if D is a domain with C^{1+} boundary, then D is a locally good Green-ian domain with respect to all ∂D. Here we note that Theorem 9.2 below is more general than Theorem 9.1.

Namely, condition (i) on the codomain of the function f in Theorem 9.1 and assumption (ii) that f is HQC are replaced with much more general assumptions (1) and (2)-(3), and it seems that using our approach we can prove a general version which is applicable to (K,K') qr mapping which in general are not injective.

Theorem 9.2 ([23]). *Suppose that*

(1) *D and G are domains with C^{1+} boundary, D is a locally good Green-ian domain with respect to all ∂D, and $f:D\xrightarrow{onto}G$.*

(2) $f\in OC^1(D)$.

(3) *Suppose in addition that G is $C^{1,\alpha}$ domain, $f=(f_1,...,f_n)$ is a C^2 vector valued function, f_i, $i=1,2,...,n$, satisfy Laplacian-gradient inequality on D.*

(4) *f is K qc on D.*

Conclusion:

(a) If (1) holds, then $M_\gamma\in L^l(D)$, for $l<l_0=\frac{p}{2-\gamma+p\gamma}$, where $\gamma=1-\alpha$.

(b) If (1), (2), (3) and (4) hold, then then f is Lipschitz continuous on D.

Since $SC^1(D)\subset OC^1(D)$ we can replace the hypothesis (3) in the previous theorem with (3') $f\in SC^1(D)$.

Question 9.2. If (1), (2), (3) and (4) hold, whether there is a unit vector fields X on D (i.e. to each x we associate a unit vector $X=X(x)$ with initial point at x) such that $|df_x(X(x))|\leqslant c$ for every $x\in V_1$?

This is true if G is C^2 domain.

Question 9.3. Suppose that
(5) f is (K, K') qr proper on D.

If (1), (2), (3) and (5) hold, whether then f is Lipschitz continuous on D?

Note that (2) and (4) imply that for every domain $Z \subset D$, the restriction of f on Z is Lipschitz with respect to corresponding k-metrics on Z and $f(Z)$.

Outline of proof of Theorem 9.2. We use harmonic coordinate for C^{1+} boundary and the formula $\chi = P_D[\chi_b] + \int\limits_D \overline{g}(x,y)\Delta\chi(y)dy$,

$$D_k\chi(x) = D_k(P_D[\chi_b])(x) + \int\limits_D D_k\overline{g}(x,y)\Delta\chi(y)dy$$

and the iteration approach related to Imbedding Lemma for Riesz potential.

Now we outline approach which can be applied to C^{1+} co-domains and which is motivated by local C^2 flattering method.In this setting, instead of local coordinate ψ described in Figure 1, we use harmonic coordinates described below.

Here we will try briefly (without technical details) to outline an approach on which we refer as local C^{1+} -coordinate method flattering the boundary (related to functions which satisfy the Laplacian-gradient inequality).

Let G be a domain in $\mathbb{R}^n$, $0 \in \partial G$, $B_0 = B(0, r_0)$, $U = B_0 \cap G$, and r_0 small enough such that $B_0 \cap \partial G$ be C^{1+} homeomorphic to unit ball in $\mathbb{R}^{n-1}$.

There is a $r_1 \in (0, r_0)$ such that a local coordinate ψ is defied on $B_1 = B(0, r_1)$ (note $B_1 \subset B_0$) and there is $B_2 = B(0, r_2)$, $r_2 < (0, r_1)$, and a $C^{1,\alpha}$ domain $V \subset U \cap B_2$ such that $0 \in \partial V$ and $\partial V \cap B_2 = \partial U \cap B_2$.

Define $\psi^* = \psi|_{\partial V}$. Since ψ is $C^{1,\alpha}$ on B_1 it is Bi-Lip on B_2 and $J_\psi(0) \neq 0$ and therefore $J_{\psi^*}(0) \neq 0$. Let H be solution of Dirichlet problem on V with boundary data ψ and $s = H_n$. We call H a local Harmonic coordinate for the domain G. The function H is $C^{1,\alpha}$ on $\overline{V}$ and therefore $J_H(0)$ exists. Here $d(x) := d_V(x) = \operatorname{dist}(x, \partial V)$. If $y \in B(x, d_V(x))$, then $|D_{ij}s(x)| \leqslant c|x-y|^{\alpha-1}$ and therefore $|D_{ij}s(x)| \leqslant c d_V^{\alpha-1}(x)$. Thus if codomain D_1 is C^2, $|D_{ij}\psi(x)|$ are bounded and if it is $C^{1,\alpha}$ in general $D_{ij}\psi$ does not exist and therefore we do not have estimate. So instead we use a local Harmonic coordinate H. Since in general $|D_{ij}H(x)|$ are not bounded we use the iteration approach related to Imbedding Lemma for Riesz potential.

Finally we use the formula $\chi = P_D[\chi_b] + \int\limits_D \overline{g}(x,y)\Delta\chi(y)dy$, $D_k\chi(x) = D_k(P_D[\chi_b])(x) + \int\limits_D D_k\overline{g}(x,y)\Delta\chi(y)dy$, for $\chi = H_n$ and $D = W$, where W is a C^{1+} locally good Greenian domain such that $f(W) \subset V$, $\partial W \cap \partial D$ is open nonempty set in ∂D and $f(\partial W \cap \partial D) \subset \partial V \cap B_2$.

Acknowledgements. We would like to thank N. Mutavdžić who made the Figure 1, and turned our attention to paper [43].

References

[1] Adamowicz, T., Shanmugalingam, N. (2010). Non-conformal Loewner type estimates for modulus of curve families, *Ann. Acad. Sci. Fenn. Math., 35*, 609–626.

[2] Chen, J., Huang, M., Rasila, A., Wang, X. (2018). On Lipschitz continuity of solutions of hyperbolic Poisson's equation, *Calc. Var. Partial Differ. Equ., 57*(1), 1–32.

[3] Dall'Acqua, A., Sweers, G. (2004). Estimates for Green function and Poisson kernels of higher order Dirichlet boundary value problems, *Journal of Differential Equations, 205*(2), 466–487.

[4] Federer, H. (1969). *Geometric Measure Theory,* Berlin etc., Springer.

[5] Gehring, F.W., (1962). Rings and quasiconformal mappings in space, *Trans. Amer. Math. Soc., 103*, 353–393.

[6] Gehring, F.W., Osgood, B.G. (1979). Uniform domains and the quasi-hyperbolic metric, *J. Anal. Math., 36*, 50–74.

[7] Gilbarg, D., Trudinger, N. (1983). *Elliptic partial Differential Equation of Second Order, Second Edition*, Berlin, Heidelberg, Springer.

[8] Golberg, A., Salimov, R., Sevost'yanov, E. (2016). Normal Families of Discrete Open Mappings with Controlled p-Module, *Contemporary Mathematics, 667*, 83–103.

[9] Gjokaj, A., Kalaj, D. (2021). *QCH mappings between unit ball and domain with $C^{1,\alpha}$ boundary*, arXiv:2005.05667 [math.AP].

[10] Heinonen, J. (2001). *Lectures on Analysis on metric spaces,* New York, Springer Science+Business Media.

[11] Hajlasz, P., Koskela, P. (2000). Sobolev met Poincare, *Mem. Amer. Math. Soc., 145*, no. 668, 1–100.

[12] Kalaj, D. (2013). A priori estimate of gradient of a solution to certain differential inequality and quasiregular mappings. *J. d'Anal. Math., 119*(1), 63–88.

[13] Heinz, E. (1956/57). On certain nonlinear elliptic differential equations and univalent mappings. *J. d' Anal. Math., 5*, 197–272.

[14] Kruglikov, V.I. (1987). Capacity of condensers and spatial mappings quasiconformal in the mean, *Math. USSR-Sb., 58*(1), 185–205.

[15] Kovtonyuk, D., Ryazanov, V., Salimov, R., Sevost'yanov, E. (2014). Toward the theory of Orlicz-Sobolev classes, *St. Petersburg Math. J., 25*(6), 929–963.

[16] Martio, O., Rickman, S., Väisälä, J. (1970). Distortion and singularities of quasiregular mappings, *Ann. Acad. Sci. Fenn. Ser. A1, 465*, 1–13.

[17] Martio, O., Ryazanov, V., Srebro, U., Yakubov, E. (2009). *Moduli in Modern Mapping Theory.* Springer Science + Business Media, New York LLC.

[18] Martio, O., Srebro, U. (1975). Automorphic quasimeromorphic mappings in $\mathbb{R}^n$, *Acta Math., 135*, 221–247.

[19] Mateljević, M., Salimov, R., Sevost'yanov, E. (2022). Hölder and Lipschitz continuity in Orlicz–Sobolev classes, distortion and harmonic mappings, *Filomat, 36*(16), 5361–5392.

[20] Ma, L. (2022). Hölder continuity of hyperbolic Poisson integral and hyperbolic Green integral, *Monatshefte für Mathematik, 199*, 595–610.

[21] Mateljević, M. (2006). Distortion of harmonic functions and harmonic quasiconformal quasi-isometry, *Revue Roum. Math. Pures Appl., 51*(5–6), 711–722.

[22] Mateljević, M., (2021). Boundary behaviour of partial derivatives for solutions to certain Laplacian-gradient inequalities and spatial qc maps, *Springer Proceedings in Mathematics & Statistics, 357*, 393–418.

[23] Mateljević, M. (2022). *Boundary Behaviour Of Partial Derivatives For Solutions To Certain Laplacian-Gradient Inequalities And Spatial Qc Maps 2,* Preprint, Comunicated at XII Symposium Mathematics and Applications, Mathematical Faculty, Belgrade, 2,3.

[24] Mateljević, M., Mutavdžić, N. (2022). *On Lipschitz continuity and smoothness up to the boundary of solutions of hyperbolic Poisson's equation*, arXiv:2208.06197v1.

[25] Mateljević, M., Vuorinen, M. (2010). On Harmonic Quasiconformal Quasi-Isometries, *Journal of Inequalities and Applications*, Article ID 178–732.

[26] Maz'ya, V. (1985). *Sobolev classes.* New York, Berlin, Springer.

[27] Mutavdžić, N. (to appear). The growth of gradients QC-mappings in n-dimensional Euclidean space with bounded Laplacian, *Kragujevac Journal of Mathematics.*

[28] Peijin, L., Ponnusamy, S. (2018). Lipschitz Continuity of Quasiconformal Mappings and of the Solutions to Second Order Elliptic PDE with Respect to the Distance Ratio Metric, *Complex Analysis and Operator Theory, 12*, 1991–2001.

[29] Rickman, S. (1993). *Quasiregular mappings*, Berlin etc., Springer-Verlag.

[30] Reshetnyak, Yu.G. (1989). *Space mappings with bounded distortion*, Transl. Math. Monogr., vol. 73, Amer. Math. Soc., Providence, RI.

[31] Ryazanov, V., Sevost'yanov. E. (2008). Toward the theory of ring Q-homeomorphisms, *Israel J. Math., 168*, 101–118.

[32] Ryazanov, V., Salimov, R., Sevost'yanov, E. (2020). On the Hölder property of mappings in domains and on boundaries, *J. Math. Sci., 246*(1), 60–74; transl. from (2019). *Ukr. Mat. Visn., 16*(3), 383–402.

[33] Saks, S. (1964). *Theory of the integral*, New York.

[34] Salimov, R.R. (2012). Estimation of the measure of the image of the ball, *Sib. Math. J., 53*(4), 739–747.

[35] Sevost'yanov, E.A. (2007). Theory of moduli and capacities, and normal families of mappings that admit branching, *Ukrainian Math. Bull., 4*(4), 573–593.

[36] Sevost'yanov, E.A. (2008). On the normality of families of space mappings with branching, *Ukrainian Math. J., 60*(10), 1618–1632.

[37] Sevost'yanov, E.A. (2010). Towards a theory of removable singularities for maps with unbounded characteristic of quasi-conformity, *Izv. Math., 74*(1), 151–165.

[38] Sevost'yanov, E.A. (2019). On the boundary behavior of some classes of mappings, *J. Math. Sci., 243*(6), 934–948.

[39] Sevost'yanov, E.A., Skvortsov, S.A. (2017). On the local behavior of the Orlicz-Sobolev classes, *Journ. of Math. Sciences, 224*(4), 563–581; transl. from (2016). *Ukr. Mat. Visn., 13*(4), 543–570.

[40] Titus, C.J., Young, G.S. (1962). The extension of interiority, with some applications, *Trans. Amer. Math. Soc., 103*(2), 329–340.

[41] Väisälä, J. (1971). *Lectures on n-Dimensional Quasiconformal Mappings*, Lecture Notes in Math., 229, Berlin etc., Springer–Verlag.

[42] Vuorinen, M. (1976). Exceptional sets and boundary behavior of quasiregular mappings in n-space, *Ann. Acad. Sci. Fenn. Ser. A 1. Math. Dissertationes, 11*, 1–44.

[43] Widman Kjell-Ove. (1967). Inequalities for the Green Function and Boundary Continuity of the Gradient of Solutions of Elliptic Differential Equations, *Mathematica Scandinavica, 21*(1), 17–37.

Contact information

Evgeny Sevost'yanov

Zhytomyr Ivan Franko State University,
Zhytomyr, Ukraine,
Institute of Applied Mathematics
and Mechanics of the NAS of Ukraine,
Slov'yans'k, Ukraine
E-Mail: esevostyanov2009@gmail.com

Miodrag Mateljevic

University of Belgrade,
Faculty of Mathematics,
Belgrade, Serbia
E-Mail: miodrag@matf.bg.ac.rs