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The Dirichlet problem for the Beltrami equations with sources

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Abstract. The present paper is devoted to the study of the Dirichlet problem $\mathrm{Re}\,\omega(z) \to \varphi(\zeta)$ as $z \to \zeta$, $z \in D, \zeta \in \partial D$, with continuous boundary data $\varphi : \partial D \to \mathbb{R}$ for Beltrami equations $\omega_{\bar{z}} = \mu(z)\omega_z + \sigma(z)$, $|\mu(z)| < 1$ a.e., with sources $\sigma : D \to \mathbb{C}$ in the case of locally uniform ellipticity. In this case, we establish a series of effective integral criteria of the type of BMO, FMO, Calderon-Zygmund, Lehto and Orlicz on singularities of the equations at the boundary for existence, representation and regularity of solutions in arbitrary bounded domains D of the complex plane $\mathbb{C}$ with no boundary component degenerated to a single point for sources σ in $L_p(D)$, $p > 2$, with compact support in D. Moreover, we prove in such domains existence, representation and regularity of weak solutions of the Dirichlet problem for the Poisson type equation $\mathrm{div}[A(z)\nabla\, u(z)] = g(z)$ whose source $g \in L_p(D)$, $p > 1$, has compact support in D and whose matrix valued coefficient $A(z)$ guarantees its locally uniform ellipticity.

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1. Introduction

Let D be a domain in the complex plane $\mathbb{C}$. We study the **Dirichlet problem**

$$\lim_{z\to\zeta} \mathrm{Re}\,\omega(z) = \varphi(\zeta) \qquad \forall\ \zeta \in \partial D\ , \tag{1.1}$$

see e.g. [5] and [48], with continuous boundary data $\varphi : \partial D \to \mathbb{R}$ in arbitrary bounded domains D with no boundary component degenerated to a single point for the **inhomogeneous Beltrami equation**

$$\omega_{\bar{z}} \ = \ \mu(z)\cdot\omega_z \ + \ \sigma(z)\ , \qquad z \in D\ , \tag{1.2}$$

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with a source $\sigma : D \to \mathbb{C}$ in L_p, $p > 2$, where $\mu : D \to \mathbb{C}$ is a measurable function with $|\mu(z)| < 1$ a.e., $\omega_{\bar{z}} = (\omega_x + i\omega_y)/2$, $\omega_z = (\omega_x - i\omega_y)/2$, $z = x + iy$, ω_x and ω_y are partial derivatives of the function ω in x and y, respectively.

Moreover, in general bounded domains D with no boundary component degenerated to a single point, we prove the corresponding theorems on existence, representation and regularity of solutions for the **classical Dirichlet problem**

$$\lim_{z \to \zeta} u(z) \;=\; \varphi(\zeta) \qquad \forall\, \zeta \in \partial D \tag{1.3}$$

with continuous boundary data $\varphi : \partial D \to \mathbb{R}$ to the **Poisson type equation**

$$\mathrm{div}[A(z)\nabla\, u(z)] = g(z) \tag{1.4}$$

with a source $g : D \to \mathbb{R}$ in $L_p(D)$, $p > 1$, see e.g. background for $g \equiv 0$ in [18].

The request on domains to have no boundary component degenerated to a single point is necessary. Indeed, consider the punctured unit disk $\mathbb{D}_0 := \mathbb{D} \setminus \{0\}$. Setting $\varphi(\zeta) \equiv 1$ on $\partial\mathbb{D}$ and $\varphi(0) = 0$, we see that φ is continuous on $\partial\mathbb{D}_0 = \partial\mathbb{D} \cup \{0\}$. Let us assume that there is a harmonic function u satisfying (1.3). Then u is bounded by the maximum principle for harmonic functions and by the classical Cauchy–Riemann theorem, see also Theorem V.4.2 in [32], the extended u is harmonic in $\mathbb{D}$. Thus, by Mean-Value-Property for harmonic functions we disprove the above assumption, see e.g. Theorem 0.2.4 in [45].

In this connection, recall that a boundary point p of a domain D in $\mathbb{R}^n, n \geq 2$, is called regular if each solution of the Dirichlet problem for the Laplace equation in D, whose Dirichlet boundary date is continuous at p, is also continuous at p. The famous Wiener criterion for regularity of a boundary point, see [49], that has been formulated in terms of so-called barrier functions, generally speaking, has no satisfactory geometric interpretation. However, there is a very simple geometric criterion of regular points in the case of $\mathbb{C}$. Namely, a point $p \in \partial D$ is regular if p belongs to a component of ∂D that is not degenerated to a single point, see Theorem 4.2.2 in [36]. The above example shows that this condition is not only sufficient but also necessary for regularity of a boundary point. Thus, results on the Dirichlet problem (1.3) to the Poisson type equations (1.4) given in the end of the paper were obtained for the most general admissible domains.

For the case $\|\mu\|_\infty < 1$, (1.2) was first introduced by L. Ahlfors and L. Bers in the paper [2], see also the Ahlfors monograph [1]. Here we study the case of locally uniform ellipticity of the equation (1.2) when its **dilatation quotient** K_μ is bounded only locally in D,

$$K_\mu(z) \ := \ \frac{1 \ + \ |\mu(z)|}{1 \ - \ |\mu(z)|}\ , \tag{1.5}$$

i.e., if $K_\mu \in L_\infty$ on each compact set in D but admits singularities at the boundary. A point $\zeta \in \partial D$ is called a **singular point of the equation** (1.2) if $K_\mu \notin L_\infty$ on each neighborhood of the point.

First of all, we show that, if D is an arbitrary simply connected domain in $\mathbb{C}$, then the Dirichlet problem (1.1) to the equation (1.2) has a solution ω in class $W^{1,2}_{\rm loc}(D)$ for a wide circle of singularities of (1.2) at the boundary. Moreover, it is unique up to an additive pure imaginary constant, and it can be represented through the so-called generalized analytic functions with sources.

Recall that the Vekua monograph [48] was devoted to **generalized analytic functions**, i.e., continuous complex valued functions $H(z)$ of one complex variable $z = x + iy$ of class $W^{1,1}_{\rm loc}$ in a domain D satisfying the equations

$$\partial_{\bar z} H \ + \ aH \ + \ b\overline{H} \ = \ S\ , \quad \partial_{\bar z} \ := (\partial_x + i\partial_y)/2 \tag{1.6}$$

with complex valued coefficients $a, b, S \in L_p(D)$, $p > 2$.

The papers [17] and [38] were devoted to boundary value problems with measurable data for the spacial case of **generalized analytic functions H with sources** $S : D \to \mathbb{C}$, when $a \equiv 0 \equiv b$,

$$\partial_{\bar z} H(z) \ = \ S(z)\ , \quad z \in D\ , \tag{1.7}$$

in regular enough domains D.

Then we establish here similar theorems on existence, representation and regularity of the so-called multi-valued solutions of the Dirichlet problem (1.1) with continuous boundary data $\varphi : \partial D \to \mathbb{R}$ to the Beltrami equations with sources (1.2) in arbitrary domains D in $\mathbb{C}$ without any boundary components degenerated to a single point. Finally, we resolve on this basis the classical Dirichlet problem (1.3) to Poisson type equations (1.4) in the general domains.

In particular, we give here a representation of the given solutions for (1.3) to (1.4) through generalized harmonic functions with sources. Recall that the paper [38] were devoted to the existence of nonclassical continuous solutions of class $W^{2,p}_{\rm loc}$ to various boundary-value problems with arbitrary boundary data that were measurable with respect to the length measure in domains with rectifiable boundaries for **generalized**

harmonic functions with sources $G : D \to \mathbb{R}$ in $L_p(D)$, $p > 2$, satisfying the Poisson equations

$$\triangle U(z) \ = \ G(z) \ . \tag{1.8}$$

Note that by the Sobolev embedding theorem, see Theorem I.10.2 in [47], such functions U belong to the class C^1. Similar results were also proved in [17] with arbitrary boundary data that were measurable with respect to the logarithmic capacity in special domains with nonrectifiable boundaries. In the case G in $L_p(D)$, $p > 1$, we will call continuous solutions of the Poisson equation (1.8) of the class $W^{2,p}_{\rm loc}$ **weak generalized harmonic functions with the sources** G.

The paper is organized as follows. Sections 2 and 3 contain the main lemma and a series of other effective integral criteria, respectively, for existence, representation and regularity of solutions of the Dirichlet problem (1.1) to the Beltrami equations with sources (1.2) in the case of arbitrary simply connected domains. Sections 4 and 5 include similar results on multi-valued solutions of the Dirichlet problem (1.1) to the equations (1.2) in the case of domains D with no boundary components degenerated to a single point. Finally, in arbitrary such domains we obtain existence, representation and regularity results for the classsical Dirichlet problem (1.3) to Poisson type equations (1.4) in Section 6.

2. The main lemma in simply connected domains

It is well known that the homogeneous Beltrami equation

$$f_{\bar{z}} = \mu(z) f_z \tag{2.1}$$

is the basic equation in analytic theory of quasiconformal and quasiregular mappings in the plane with numerous applications in nonlinear elasticity, gas flow, hydrodynamics and other sections of natural sciences. Note that continuous functions f with generalized derivative by Sobolev $f_{\bar{z}} = 0$ are analytic functions, see e.g. Lemma 1 in [2], that corresponds to the case $\mu(z) \equiv 0$ in (2.1).

The equation (2.1) is called **degenerate** if $\operatorname{ess\,sup} K_\mu(z) = \infty$. It is known that if K_μ is bounded, then the equation has homeomorphic solutions in $W^{1,2}_{\rm loc}$, see e.g. monographs [1,6] and [28], called **quasiconformal mappings**. Recently, a series of effective criteria for existence of homeomorphic solutions in $W^{1,1}_{\rm loc}$ have been also established for the degenerate Beltrami equations, see e.g. historic comments with relevant references in monographs [4,19] and [30].

These criteria were formulated both in terms of K_μ and the more refined quantity that takes into account not only the modulus of the complex coefficient μ but also its argument

$$K^T_\mu(z,z_0) \ := \ \frac{\left|1-\frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1-|\mu(z)|^2} \tag{2.2}$$

called **tangent dilatation quotient** of Beltrami equations with respect to a point $z_0\in\mathbb{C}$, see e.g. [3, 7, 8, 11, 19, 27, 30] and [40–44]. Note that

$$K^{-1}_\mu(z)\leqslant K^T_\mu(z,z_0)\leqslant K_\mu(z) \qquad \forall\, z\in D\,,\ z_0\in\mathbb{C}\,. \tag{2.3}$$

Let D be a domain in the complex plane $\mathbb{C}$. A function $f:D\to\mathbb{C}$ in the Sobolev class $W^{1,1}_{\rm loc}$ is called a **regular solution** of the Beltrami equation (2.1) if f satisfies (2.1) a.e. and its Jacobian $J_f(z)=|f_z|^2-|f_{\bar z}|^2>0$ a.e. By Lemma 3 and Remark 2 in [43], we have the following statement on the existence of regular homeomorphic solutions f in $\mathbb{C}$ for the Beltrami equation (2.1).

Proposition 1. *Let $\mu:\mathbb{C}\to\mathbb{C}$ be a measurable function with $|\mu(z)|<1$ a.e. and $K_\mu\in L_{1,{\rm loc}}(\mathbb{C})$. Suppose that, for each $z_0\in\mathbb{C}$ with some $\varepsilon_0=\varepsilon(z_0)>0$,*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z,z_0)\cdot\psi^2_{z_0,\varepsilon}(|z-z_0|)\,dm(z) \ = \ o(I^2_{z_0}(\varepsilon)) \quad as\ \varepsilon\to 0 \tag{2.4}$$

for a family of measurable functions $\psi_{z_0,\varepsilon}:(0,\varepsilon_0)\to(0,\infty)$, $\varepsilon\in(0,\varepsilon_0)$, with

$$I_{z_0}(\varepsilon)\ :=\ \int\limits_{\varepsilon}^{\varepsilon_0}\psi_{z_0,\varepsilon}(t)\ dt\ <\ \infty \qquad \forall\,\varepsilon\in(0,\varepsilon_0)\,. \tag{2.5}$$

Then the Beltrami equation (2.1) *has a regular homeomorphic solution f^μ.*

Here $dm(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$ and by (2.3) K^T_μ can be replaced by K_μ. We call such solutions f^μ of (2.1) $\mu-$**conformal mappings**.

Lemma 1. *Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma\in L_p(D)$, $p>2$, with compact support in D, $\mu:D\to\mathbb{C}$ be a measurable function with $|\mu(z)|<1$ a.e., K_μ be locally bounded in D, $K_\mu\in L_1(D)$ and conditions* (2.4) *and* (2.5) *hold for all $z_0\in\partial D$.*

Then the Beltrami equation (1.2) *with the source* σ *has a locally Hölder continuous solution* ω *in the class* $W^{1,2}_{\rm loc}$ *of the Dirichlet problem* (1.1) *in* D *for each continuous function* $\varphi : \partial D \to \mathbb{R}$ *that is unique up to an additive pure imaginary constant.*

Moreover, $\omega = h \circ f$, *where* $f : \mathbb{C} \to \mathbb{C}$ *is a* $\mu-$*conformal mapping with* μ *extended by zero outside of* D *and* $h : D_* \to \mathbb{C}$ *is a generalized analytic function in* $D_* := f(D)$ *with the source* S *of the class* $L_{p_*}(D_*)$ *for some* $p_* \in (2,p)$,

$$S \ := \ \sigma \cdot \frac{f_z}{J} \circ f^{-1} \ , \tag{2.6}$$

where $J = |f_z|^2 - |f_{\bar z}|^2$ *is the Jacobian of* f, *that satisfies the Dirichlet condition*

$$\lim_{w\to\zeta} \ {\rm Re}\, h(w) \ = \ \varphi_*(\zeta) \quad \forall\, \zeta \in \partial D_* \ , \quad \textit{where } \varphi_* := \varphi \circ f^{-1}|_{\partial D_*} . \tag{2.7}$$

We assume here and further that the dilatation quotients $K^T_\mu(z,z_0)$ and $K_\mu(z)$ are extended by 1 outside of the domain D.

Remark 1. In tern, the generalized analytic function h with the source S by Theorem 1.16 in [48] has the representation $h = \mathcal{A} + H$, where

$$H(w) \ = \ -\frac{1}{\pi}\int\limits_{D_*} \frac{S(\zeta)}{\zeta - w}\, dm(\zeta) \ , \quad w \in \mathbb{C} \ , \tag{2.8}$$

with $H_{\overline{w}} = S$, and $\mathcal{A}$ is a holomorphic function in D_* with the Dirichlet condition

$$\lim_{w\to\zeta} \ {\rm Re}\, \mathcal{A}(w) \ = \ \varphi^*(\zeta) \quad \forall\, \zeta \in \partial D_* \ , \quad \text{where } \varphi^* := \varphi_* - {\rm Re}\, H|_{\partial D_*} \ . \tag{2.9}$$

Note that H is α_*-Hölder continuous in D_* with $\alpha_* = 1 - 2/p_*$ by Theorem 1.19 and $H|_{D_*} \in W^{1,p_*}(D_*)$ by Theorems 1.36 and 1.37 in [48]. Note also that f and f^{-1} are locally quasiconformal mappings in D and D_*, respectively.

Proof. Let us first show the uniqueness of the desired solution. Indeed, if ω_1 and ω_2 are such solutions, then $\Omega := \omega_2 - \omega_1$ is such a solution for the Dirichlet problem with zero boundary date to the homogeneous Beltrami equation (2.1). Consider the function $\Omega^* := \Omega \circ f^{-1}$. First of all, note that $f^* := f^{-1}|_{D_*}$ is a locally quasiconformal mapping and, in particular, $f^* \in W^{1,2}_{\rm loc}(D_*)$. Hence

$$\Omega^*_{\overline{w}} \ = \ \Omega_z {\circ} f^* {\cdot} f^*_{\overline{w}} + \Omega_{\overline{z}} {\circ} f^* {\cdot} \overline{f^*_w} \ = \ \Omega_z {\circ} f^* {\cdot} \left[\ f^*_{\overline{w}} \ + \ \mu \circ f^* \cdot \overline{f^*_w}\ \right] \quad \text{a.e. in } D_*$$

and $\Omega^* \in W^{1,1}_{\rm loc}$ by Lemma III.6.4 in [28]. Consequently, $\Omega^*_{\overline{w}} = 0$ a.e. in D_* by I.C(3) in [1]. Thus, the function Ω^* is analytic in D_* by Lemma 1 in [2]. Its real part u satisfies zero Dirichlet condition and by the maximum principle for harmonic functions $u \equiv 0$ in D_*. Thus, Ω is a pure imaginary constant.

Now, let us prove that in (2.6) $S \in L_{p_*}(D_*)$ for some $p_* \in (2,p)$. Indeed, let $\tilde{D}$ be a subdomain of D with its closure in D, containing the support of σ. Setting $\tilde{\mu} \equiv \mu$ on $\tilde{D}$ and zero for the rest points of $\mathbb{C}$, we see that $K_{\tilde{\mu}} \in L_\infty(\mathbb{C})$. Consequently, we obtain that the function $\tilde{S} : \tilde{D}_* \to \mathbb{C}$, $\tilde{D}_* := f^{\tilde{\mu}}(D_*)$,

$$\tilde{S} \ := \ \left(\frac{f_z^{\tilde{\mu}}}{J^{\tilde{\mu}}} \cdot \sigma \right) \circ \left(f^{\tilde{\mu}} \right)^{-1} \ , \tag{2.10}$$

where $f^{\tilde{\mu}} : \mathbb{C} \to \mathbb{C}$ is the $\tilde{\mu}-$ conformal mapping from Theorem B in [16] and $J^{\tilde{\mu}}$ is the Jacobian of $f^{\tilde{\mu}}$, belongs to class $L_{p_*}(\tilde{D}_*)$ for some $p_* \in (2,p)$ by Remark 2 to Lemma 1 in [16]. However, $f^{\tilde{\mu}}|_{\tilde{D}} = C \circ f|_{\tilde{D}}$, where C is a conformal mapping on $D'_* := f(\tilde{D})$ because both f and $f^{\tilde{\mu}}$ are quasiconformal mappings on $\tilde{D}$ with the same complex characteristic μ there. Consequently,

$$S \ = \ \left(\tilde{S} \cdot \overline{C'} \right) \circ C \tag{2.11}$$

on $D'_* \subset D_*$ containing the compact support of the function $\Sigma := \sigma \circ f^{-1}$. Thus, really $S \in L_{p_*}(D_*)$ for the same $p_* \in (2,p)$.

Next, let $\varphi^* := \left(\varphi \circ f^{-1} - \operatorname{Re} H \right)|_{\partial D_*}$, where H is the generalized analytic function (2.8) with the source S. Then by Corollary 4.1.8 and Theorem 4.2.1 in [36] there is a harmonic function $u : D_* \to \mathbb{R}$ satisfying the Dirichlet condition

$$\lim_{w \to \zeta} u(w) \ = \ \varphi^*(\zeta) \qquad \forall\, \zeta \in \partial D_* \tag{2.12}$$

because D_* is a bounded simple connected domain that, of course, has at least 2 boundary points. On the other hand, there is its conjugate harmonic function $v : D_* \to \mathbb{R}$ such that $\mathcal{A} := u + iv : D_* \to \mathbb{C}$ forms a holomorphic function again because of the domain D_* is simply connected, see e.g. arguments in the beginning of the book [26]. Thus, the function $\omega := h \circ f$, where $h := \mathcal{A} + H$, gives the desired solution of the Dirichlet problem (1.1) in D to the Beltrami equation (1.2) with the source σ of the class $W^{1,1}_{\rm loc}$ by Lemma III.6.4 in [28], see also Remark 1. Arguing as in the previous item with the application of the auxiliary quasiconformal mapping $f^{\tilde{\mu}} : \mathbb{C} \to \mathbb{C}$, it is easy to prove on the basis of Lemma 1 in [16] that $\omega \in W^{1,2}_{\rm loc}$. Finally, the given solution ω is locally

Hölder continuous because the function h and the mapping f are so, see Remark 1. □

Remark 2. Note that if the family of the functions $\psi_{z_0,\varepsilon}(t) \equiv \psi_{z_0}(t)$, $z_0 \in \partial D$, in Lemma 1 is independent on the parameter ε, then the condition (2.4) implies that $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. This follows immediately from arguments by contradiction, apply for it (2.3) and the condition $K_\mu \in L_1(D)$. Note also that (2.4) holds, in particular, if, for some $\varepsilon_0 = \varepsilon(z_0)$,

$$\int\limits_{|z-z_0|<\varepsilon_0} K_\mu^T(z,z_0) \cdot \psi_{z_0}^2(|z-z_0|)\, dm(z) < \infty \qquad \forall\; z_0 \in \partial D \tag{2.13}$$

and $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In other words, for the solvability of the Dirichlet problem (1.1) in D for the Beltrami equations with sources (1.2) for all continuous boundary functions φ, it is sufficient that the integral in (2.13) converges for some nonnegative function $\psi_{z_0}(t)$ that is locally integrable over $(0,\varepsilon_0]$ but has a nonintegrable singularity at 0. The functions $\log^\lambda(e/|z-z_0|)$, $\lambda \in (0,1)$, $z \in \mathbb{D}$, $z_0 \in \overline{\mathbb{D}}$, and $\psi(t) = 1/(t\ \log(e/t))$, $t \in (0,1)$, show that the condition (2.13) is compatible with the condition $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Furthermore, the condition (2.4) shows that it is sufficient for the solvability of the Dirichlet problem even if the integral in (2.13) is divergent in a controlled way.

3. The main criteria in simply connected domains

Lemma 1 makes it to be possible to derive a number of effective integral criteria for solvability of the Dirichlet problem to the Beltrami equations with sources.

Recall first that a real-valued function u in a domain D in $\mathbb{C}$ is said to be of **bounded mean oscillation** in D, abbr. $u \in \mathrm{BMO}(D)$, if $u \in L^1_{\mathrm{loc}}(D)$ and

$$\|u\|_* := \sup_B \frac{1}{|B|} \int\limits_B |u(z) - u_B|\, dm(z) < \infty\,, \tag{3.1}$$

where the supremum is taken over all discs B in D and

$$u_B = \frac{1}{|B|} \int\limits_B u(z)\, dm(z)\,.$$

We write $u \in \mathrm{BMO_{loc}}(D)$ if $u \in \mathrm{BMO}(U)$ for each relatively compact subdomain U of D. We also write sometimes for short BMO and $\mathrm{BMO_{loc}}$, respectively.

The class BMO was introduced by John and Nirenberg (1961) in the paper [25] and soon became an important concept in harmonic analysis, partial differential equations and related areas, see e.g. [20] and [37].

A function φ in BMO is said to have **vanishing mean oscillation**, abbr. $\varphi \in$ VMO, if the supremum in (3.1) taken over all balls B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$. VMO has been introduced by Sarason in [46]. There are a number of papers devoted to the study of partial differential equations with coefficients of the class VMO, see e.g. [10, 24, 29, 33, 34] and [35].

Remark 3. Note that $W^{1,2}(D) \subset VMO(D)$, see e.g. [9].

Following [21], we say that a function $\varphi : D \to \mathbb{R}$ has **finite mean oscillation** at a point $z_0 \in D$, abbr. $\varphi \in \mathrm{FMO}(z_0)$, if

$$\overline{\lim_{\varepsilon\to 0}} \; \fint_{B(z_0,\varepsilon)} |\varphi(z) - \widetilde{\varphi}_\varepsilon(z_0)| \, dm(z) < \infty \,, \tag{3.2}$$

where

$$\widetilde{\varphi}_\varepsilon(z_0) = \fint_{B(z_0,\varepsilon)} \varphi(z) \, dm(z) \tag{3.3}$$

is the mean value of the function $\varphi(z)$ over the disk $B(z_0, \varepsilon) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$. Note that the condition (3.2) includes the assumption that φ is integrable in some neighborhood of the point z_0. We say also that a function $\varphi : D \to \mathbb{R}$ is of **finite mean oscillation in** D, abbr. $\varphi \in \mathrm{FMO}(D)$ or simply $\varphi \in$ FMO, if $\varphi \in \mathrm{FMO}(z_0)$ for all points $z_0 \in D$. We write $\varphi \in \mathrm{FMO}(\overline{D})$ if φ is given in a domain G in $\mathbb{C}$ such that $\overline{D} \subset G$ and $\varphi \in \mathrm{FMO}(G)$.

The following statement is obvious by the triangle inequality.

Proposition 2. *If, for a collection of numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to 0}} \; \fint_{B(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \, dm(z) < \infty \,, \tag{3.4}$$

then φ is of finite mean oscillation at z_0.

In particular, choosing here $\varphi_\varepsilon \equiv 0$, $\varepsilon \in (0, \varepsilon_0]$ in Proposition 1, we obtain the following.

Corollary 1. *If, for a point $z_0 \in D$,*

$$\overline{\lim_{\varepsilon\to 0}} \; \fint_{B(z_0,\varepsilon)} |\varphi(z)| \, dm(z) < \infty \,, \tag{3.5}$$

then φ has finite mean oscillation at z_0.

Recall that a point $z_0 \in D$ is called a **Lebesgue point** of a function $\varphi : D \to \mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon \to 0} \quad \fint_{B(z_0,\varepsilon)} |\varphi(z) - \varphi(z_0)| \, dm(z) = 0 \, . \tag{3.6}$$

It is known that, almost every point in D is a Lebesgue point for every function $\varphi \in L_1(D)$. Thus, we have by Proposition 1 the next corollary.

Corollary 2. *Every locally integrable function $\varphi : D \to \mathbb{R}$ has a finite mean oscillation at almost every point in D.*

Remark 4. Note that the function $\varphi(z) = \log(1/|z|)$ belongs to BMO in the unit disk Δ, see, e.g., [37], p. 5, and hence also to FMO. However, $\widetilde{\varphi}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that condition (3.5) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0. Clearly, $\mathrm{BMO}(D) \subset \mathrm{BMO}_{\mathrm{loc}}(D) \subset \mathrm{FMO}(D)$ and as well-known $\mathrm{BMO}_{\mathrm{loc}} \subset L^p_{\mathrm{loc}}$ for all $p \in [1, \infty)$, see, e.g., [25] or [37]. However, FMO is not a subclass of L^p_{loc} for any $p > 1$ but only of L^1_{loc}. Thus, the class FMO is much more wider than $\mathrm{BMO}_{\mathrm{loc}}$.

Versions of the next lemma has been first proved for the class BMO in [40]. For the FMO case, see the papers [21,39,41,42] and the monographs [19] and [30].

Lemma 2. *Let D be a domain in $\mathbb{C}$ and let $\varphi : D \to \mathbb{R}$ be a non-negative function of the class* FMO(z_0) *for some $z_0 \in D$. Then*

$$\int\limits_{\varepsilon < |z - z_0| < \varepsilon_0} \frac{\varphi(z) \, dm(z)}{\left(|z - z_0| \log \frac{1}{|z - z_0|}\right)^2} = O\left(\log \log \frac{1}{\varepsilon}\right) \qquad as \quad \varepsilon \to 0 \tag{3.7}$$

for some $\varepsilon_0 \in (0, \delta_0)$ where $\delta_0 = \min(e^{-e}, d_0)$, $d_0 = \sup\limits_{z \in D} |z - z_0|$.

Recall that we assume further that the dilatation quotients $K^T_\mu(z, z_0)$ and $K_\mu(z)$ are extended by 1 outside of the domain D.

Choosing $\psi(t) = 1/\left(t \log(1/t)\right)$ in Lemma 1, see also Remark 1, we obtain by Lemma 2 the following result with the FMO type criterion.

Theorem 1. *Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in L_p(D)$, $p > 2$, with compact support, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D, $K_\mu \in L_1(D)$, $K^T_\mu(z, z_0) \leqslant Q_{z_0}(z)$ a.e. in U_{z_0} for each point $z_0 \in \partial D$, a neighborhood U_{z_0} of z_0, a function $Q_{z_0} : U_{z_0} \to [0, \infty]$ in the class* FMO(z_0).

Then the Beltrami equation (1.2) *with the source* σ *has a locally Hölder continuous solution* ω *in the class* $W^{1,2}_{\rm loc}$ *of the Dirichlet problem* (1.1) *in* D *for each continuous function* $\varphi : \partial D \to \mathbb{R}$ *that is unique up to an additive pure imaginary constant.*

Moreover, $\omega = h \circ f$, $h := \mathcal{A} + H$, *where* $f : \mathbb{C} \to \mathbb{C}$ *is a* $\mu-$*conformal mapping with* μ *extended by zero outside of* D, $H : D_* \to \mathbb{C}$ *is a generalized analytic function in* $D_* := f(D)$ *with the source* S *calculated in* (2.6) *and* $\mathcal{A}$ *is a holomorphic function in* D_* *with the Dirichlet condition* (2.9).

By Corollary 1 we obtain the following nice consequence of Theorem 1, where $B(z_0, \varepsilon)$ denote the infinitesimal disks $\{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ centered at $z_0 \in \partial D$.

Corollary 3. *Let* D *be a bounded simply connected domain in* $\mathbb{C}$, $\sigma \in L_p(D)$, $p > 2$, *with compact support,* $\mu : D \to \mathbb{C}$ *be a measurable function with* $|\mu(z)| < 1$ *a.e.,* K_μ *be locally bounded in* D, $K_\mu \in L_1(D)$ *and, for each point* $z_0 \in \partial D$,

$$\overline{\lim_{\varepsilon \to 0}} \;\; -\!\!\!\!\!\!\int_{B(z_0,\varepsilon)} K^T_\mu(z, z_0)\, dm(z) \; < \; \infty \; . \tag{3.8}$$

Then all the conclusions of Theorem 1 on solutions for the Dirichlet problem (1.1) *with arbitrary continuous boundary data* $\varphi : \partial D \to \mathbb{R}$ *to the Beltrami equation* (1.2) *with the source* σ *hold.*

Since $K^T_\mu(z, z_0) \leqslant K_\mu(z)$ for all z and $z_0 \in \mathbb{C}$, we also obtain the following consequences of Theorem 1 with the BMO type criterion.

Corollary 4. *Let* D *be a bounded simply connected domain in* $\mathbb{C}$, $\sigma \in L_p(D)$, $p > 2$, *with compact support,* $\mu : D \to \mathbb{C}$ *be a measurable function with* $|\mu(z)| < 1$ *a.e.,* K_μ *be locally bounded in* D *and* K_μ *have a dominant* $Q \in {\rm BMO}_{\rm loc}$ *in a neighborhood of* ∂D. *Then the conclusions of Theorem 1 hold.*

Remark 5. In particular, the conclusions of Theorem 1 hold if $Q \in {\rm W}^{1,2}_{\rm loc}$ in a neighborhood of ∂D, because of $W^{1,2}_{\rm loc} \subset {\rm VMO}_{\rm loc}$, see e.g. [9].

Corollary 5. *Let* D *be a bounded simply connected domain in* $\mathbb{C}$, $\sigma \in L_p(D)$, $p > 2$, *with compact support,* $\mu : D \to \mathbb{C}$ *be a measurable function with* $|\mu(z)| < 1$ *a.e.,* K_μ *be locally bounded in* D *and* K_μ *have a dominant* $Q \in {\rm FMO}$ *in a neighborhood of* ∂D. *Then the conclusions of Theorem 1 hold.*

Similarly, choosing in Lemma 1 the function $\psi(t) = 1/t$, see also Remark 1, we come to the next statement with the Calderon–Zygmund type criterion.

Theorem 2. *Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in L_p(D)$, $p > 2$, with compact support, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D, $K_\mu \in L_1(D)$ and, for each point $z_0 \in \partial D$ and $\varepsilon_0 = \varepsilon(z_0) > 0$,*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z, z_0)\, \frac{dm(z)}{|z - z_0|^2} = o\left(\left[\log \frac{1}{\varepsilon}\right]^2\right) \qquad as\ \varepsilon \to 0\ . \tag{3.9}$$

Then the Beltrami equation (1.2) with the source σ has a locally Hölder continuous solution ω in the class $W^{1,2}_{\rm loc}$ of the Dirichlet problem (1.1) in D for each continuous function $\varphi : \partial D \to \mathbb{R}$ that is unique up to an additive pure imaginary constant.

Moreover, $\omega = h \circ f$, $h := \mathcal{A} + H$, where $f : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping with μ extended by zero outside of D, $H : D_* \to \mathbb{C}$ is a generalized analytic function in $D_* := f(D)$ with the source S calculated in (2.6) and $\mathcal{A}$ is a holomorphic function in D_* with the Dirichlet condition (2.9).

Remark 6. Choosing in Lemma 1 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (3.9) by the conditions

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{K^T_\mu(z, z_0)\, dm(z)}{\left(|z - z_0| \log \frac{1}{|z-z_0|}\right)^2} = o\left(\left[\log\log \frac{1}{\varepsilon}\right]^2\right) \qquad \forall\ z_0 \in \partial D \tag{3.10}$$

as $\varepsilon \to 0$ for some $\varepsilon_0 = \varepsilon(z_0) > 0$. More generally, we would be able to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t \log 1/t \cdot \log\log 1/t \cdot \ldots \cdot \log \ldots \log 1/t)$.

Choosing in Lemma 1 the functional parameter $\psi_{z_0}(t) := 1/[tk^T_\mu(z_0, t)]$, where $k^T_\mu(z_0, r)$ is the integral mean of $K^T_\mu(z, z_0)$ over the circle $S(z_0, r) := \{z \in \mathbb{C} : |z - z_0| = r\}$, we obtain the Lehto type criterion.

Theorem 3. *Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in L_p(D)$, $p > 2$, with compact support, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D, $K_\mu \in L_1(D)$ and, for each point $z_0 \in \partial D$ and $\varepsilon_0 = \varepsilon(z_0) > 0$,*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{rk^T_\mu(z_0, r)} \ = \ \infty\ . \tag{3.11}$$

Then the Beltrami equation (1.2) with the source σ has a locally Hölder continuous solution ω in the class $W^{1,2}_{\rm loc}$ of the Dirichlet prob-

lem (1.1) *in* D *for each continuous function* $\varphi : \partial D \to \mathbb{R}$ *that is unique up to an additive pure imaginary constant.*

Moreover, $\omega = h \circ f$, $h := \mathcal{A} + H$, *where* $f : \mathbb{C} \to \mathbb{C}$ *is a* $\mu-$*conformal mapping with* μ *extended by zero outside of* D, $H : D_* \to \mathbb{C}$ *is a generalized analytic function in* $D_* := f(D)$ *with the source* S *calculated in* (2.6) *and* $\mathcal{A}$ *is a holomorphic function in* D_* *with the Dirichlet condition* (2.9).

Corollary 6. *Let* D *be a bounded simply connected domain in* $\mathbb{C}$, $\sigma \in L_p(D)$, $p > 2$, *with compact support,* $\mu : D \to \mathbb{C}$ *be a measurable function with* $|\mu(z)| < 1$ *a.e.,* K_μ *be locally bounded in* D, $K_\mu \in L_1(D)$ *and, for each point* $z_0 \in \partial D$,

$$k_\mu^T(z_0, \varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \qquad as\ \varepsilon \to 0\ . \tag{3.12}$$

Then all conclusions of Theorem 3 on solutions for the Dirichlet problem (1.1) *with arbitrary continuous boundary data* $\varphi : \partial D \to \mathbb{R}$ *to the Beltrami equation* (1.2) *with the source* σ *hold.*

Remark 7. In particular, the conclusions of Theorem 3 hold if

$$K_\mu^T(z, z_0) = O\left(\log \frac{1}{|z - z_0|}\right) \qquad \text{as} \quad z \to z_0 \quad \forall\ z_0 \in \partial D\,. \tag{3.13}$$

Moreover, the condition (3.12) can be replaced by the series of weaker conditions

$$k_\mu^T(z_0, \varepsilon) = O\left(\left[\log \frac{1}{\varepsilon} \cdot \log\log \frac{1}{\varepsilon} \cdot \ldots \cdot \log \ldots \log \frac{1}{\varepsilon}\right]\right) \qquad \forall\ z_0 \in \partial D\ . \tag{3.14}$$

To get another criterion, we need a couple of auxiliary statements. The first of them can be found e.g. as Theorem 3.2 in [44].

Proposition 3. *Let* $Q : \mathbb{D} \to [0, \infty]$ *be a measurable function such that*

$$\int\limits_{\mathbb{D}} \Phi(Q(z))\, dm(z) < \infty \tag{3.15}$$

where $\Phi : [0, \infty] \to [0, \infty]$ *is a non-decreasing convex function such that*

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \tag{3.16}$$

for some $\delta > \Phi(+0)$. *Then*

$$\int\limits_0^1 \frac{dr}{rq(r)} = \infty \tag{3.17}$$

where $q(r)$ *is the average of the function* $Q(z)$ *over the circle* $|z| = r$.

Above we used the following notions of the inverse function for monotone functions. Namely, for every non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$ the inverse function $\Phi^{-1} : [0,\infty] \to [0,\infty]$ can be well-defined by setting

$$\Phi^{-1}(\tau) \ := \ \inf_{\Phi(t)\geq\tau} t \tag{3.18}$$

Here inf is equal to ∞ if the set of $t \in [0,\infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function Φ^{-1} is non-decreasing, too. It is also evident immediately by the definition that $\Phi^{-1}(\Phi(t)) \leq t$ for all $t \in [0,\infty]$ with the equality except intervals of constancy of the function $\Phi(t)$.

Recall connections between integral conditions, see e.g. Theorem 2.5 in [44].

Proposition 4. *Let* $\Phi : [0,\infty] \to [0,\infty]$ *be a non-decreasing function and set*

$$H(t) \ = \ \log\Phi(t) \ . \tag{3.19}$$

Then the equality

$$\int\limits_\Delta^\infty H'(t)\,\frac{dt}{t} = \infty, \tag{3.20}$$

implies the equality

$$\int\limits_\Delta^\infty \frac{dH(t)}{t} = \infty\,, \tag{3.21}$$

and (3.21) *is equivalent to*

$$\int\limits_\Delta^\infty H(t)\,\frac{dt}{t^2} = \infty \tag{3.22}$$

for some $\Delta > 0$, *and* (3.22) *is equivalent to each of the equalities*

$$\int\limits_0^{\delta_*} H\left(\frac{1}{t}\right)\,dt = \infty \tag{3.23}$$

for some $\delta_* > 0$,

$$\int\limits_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} = \infty \tag{3.24}$$

for some $\Delta_* > H(+0)$ *and to* (3.16) *for some* $\delta > \Phi(+0)$.

Moreover, (3.20) *is equivalent to* (3.21) *and to hence* (3.20)–(3.24) *as well as to* (3.16) *are equivalent to each other if* Φ *is in addition absolutely continuous. In particular, all the given conditions are equivalent if* Φ *is convex and non-decreasing.*

Note that the integral in (3.21) is understood as the Lebesgue–Stieltjes integral and the integrals in (3.20) and (3.22)–(3.24) as the ordinary Lebesgue integrals. It is necessary to give one more explanation. From the right hand sides in the conditions (3.20)–(3.24) we have in mind $+\infty$. If $\Phi(t) = 0$ for $t \in [0, t_*,$ then $H(t) = -\infty$ for $t \in [0, t_*]$ and we complete the definition $H'(t) = 0$ for $t \in [0, t_*]$. Note, the conditions (3.21) and (3.22) exclude that t_* belongs to the interval of integrability because in the contrary case the left hand sides in (3.21) and (3.22) are either equal to $-\infty$ or indeterminate. Hence we may assume in (3.20)–(3.23) that $\delta > t_0$, correspondingly, $\Delta < 1/t_0$ where $t_0 := \sup\limits_{\Phi(t)=0} t$, and set $t_0 = 0$ if $\Phi(0) > 0$. The most interesting condition (3.22) can be written in the form:

$$\int\limits_{\Delta}^{\infty} \log\, \Phi(t)\, \frac{dt}{t^2} \;=\; +\infty \qquad \text{for some } \Delta > 0\ . \tag{3.25}$$

Combining Proposition 3 and 4 with Theorems 3 we obtain the following significant result with the Orlicz type criterion.

Theorem 4. *Let* D *be a bounded simply connected domain in* $\mathbb{C}$, $\sigma \in L_p(D)$, $p > 2$, *with compact support,* $\mu : D \to \mathbb{C}$ *be a measurable function with* $|\mu(z)| < 1$ *a.e.,* K_μ *be locally bounded in* D, $K_\mu \in L_1(D)$ *and, for each point* $z_0 \in \partial D$ *and a neighborhood* U_{z_0} *of* z_0,

$$\int\limits_{U_{z_0}} \Phi_{z_0}\left(K^T_\mu(z, z_0)\right)\, dm(z) < \infty\ , \tag{3.26}$$

where $\Phi_{z_0} : (0, \infty] \to (0, \infty]$ *is a convex non-decreasing function such that*

$$\int\limits_{\Delta(z_0)}^{\infty} \log\, \Phi_{z_0}(t)\, \frac{dt}{t^2} \;=\; +\infty \quad \textit{for some } \Delta(z_0) > 0\ . \tag{3.27}$$

Then the Beltrami equation (1.2) with the source σ has a locally Hölder continuous solution ω in the class $W^{1,2}_{\rm loc}$ of the Dirichlet problem (1.1) in D for each continuous function $\varphi : \partial D \to \mathbb{R}$ that is unique up to an additive pure imaginary constant.

Moreover, $\omega = h \circ f$, $h := \mathcal{A} + H$, where $f : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping with μ extended by zero outside of D, $H : D_* \to \mathbb{C}$ is a generalized analytic function in $D_* := f(D)$ with the source S calculated in (2.6) and $\mathcal{A}$ is a holomorphic function in D_* with the Dirichlet condition (2.9).

Corollary 7. *Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in L_p(D)$, $p > 2$, with compact support, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D, $K_\mu \in L_1(D)$ and, for each point $z_0 \in \partial D$, a neighborhood U_{z_0} of z_0 and $\alpha(z_0) > 0$,*

$$\int\limits_{U_{z_0}} e^{\alpha(z_0) K^T_\mu(z,z_0)}\, dm(z) < \infty \ . \tag{3.28}$$

Then all conclusions of Theorem 4 on solutions for the Dirichlet problem (1.1) with continuous data $\varphi : \partial D \to \mathbb{R}$ to the Beltrami equation (1.2) with the source σ hold.

Corollary 8. *Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in L_p(D)$, $p > 2$, with compact support, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D and, for a neighborhood U of ∂D,*

$$\int\limits_{U} \Phi\left(K_\mu(z)\right) dm(z) < \infty \ , \tag{3.29}$$

where $\Phi : (0, \infty] \to (0, \infty]$ is a convex non-decreasing function with, for $\delta > 0$,

$$\int\limits_{\delta}^{\infty} \log\, \Phi(t)\, \frac{dt}{t^2} \ = \ +\infty \ . \tag{3.30}$$

Then all conclusions of Theorem 4 on solutions for the Dirichlet problem (1.1) with continuous data $\varphi : \partial D \to \mathbb{R}$ to the Beltrami equation (1.2) with the source σ hold.

Remark 8. By Theorems 2.5 and 5.1 in [44], condition (3.30) is not only sufficient but also necessary to have the regular solutions of the Dirichlet problem (1.1) in D for arbitrary Beltrami equations with sources (1.2), satisfying the integral constraints (3.29), for all continuous functions $\varphi : \partial D \to \mathbb{R}$ because such solutions have the representation through regular homeomorphic solutions $f = f^{\mu}$ of the homogeneous Beltrami equation (2.1) from Proposition 1.

Corollary 9. *Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in L_p(D)$, $p > 2$, with compact support, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D and, for a neighborhood U of ∂D and $\alpha > 0$,*

$$\int\limits_U e^{\alpha K_\mu(z)}\, dm(z) \ < \ \infty\ . \tag{3.31}$$

Then all conclusions of Theorem 4 on solutions for the Dirichlet problem (1.1) *with continuous data $\varphi : \partial D \to \mathbb{R}$ to the Beltrami equation* (1.2) *with the source σ hold.*

4. The main lemma in general domains

In this section we obtain criteria for the existence, representation and regularity of the so-called multi-valued solutions ω of the Dirichlet problem (1.1) to the Beltrami equations with sources (1.2) in the spirit of the theory of multi-valued analytic functions in arbitrary bounded domains D in $\mathbb{C}$ with no boundary component degenerated to a single point. Simple examples show that such domains form the most wide class of domains for which the problem is always solvable for any continuous boundary data.

We say that a locally Hölder continuous function $\omega : B(z_0, \varepsilon_0) \to \mathbb{C}$, where $B(z_0, \varepsilon_0) \subseteq D$, is a **local regular solution of the equation** (1.2) if $\omega \in W^{1,2}_{\rm loc}$ and ω satisfies (1.2) a.e. in $B(z_0, \varepsilon_0)$. Local regular solutions $\omega_0 : B(z_0, \varepsilon_0) \to \mathbb{C}$ and $\omega_* : B(z_*, \varepsilon_*) \to \mathbb{C}$ of the equation (1.2) will be called extension of each to other if there is a finite chain of its local regular solutions $\omega_i : B(z_i, \varepsilon_i) \to \mathbb{C}$, $i = 1, \ldots, m$, such that $\omega_1 = \omega_0$, $\omega_m = \omega_*$ and $\omega_i(z) = \omega_{i+1}(z)$ for all points $z \in E_i := B(z_i, \varepsilon_i) \cap B(z_{i+1}, \varepsilon_{i+1}) \neq \emptyset$, $i = 1, \ldots, m-1$.

A collection of local regular solutions $\omega_j : B(z_j, \varepsilon_j) \to \mathbb{C}$, $j \in J$, will be called a **regular multi-valued solution** of the equation (1.2) in D if the collection of the disks $B(z_j, \varepsilon_j)$ cover the domain D and ω_j are extensions of each to other through this collection and the collection is maximal by inclusion.

A regular multi-valued solution of the equation (1.2) will be called a **regular multi-valued solution of the Dirichlet problem** (1.1) to (1.2) in D if $u(z) = {\rm Re}\, \omega(z) = {\rm Re}\, \omega_j(z)$, $z \in B(z_j, \varepsilon_j)$, $j \in J$, is a single-valued function in D satisfying the Dirichlet condition $\lim\limits_{z\in\zeta} u(z) = \varphi(\zeta)$ for all $\zeta \in \partial D$.

Lemma 3. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $\sigma \in L_p(D)$, $p > 2$, with compact support in D, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D, $K_\mu \in L_1(D)$ and, for each point $z_0 \in \partial D$ and $\varepsilon_0 = \varepsilon(z_0) > 0$,*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z, z_0) \cdot \psi^2_{z_0,\varepsilon}(|z - z_0|)\, dm(z) = o(I^2_{z_0}(\varepsilon)) \quad as\ \varepsilon \to 0\ , \tag{4.1}$$

where $\psi_{z_0,\varepsilon} : (0, \varepsilon_0) \to (0,\infty)$ is a family of measurable functions such that

$$I_{z_0}(\varepsilon) :\, = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\, dt < \infty \qquad \forall\ \varepsilon \in (0, \varepsilon_0)\,. \tag{4.2}$$

Then the Beltrami equation (1.2) with the source σ has a regular multi-valued solution ω of the Dirichlet problem (1.1) in D for each continuous function $\varphi : \partial D \to \mathbb{R}$ that is unique up to an additive pure imaginary constant.

Moreover, $\omega = h \circ f$, $h := \mathcal{A} + H$, where $f : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping with μ extended by zero outside of D, $H : D_* \to \mathbb{C}$ is a generalized analytic function in $D_* := f(D)$ with the source S calculated in (2.6) and $\mathcal{A}$ is a multi-valued analytic function in D_* with a single valued real part satisfying the Dirichlet condition (2.9).

Proof. By Proposition 1 there is a $\mu-$conformal mapping $f : \mathbb{C} \to \mathbb{C}$ with μ extended by zero outside of D, which is locally quasiconformal in D. Arguing locally as in the first item of the proof to Lemma 1, we first show that the desired solution is unique up to an additive pure imaginary constant. Moreover, by the second item of the proof to Lemma 1, the function S described in (2.6) belongs to the class $L_{p_*}(D_*)$ in the domain $D_* = f(D)$ with $p_* \in (2, p)$.

Note also that D_* is also a bounded domain with no boundary components degenerated to a single point. Let $\varphi^* := \big(\varphi \circ f^{-1} - \operatorname{Re} H\big)\,|_{\partial D_*}$, where H is the generalized analytic function (2.8) with the source S. Then by Corollary 4.1.8 and Theorem 4.2.2 in [36] there is a harmonic function $u : D_* \to \mathbb{R}$ satisfying the Dirichlet condition

$$\lim_{w\to\zeta} u(w) \ = \ \varphi^*(\zeta) \qquad \forall\ \zeta \in \partial D_*\ . \tag{4.3}$$

Now, let $B_0 = B(z_0, r_0)$ be a disk in the domain D. Then $\mathfrak{B}_0 := f(B_0)$ is a simply connected subdomain of the domain D_* where there is a conjugate harmonic function v determined up to an additive constant such that $u+iv$ is a single–valued analytic function. Let us denote through $\mathcal{A}_0$ the holomorphic function corresponding to the choice of such a harmonic function v_0 in $\mathfrak{B}_0$ with the normalization $v_0(f(z_0))=0$. Thereby we have determined the initial element of a multi-valued analytic function. The function $\mathcal{A}_0$ can be extended to, generally speaking multi-valued, analytic function $\mathcal{A}$ along any path in D_* because u is given in the whole domain D_*.

Thus, $\omega := h\circ f$, $h=\mathcal{A}+H$, is a continuous multi-valued solution of the Dirichlet problem (1.1) in D for the Beltrami equation with the source σ (1.2) of the class $W^{1,1}_{\rm loc}$ by Lemma III.6.4 in [28], see also Remark 1. Arguing as in the second item of the proof to Lemma 1 with the application of the auxiliary quasiconformal mapping $f^{\tilde{\mu}}:\mathbb{C}\to\mathbb{C}$, it is easy to prove on the basis of Lemma 1 in [16] that $\omega\in W^{1,2}_{\rm loc}$. Finally, the given solution ω is locally Hölder continuous because the function h and the mapping f are so, see Remark 1. □

Remark 9. Note that if the family of the functions $\psi_{z_0,\varepsilon}(t)\equiv\psi_{z_0}(t)$, $z_0\in\partial D$, in Lemma 3 is independent on the parameter ε, then the condition (4.1) implies that $I_{z_0}(\varepsilon)\to\infty$ as $\varepsilon\to 0$. This follows immediately from arguments by contradiction, apply for it (2.3) and the condition $K_\mu\in L_1(D)$. Note also that (4.1) holds, in particular, if, for some $\varepsilon_0=\varepsilon(z_0)$,

$$\int\limits_{|z-z_0|<\varepsilon_0} K^T_\mu(z,z_0)\cdot\psi^2_{z_0}(|z-z_0|)\,dm(z)<\infty \qquad \forall\ z_0\in\partial D \tag{4.4}$$

and $I_{z_0}(\varepsilon)\to\infty$ as $\varepsilon\to 0$. In other words, for the existence of a regular multi-valued solutions of the Dirichlet problem (1.1) in D for the Beltrami equations with sources (1.2) for all continuous boundary functions φ, it is sufficient that the integral in (4.4) converges for some nonnegative function $\psi_{z_0}(t)$ that is locally integrable over $(0,\varepsilon_0]$ but has a nonintegrable singularity at 0. The functions $\log^\lambda(e/|z-z_0|)$, $\lambda\in(0,1)$, $z\in\mathbb{D}$, $z_0\in\overline{\mathbb{D}}$, and $\psi(t)=1/(t\ \log(e/t))$, $t\in(0,1)$, show that the condition (4.4) is compatible with the condition $I_{z_0}(\varepsilon)\to\infty$ as $\varepsilon\to 0$. Furthermore, the condition (4.1) shows that it is sufficient for the existence of regular multi-valued solutions of the Dirichlet problem (1.1) in D for the Beltrami equations with sources (1.2) for all continuous boundary functions φ even if the integral in (4.4) is divergent in a controlled way.

5. The main criteria in general domains

Arguing as in Section 3, we derive from Lemma 3 the following consequences.

Theorem 5. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $\sigma \in L_p(D)$, $p > 2$, have compact support in D, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D, $K_\mu \in L_1(D)$, $K^T_\mu(z, z_0) \leqslant Q_{z_0}(z)$ a.e. in U_{z_0} for each point $z_0 \in \partial D$, a neighborhood U_{z_0} of z_0 and a function $Q_{z_0} : U_{z_0} \to [0, \infty]$ in the class* $\mathrm{FMO}(z_0)$.

Then the Beltrami equation (1.2) *with the source σ has a regular multi-valued solution ω of the Dirichlet problem* (1.1) *in D for each continuous function $\varphi : \partial D \to \mathbb{R}$ that is unique up to an additive pure imaginary constant.*

Moreover, $\omega = h \circ f$, $h := \mathcal{A} + H$, where $f : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping with μ extended by zero outside of D, $H : D_* \to \mathbb{C}$ is a generalized analytic function in $D_* := f(D)$ with the source S calculated in (2.6) *and $\mathcal{A}$ is a multi-valued analytic function in D_* with a single valued real part satisfying the Dirichlet condition* (2.9).

Corollary 10. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $\sigma \in L_p(D)$, $p > 2$, have compact support in D, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D, $K_\mu \in L_1(D)$ and, for each point $z_0 \in \partial D$,*

$$\overline{\lim_{\varepsilon \to 0}} \ \fint_{B(z_0,\varepsilon)} K^T_\mu(z, z_0)\, dm(z) < \infty \ . \tag{5.1}$$

Then all the conclusions of Theorem 5 hold.

Corollary 11. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $\sigma \in L_p(D)$, $p > 2$, have compact support in D, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D and K_μ have a dominant $Q \in \mathrm{BMO}_{\mathrm{loc}}$ in a neighborhood of ∂D. Then all the conclusions of Theorem 5 hold.*

Remark 10. In particular, the conclusions of Theorem 5 hold if $Q \in \mathrm{W}^{1,2}_{\mathrm{loc}}$ in a neighborhood of ∂D, because of $W^{1,2}_{\mathrm{loc}} \subset \mathrm{VMO}_{\mathrm{loc}}$.

Corollary 12. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $\sigma \in L_p(D)$, $p > 2$, have compact support in D, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D and K_μ have a dominant $Q \in \mathrm{FMO}$ in a neighborhood of ∂D. Then all the conclusions of Theorem 5 hold.*

Theorem 6. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $\sigma \in L_p(D)$, $p > 2$, have compact support in D, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D, $K_\mu \in L_1(D)$ and, for each point $z_0 \in \partial D$ and $\varepsilon_0 = \varepsilon(z_0) > 0$,*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z, z_0)\, \frac{dm(z)}{|z - z_0|^2} = o\left(\left[\log \frac{1}{\varepsilon}\right]^2\right) \qquad \text{as } \varepsilon \to 0\ . \tag{5.2}$$

Then the Beltrami equation (1.2) *with the source σ has a regular multi-valued solution ω of the Dirichlet problem* (1.1) *in D for each continuous function $\varphi : \partial D \to \mathbb{R}$ that is unique up to an additive pure imaginary constant.*

Moreover, $\omega = h \circ f$, $h := \mathcal{A} + H$, where $f : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping with μ extended by zero outside of D, $H : D_* \to \mathbb{C}$ is a generalized analytic function in $D_* := f(D)$ with the source S calculated in (2.6) *and $\mathcal{A}$ is a multi-valued analytic function in D_* with a single valued real part satisfying the Dirichlet condition* (2.9).

Remark 11. Choosing in Lemma 3 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (5.2) by the conditions

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{K^T_\mu(z, z_0)\, dm(z)}{\left(|z - z_0| \log \frac{1}{|z-z_0|}\right)^2} = o\left(\left[\log\log \frac{1}{\varepsilon}\right]^2\right) \qquad \forall\ z_0 \in \partial D \tag{5.3}$$

as $\varepsilon \to 0$ for some $\varepsilon_0 = \varepsilon(z_0) > 0$. More generally, we would be able to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t \log 1/t \cdot \log\log 1/t \cdot \ldots \cdot \log \ldots \log 1/t)$.

Theorem 7. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $\sigma \in L_p(D)$, $p > 2$, have compact support in D, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D, $K_\mu \in L_1(D)$ and, for each point $z_0 \in \partial D$ and for some $\varepsilon_0 = \varepsilon(z_0) > 0$,*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{r k^T_\mu(z_0, r)}\ =\ \infty\ , \tag{5.4}$$

where $k^T_\mu(z_0, r)$ is the mean value of $K^T_\mu(z_0, r)$ over the circles $S(z_0, r)$.

Then the Beltrami equation (1.2) *with the source σ has a regular multi-valued solution ω of the Dirichlet problem* (1.1) *in D for each continuous function $\varphi : \partial D \to \mathbb{R}$ that is unique up to an additive pure imaginary constant.*

Moreover, $\omega = h\circ f$, $h := \mathcal{A}+H$, where $f:\mathbb{C}\to\mathbb{C}$ is a $\mu-$conformal mapping with μ extended by zero outside of D, $H : D_*\to\mathbb{C}$ is a generalized analytic function in $D_* := f(D)$ with the source S calculated in (2.6) and $\mathcal{A}$ is a multi-valued analytic function in D_* with a single valued real part satisfying the Dirichlet condition (2.9).

Corollary 13. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $\sigma\in L_p(D)$, $p>2$, have compact support in D, $\mu : D\to\mathbb{C}$ be a measurable function with $|\mu(z)|<1$ a.e., K_μ be locally bounded in D, $K_\mu\in L_1(D)$ and, for each point $z_0\in\partial D$,*

$$k_\mu^T(z_0,\varepsilon) = O\left(\log\frac{1}{\varepsilon}\right) \qquad as\ \varepsilon\to 0\ . \tag{5.5}$$

Then all conclusions of Theorem 7 on regular multi-valued solutions for the Dirichlet problem (1.1) with arbitrary continuous boundary data $\varphi : \partial D\to\mathbb{R}$ to the Beltrami equation (1.2) with the source σ hold.

Remark 12. In particular, the conclusions of Theorem 7 hold if

$$K_\mu^T(z,z_0) = O\left(\log\frac{1}{|z-z_0|}\right) \qquad \text{as} \quad z\to z_0 \quad \forall\ z_0\in\partial D\,. \tag{5.6}$$

Moreover, the condition (5.5) can be replaced by the series of weaker conditions

$$k_\mu^T(z_0,\varepsilon) = O\left(\left[\log\frac{1}{\varepsilon}\cdot\log\log\frac{1}{\varepsilon}\cdot\ldots\cdot\log\ldots\log\frac{1}{\varepsilon}\right]\right) \qquad \forall\ z_0\in\partial D\ . \tag{5.7}$$

Theorem 8. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $\sigma\in L_p(D)$, $p>2$, have compact support in D, $\mu : D\to\mathbb{C}$ be a measurable function with $|\mu(z)|<1$ a.e., K_μ be locally bounded in D, $K_\mu\in L_1(D)$ and, for each point $z_0\in\partial D$ and a neighborhood U_{z_0} of z_0,*

$$\int\limits_{U_{z_0}} \Phi_{z_0}\left(K_\mu^T(z,z_0)\right)\, dm(z) < \infty\ , \tag{5.8}$$

where $\Phi_{z_0} : (0,\infty]\to(0,\infty]$ is a convex non-decreasing function such that

$$\int\limits_{\Delta(z_0)}^{\infty} \log\,\Phi_{z_0}(t)\,\frac{dt}{t^2} \ = \ +\infty \quad for\ some\ \Delta(z_0)>0\ . \tag{5.9}$$

Then the Beltrami equation (1.2) *with the source* σ *has a regular multi-valued solution* ω *of the Dirichlet problem* (1.1) *in* D *for each continuous function* $\varphi : \partial D \to \mathbb{R}$ *that is unique up to an additive pure imaginary constant.*

Moreover, $\omega = h \circ f$, $h := \mathcal{A} + H$, *where* $f : \mathbb{C} \to \mathbb{C}$ *is a* $\mu-$*conformal mapping with* μ *extended by zero outside of* D, $H : D_* \to \mathbb{C}$ *is a generalized analytic function in* $D_* := f(D)$ *with the source* S *calculated in* (2.6) *and* $\mathcal{A}$ *is a multi-valued analytic function in* D_* *with a single valued real part satisfying the Dirichlet condition* (2.9)*.*

Corollary 14. *Let* D *be a bounded domain in* $\mathbb{C}$ *with no boundary component degenerated to a single point,* $\sigma \in L_p(D)$, $p > 2$, *have compact support in* D, $\mu : D \to \mathbb{C}$ *be a measurable function with* $|\mu(z)| < 1$ *a.e.,* K_μ *be locally bounded in* D, $K_\mu \in L_1(D)$ *and, for each point* $z_0 \in \partial D$ *and a neighborhood* U_{z_0} *of* z_0,

$$\int\limits_{U_{z_0}} e^{\alpha(z_0) K_\mu^T(z,z_0)}\, dm(z) < \infty \qquad \text{for some } \alpha(z_0) > 0\ . \tag{5.10}$$

Then all the conclusions of Theorem 8 on regular multi-valued solutions for the Dirichlet problem (1.1) *with continuous data* $\varphi : \partial D \to \mathbb{R}$ *to the Beltrami equation* (1.2) *with the source* σ *hold.*

Corollary 15. *Let* D *be a bounded domain in* $\mathbb{C}$ *with no boundary component degenerated to a single point,* $\sigma \in L_p(D)$, $p > 2$, *have compact support in* D, $\mu : D \to \mathbb{C}$ *be a measurable function with* $|\mu(z)| < 1$ *a.e.,* K_μ *be locally bounded in* D *and, for a neighborhood* U *of* ∂D,

$$\int\limits_{U} \Phi\left(K_\mu(z)\right)\, dm(z) < \infty\ , \tag{5.11}$$

where $\Phi : (0,\infty] \to (0,\infty]$ *is a convex non-decreasing function with, for* $\delta > 0$,

$$\int\limits_{\delta}^{\infty} \log\, \Phi(t)\, \frac{dt}{t^2} \;=\; +\infty\ . \tag{5.12}$$

Then all the conclusions of Theorem 8 on regular multi-valued solutions for the Dirichlet problem (1.1) *with continuous data* $\varphi : \partial D \to \mathbb{R}$ *to the Beltrami equation* (1.2) *with the source* σ *hold.*

Remark 13. The condition (5.12) is not only sufficient but also necessary to have the regular multi-valued solutions of the Dirichlet problem

(1.1) for arbitrary Beltrami equations with sources (1.2), satisfying the integral constraints (5.11), for all continuous functions $\varphi : \partial D \to \mathbb{R}$, see arguments in Remark 8.

Corollary 16. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $\sigma \in L_p(D)$, $p > 2$, have compact support in D, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D and, for a neighborhood U of ∂D and some $\alpha > 0$,*

$$\int\limits_U e^{\alpha K_\mu(z)}\, dm(z) \ < \ \infty \ . \tag{5.13}$$

Then all the conclusions of Theorem 8 on regular multivalued solutions for the Dirichlet problem (1.1) *with continuous data $\varphi : \partial D \to \mathbb{R}$ to the Beltrami equation* (1.2) *with the source σ hold.*

6. Dirichlet problem for Poisson type equations

Let us denote by $\mathbb{S}^{2\times 2}$ the collection of all 2×2 matrices with real entries

$$A \ = \ \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] \ , \tag{6.1}$$

which are symmetric, i.e., $a_{12} = a_{21}$, with $\det A = 1$ and **ellipticity condition** $\det(I + A) > 0$, where I is the unit 2×2 matrix. The latter condition means in terms of entries of A that $(1 + a_{11})(1 + a_{22}) > a_{12}a_{21}$.

Now, let us consider in a domain D of the complex plane $\mathbb{C}$ the Poisson type equations (1.4), where $A : D \to \mathbb{S}^{2\times 2}$ is a measurable matrix valued function whose elements $a_{ij}(z)$, $i, j = 1, 2$ are measurable and locally bounded and first the source $g : D \to \mathbb{R}$ is a scalar function in $L_{1,\text{loc}}$.

It is well–known, see Theorem 16.1.6 in [4], that nonhomogeneous Beltrami equations (1.2) with locally bounded K_μ are closely connected with the Poisson type equations (1.4), where $A : D \to \mathbb{S}^{2\times 2}$ is the measurable matrix valued function

$$A(z) \ := \ \left[\begin{array}{cc} \frac{|1-\mu(z)|^2}{1-|\mu(z)|^2} & \frac{-2\mathrm{Im}\,\mu(z)}{1-|\mu(z)|^2} \\ \frac{-2\mathrm{Im}\,\mu(z)}{1-|\mu(z)|^2} & \frac{|1+\mu(z)|^2}{1-|\mu(z)|^2} \end{array} \right] \ , \tag{6.2}$$

whose entries $a_{ij}(z)$ are dominated by $K_\mu(z)$ and, thus, they are locally bounded.

Vice versa, locally uniform elliptic (1.4) with measurable $A : D \to \mathbb{S}^{2\times 2}$ just correspond to nonhomogeneous Beltrami equations (1.2) with coefficients

$$\mu_A \ := \ -\frac{a_{11} - a_{22} + i(a_{12} + a_{21})}{2 + a_{11} + a_{22}} \ , \tag{6.3}$$

whose dilatation quotients K_{μ_A} are locally bounded.

Given such a matrix function A and a μ-conformal mapping $f^\mu : D \to \mathbb{C}$, we have already seen in Lemma 1 of [12], by direct computations, that if a function T and the entries of A are sufficiently smooth, then

$$\mathrm{div}\,[A(z)\nabla\,(T(f^\mu(z)))] \;=\; J(z)\triangle T(f^\mu(z))\;. \tag{6.4}$$

In the case $T \in W^{1,2}_{\mathrm{loc}}$, we understand equality (6.4) in the distributional sense, see Proposition 3.1 in [13], i.e., for all $\psi \in W^{1,2}_0(D)$,

$$\int\limits_D \langle A\nabla(T\circ f^\mu), \nabla\psi\rangle\; dm_z \;=\; \int\limits_D J(z)\langle M^{-1}((\nabla T)\circ f^\mu), \nabla\psi\rangle\; dm_z\;. \tag{6.5}$$

Here M is the Jacobian matrix of the mapping f^μ and J is its Jacobian.

Later on, we use the **logarithmic (Newtonian) potential of sources** $G \in L_1(\mathbb{C})$ with compact supports given by the formula:

$$\mathcal{N}^G(z) \;:=\; \frac{1}{2\pi}\int\limits_{\mathbb{C}} \ln|z-w|\,G(w)\; dm(w)\;. \tag{6.6}$$

By Lemmas 3 in [14] and Theorem 2 in [15], we have its basic properties:

Proposition 5. *Let $G : \mathbb{C} \to \mathbb{R}$ have compact support. If $G \in L_1(\mathbb{C})$, then $\mathcal{N}^G \in L_{r,\mathrm{loc}}(\mathbb{C})$ for all $r \in [1,\infty)$, $\mathcal{N}^G \in W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ for all $p \in [1,2)$, moreover, there exist generalized derivatives by Sobolev $\frac{\partial^2 N^G}{\partial z\partial\overline{z}}$ and $\frac{\partial^2 N^G}{\partial\overline{z}\partial z}$ satisfying the equalities*

$$4\cdot\frac{\partial^2 N^G}{\partial z\partial\overline{z}} \;=\; \triangle\mathcal{N}^G \;=\; 4\cdot\frac{\partial^2 N^G}{\partial\overline{z}\partial z} \;=\; G \quad a.e. \tag{6.7}$$

Furthermore, if $G \in L_{p'}(\mathbb{C})$ for some $p' > 1$, then $\mathcal{N}^G \in W^{2,p'}_{\mathrm{loc}}(\mathbb{C})$, moreover, $\mathcal{N}^G \in W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ for some $p > 2$ and, consequently, $\mathcal{N}^G \in C^\alpha_{\mathrm{loc}}(\mathbb{C})$ with $\alpha = 1-2/p$. Finally, if $G \in L_{p'}(\mathbb{C})$ for some $p' > 2$, then $\mathcal{N}^G \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{C})$ with $\alpha = 1-2/p'$.

As it was before, we assume here that the dilatations $K^T_{\mu_A}(z,z_0)$ and $K_{\mu_A}(z)$ are extended by 1 outside of the domain D.

Lemma 4. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $g \in L_{p'}(D)$, $p' > 1$, with compact support, and $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function with locally Hölder continuous entries.*

Suppose also that $K_{\mu_A}\in L^1(D)$ is locally bounded and, for each $z_0\in\partial D$,

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_{\mu_A}(z,z_0)\cdot\psi^2_{z_0,\varepsilon}(|z-z_0|)\,dm(z)=o(I^2_{z_0}(\varepsilon))\qquad as\ \varepsilon\to 0\ , \tag{6.8}$$

where $\varepsilon_0=\varepsilon(z_0)>0$ and $\psi_{z_0,\varepsilon}:(0,\varepsilon_0)\to(0,\infty)$ are measurable functions with

$$I_{z_0}(\varepsilon)\colon=\int\limits_{\varepsilon}^{\varepsilon_0}\psi_{z_0,\varepsilon}(t)\,dt<\infty\qquad\forall\ \varepsilon\in(0,\varepsilon_0)\,. \tag{6.9}$$

Then the Poisson type equation (1.4) has the unique weak continuous solution u in the class $C^{\alpha}_{\rm loc}\cap W^{1,p}_{\rm loc}\cap W^{2,p'}_{\rm loc}$, $\alpha=1-2/p$, for some $p>2$ of the Dirichlet problem (1.3) in D for each continuous function $\varphi:\partial D\to\mathbb{R}$.

Moreover, $u=U\circ f$, where $f:\mathbb{C}\to\mathbb{C}$ is a μ_A-conformal mapping with μ_A extended by zero outside of D, U is a weak generalized harmonic function with the source G of the class $L_{p'}(D_*)$ in the domain $D_*:=f(D)$,

$$G\ :=\ \frac{g}{J}\circ f^{-1}\ ,\quad J(z):=|f_z|^2-|f_{\overline{z}}|^2\ , \tag{6.10}$$

of the class $C^{\alpha}_{\rm loc}\cap W^{1,p}_{\rm loc}\cap W^{2,p'}_{\rm loc}$ in D_*, satisfying the Dirichlet condition

$$\lim_{w\to\zeta}\ U(w)=\varphi_*(\zeta)\quad\forall\ \zeta\in\partial D_*\ ,\quad \varphi_*:=\varphi\circ f^{-1}|_{\partial D_*}\ . \tag{6.11}$$

Here u is called a **weak solution** of the Poisson type equation (1.4) if

$$\int\limits_D\{\langle A(z)\nabla u(z),\nabla\psi\rangle\ +\ \Sigma(z)\,Q(u(z))\,\psi(z)\}\ dm_z\ =\ 0\quad\ \forall\ \psi\in C^1_0(D)\ . \tag{6.12}$$

Remark 14. In turn, by the proof below, $U:=\mathcal{H}+\mathcal{N}^G$, where $\mathcal{H}$ is the unique harmonic function in D_*, satisfying the Dirichlet condition

$$\lim_{w\to\zeta}\ \mathcal{H}(w)=\varphi^*(\zeta)\quad\forall\ \zeta\in\partial D_*\ ,\quad \varphi^*:=\varphi_*-\mathcal{N}^G|_{\partial D_*}\ . \tag{6.13}$$

Furthermore, arguing similarly to the proof, we obtain also by Proposition 5 that if $g\in L_{p'}(D)$ for some $p'>2$, then in addition $u\in C^{1,\alpha'}_{\rm loc}(D)$ with $\alpha'=1-2/p'$. In the case, the function U is a generalized harmonic function with the source $G\in L_{p'}(D_*)$ of the class $C^{1,\alpha'}_{\rm loc}\cap W^{2,p'}_{\rm loc}$ in the domain D_*.

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be a μ_A-conformal mapping from Proposition 1 with the complex coefficient μ_A in $\mathbb{C}$ extended by zero outside of D. Since entries of A are locally Hölder continuous, the mapping $f|_D$ is smooth, see e.g. [22] and [23], and, moreover, see e.g. Theorem V.7.1 in [28], its continuous Jacobian

$$J(z) \ = \ |f_z|^2 \, - \, |f_{\bar{z}}|^2 \ > \ 0 \qquad \forall \, z \in D \ . \tag{6.14}$$

Consequently, f^{-1} is also smooth in the domain $D_* := f(D)$, see e.g. formulas I.C(3) in [1]. Thus, by the replacement of variables, see e.g. the point (vi) of Theorem 5 in [2], the function G in (6.10) belongs to the class $L_{p'}(D_*)$ because of G has compact support in D_* by hypotheses of the lemma and in view of homeomorphism of f, and because of the continuous function $J^{-1} \circ f^{-1}$ is bounded over the support of G.

Next, the domain D_* is bounded and has no boundary component degenerated to a single point because of D is so by hypotheses of the lemma and because of the mapping f is a homeomorphism of $\mathbb{C}$ into itself. Thus, by Corollary 4.1.8 and Theorem 4.2.2 in [36], there is the unique harmonic function $\mathcal{H} : D \to \mathbb{R}$, satisfying the Dirichlet condition (6.13). Thus, by Proposition 5 $U := \mathcal{H} + \mathcal{N}^G$ is a weak generalized harmonic function with the source G of the class $L_{p'}(D_*)$ in the domain D_*, satisfying the Dirichlet condition (6.11). Note that again by Proposition 5 $U \in W^{2,p'}_{\rm loc}(\mathbb{C})$, moreover, $U \in W^{1,p}_{\rm loc}(\mathbb{C})$ for some $p > 2$ and, consequently, $U \in C^{\alpha}_{\rm loc}(\mathbb{C})$ with $\alpha = 1 - 2/p$.

Finally, by Proposition 3.1 in [13], see (6.5), the function $u := U \circ f$ gives the desired solution of the Poisson type equation (1.4) because $f|_D$ is a local quasi-isometry in D of the class C^1, see e.g. 1.1.7 in [31], and such a solution is unique. □

Remark 15. Note that if the family of the functions $\psi_{z_0,\varepsilon}(t) \equiv \psi_{z_0}(t)$ is independent on the parameter ε, then the condition (6.8) implies that $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. This follows immediately from arguments by contradiction, apply for it (2.3) and the condition $K_{\mu_A} \in L^1(D)$. Note also that (6.8) holds, in particular, if, for some $\varepsilon_0 = \varepsilon(z_0)$,

$$\int\limits_{|z-z_0|<\varepsilon_0} K^T_{\mu_A}(z, z_0) \cdot \psi^2_{z_0}(|z - z_0|) \, dm(z) < \infty \qquad \forall \; z_0 \in \partial D \tag{6.15}$$

and $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In other words, for the existence of regular enough weak solutions of the Dirichlet problem (1.3) in D to the Poisson type equation (1.4) with arbitrary continuous boundary functions φ, it is sufficient that the integral in (6.15) converges for some nonnegative function $\psi_{z_0}(t)$ that is locally integrable over $(0, \varepsilon_0]$ but has a nonintegrable singularity at 0. The functions $\log^{\lambda}(e/|z-z_0|)$, $\lambda \in (0,1)$, $z \in \mathbb{D}$, $z_0 \in \overline{\mathbb{D}}$, and $\psi(t) = 1/(t\ \log(e/t))$, $t \in (0,1)$, show that the condition (6.15) is compatible with the condition $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Furthermore, the condition (6.8) in Lemma 4 shows that, for the existence of such solutions of the Dirichlet problem (1.3) to the Poisson type equation (1.4), it is sufficient even that the integral in (6.15) to be divergent in a controlled way.

Similarly to Section 3, we derive from Lemma 4 the next series of results.

Theorem 9. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $g \in L_{p'}(D)$, $p' > 1$, with compact support, and $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function with locally Hölder continuous entries, $K_{\mu_A} \in L^1(D)$ be locally bounded and, for each $z_0 \in \partial D$, $K^T_{\mu_A}(z, z_0) \leqslant Q_{z_0}(z)$ in its neighborhood U_{z_0} for a function $Q_{z_0} : U_{z_0} \to [0,\infty]$ in the class* FMO(z_0).

Then the Poisson type equation (1.4) *has the unique weak continuous solution u in the class $C^{\alpha}_{\rm loc} \cap W^{1,p}_{\rm loc} \cap W^{2,p'}_{\rm loc}$, $\alpha = 1-2/p$, for some $p > 2$ of the Dirichlet problem* (1.3) *in D for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Moreover, $u = U \circ f$, $U := \mathcal{H} + \mathcal{N}^G$, where $f : \mathbb{C} \to \mathbb{C}$ is a μ_A-conformal mapping with μ_A extended by zero outside of D, U is a weak generalized harmonic function in $D_* := f(D)$ with the source $G \in L_{p'}(D_*)$ calculated in (6.10) *and $\mathcal{H}$ is the unique harmonic function with the Dirichlet condition* (6.13).

Corollary 17. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $g \in L_{p'}(D)$, $p' > 1$, with compact support, and $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function with locally Hölder continuous entries, $K_{\mu_A} \in L^1(D)$ be locally bounded and, for each $z_0 \in \partial D$,*

$$\overline{\lim_{\varepsilon \to 0}} \ \fint_{B(z_0,\varepsilon)} K^T_{\mu_A}(z, z_0)\, dm(z) < \infty \ . \tag{6.16}$$

Then all the conclusions of Theorem 9 on solutions of the Dirichlet problem (1.3) *with continuous data $\varphi : \partial D \to \mathbb{R}$ to the Poisson type equation* (1.4) *hold.*

Corollary 18. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $g \in L_{p'}(D)$, $p' > 1$, with compact support, and $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function with locally Hölder continuous entries, K_{μ_A} be locally bounded in D and have a dominant*

$Q : U \to [1,\infty)$ of the class $\mathrm{BMO}_{\mathrm{loc}}(U)$ in a neighborhood U of ∂D. Then all the conclusions of Theorem 9 hold.

Remark 16. In particular, the conclusions of Theorem 9 hold if $Q \in \mathrm{W}^{1,2}_{\mathrm{loc}}$.

Corollary 19. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $g \in L_{p'}(D)$, $p' > 1$, with compact support, and $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function with locally Hölder continuous entries, K_{μ_A} be locally bounded in D and have a dominant $Q : U \to [1,\infty)$ of the class $\mathrm{FMO}(U)$ in a neighborhood U of ∂D. Then all the conclusions of Theorem 9 hold.*

Theorem 10. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $g \in L_{p'}(D)$, $p' > 1$, with compact support, and $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function with locally Hölder continuous entries, $K_{\mu_A} \in L^1(D)$ be locally bounded in D and, for each $z_0 \in \partial D$, $\varepsilon_0 = \varepsilon(z_0) > 0$,*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_{\mu_A}(z,z_0)\,\frac{dm(z)}{|z-z_0|^2} = o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \qquad \text{as } \varepsilon \to 0\ . \quad (6.17)$$

Then the Poisson type equation (1.4) has the unique weak continuous solution u in the class $C^{\alpha}_{\mathrm{loc}} \cap W^{1,p}_{\mathrm{loc}} \cap W^{2,p'}_{\mathrm{loc}}$, $\alpha = 1 - 2/p$, for some $p > 2$ of the Dirichlet problem (1.3) in D for each continuous function $\varphi : \partial D \to \mathbb{R}$.

Moreover, $u = U \circ f$, $U := \mathcal{H} + \mathcal{N}^G$, where $f : \mathbb{C} \to \mathbb{C}$ is a μ_A-conformal mapping with μ_A extended by zero outside of D, U is a weak generalized harmonic function in $D_* := f(D)$ with the source $G \in L_{p'}(D_*)$ calculated in (6.10) and $\mathcal{H}$ is the unique harmonic function with the Dirichlet condition (6.13).

Remark 17. Choosing in Lemma 4 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (6.17) by

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{K^T_{\mu_A}(z,z_0)\,dm(z)}{\left(|z-z_0|\log\frac{1}{|z-z_0|}\right)^2} = o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right) \qquad (6.18)$$

In general, we are able to give here the whole scale of the corresponding conditions using functions $\psi(t)$ of the form $1/(t \log 1/t \cdot \log\log 1/t \cdot \ldots \cdot \log \ldots \log 1/t)$.

Theorem 11. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $g \in L_{p'}(D)$, $p' > 1$, with* *compact support, and* $A : D \to \mathbb{S}^{2\times 2}$ *be a matrix function with locally Hölder continuous entries,* $K_{\mu_A} \in L^1(D)$ *be locally bounded and, for each* $z_0 \in \partial D$, $\varepsilon_0 = \varepsilon(z_0) > 0$,

$$\int\limits_0^{\varepsilon_0} \frac{dr}{r k^T_{\mu_A}(z_0, r)} \;=\; \infty\ , \tag{6.19}$$

where $k^T_{\mu_A}(z_0, r)$ *is the integral mean of* $K^T_{\mu_A}(z, z_0)$ *over the circle* $S(z_0, r)$.

Then the Poisson type equation (1.4) *has the unique weak continuous solution* u *in the class* $C^\alpha_{\rm loc} \cap W^{1,p}_{\rm loc} \cap W^{2,p'}_{\rm loc}$, $\alpha = 1 - 2/p$, *for some* $p > 2$ *of the Dirichlet problem* (1.3) *in* D *for each continuous function* $\varphi : \partial D \to \mathbb{R}$.

Moreover, $u = U \circ f$, $U := \mathcal{H} + \mathcal{N}^G$, *where* $f : \mathbb{C} \to \mathbb{C}$ *is a* μ_A-*conformal mapping with* μ_A *extended by zero outside of* D, U *is a weak generalized harmonic function in* $D_* := f(D)$ *with the source* $G \in L_{p'}(D_*)$ *calculated in* (6.10) *and* $\mathcal{H}$ *is the unique harmonic function with the Dirichlet condition* (6.13).

Corollary 20. *Let* D *be a bounded domain in* $\mathbb{C}$ *with no boundary component degenerated to a single point,* $g \in L_{p'}(D)$, $p' > 1$, *with compact support, and* $A : D \to \mathbb{S}^{2\times 2}$ *be a matrix function with locally Hölder continuous entries,* $K_{\mu_A} \in L^1(D)$ *be locally bounded and, for each* $z_0 \in \partial D$,

$$k^T_{\mu_A}(z_0, \varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \qquad as\ \varepsilon \to 0\ . \tag{6.20}$$

Then all the conclusions of Theorem 11 on solutions of the Dirichlet problem (1.3) *with continuous data* $\varphi : \partial D \to \mathbb{R}$ *to the Poisson type equation* (1.4) *hold.*

Remark 18. In particular, all the conclusions of Theorem 11 hold if, for each point $z_0 \in \partial D$,

$$K^T_{\mu_A}(z, z_0) = O\left(\log \frac{1}{|z - z_0|}\right) \qquad \text{as} \quad z \to z_0\ . \tag{6.21}$$

Moreover, (6.20) can be replaced by the whole series of weaker condition

$$k^T_{\mu_A}(z_0, \varepsilon) = O\left(\left[\log \frac{1}{\varepsilon} \cdot \log\log \frac{1}{\varepsilon} \cdot \ldots \cdot \log \ldots \log \frac{1}{\varepsilon}\right]\right) \qquad \forall\ z_0 \in \partial D\ . \tag{6.22}$$

Theorem 12. *Let* D *be a bounded domain in* $\mathbb{C}$ *with no boundary component degenerated to a single point,* $g \in L_{p'}(D)$, $p' > 1$, *with* *compact support, and $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function with locally Hölder continuous entries, $K_{\mu_A} \in L^1(D)$ be locally bounded and, for each $z_0 \in \partial D$ and a neighborhood U_{z_0},*

$$\int\limits_{U_{z_0}} \Phi_{z_0}\left(K^T_{\mu_A}(z,z_0)\right)\, dm(z) \;<\; \infty\ , \tag{6.23}$$

where $\Phi_{z_0} : [0,\infty] \to [0,\infty]$ is a convex non-decreasing function such that, for some $\Delta(z_0) > 0$,

$$\int\limits_{\Delta(z_0)}^{\infty} \log\, \Phi_{z_0}(t)\, \frac{dt}{t^2} \;=\; +\infty\ . \tag{6.24}$$

Then the Poisson type equation (1.4) *has the unique weak continuous solution u in the class $C^{\alpha}_{\rm loc} \cap W^{1,p}_{\rm loc} \cap W^{2,p'}_{\rm loc}$, $\alpha = 1-2/p$, for some $p > 2$ of the Dirichlet problem* (1.3) *in D for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

Moreover, $u = U \circ f$, $U := \mathcal{H} + \mathcal{N}^G$, where $f : \mathbb{C} \to \mathbb{C}$ is a μ_A-conformal mapping with μ_A extended by zero outside of D, U is a weak generalized harmonic function in $D_* := f(D)$ with the source $G \in L_{p'}(D_*)$ calculated in (6.10) *and $\mathcal{H}$ is the unique harmonic function with the Dirichlet condition* (6.13).

Corollary 21. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $g \in L_{p'}(D)$, $p' > 1$, with compact support, and $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function with locally Hölder continuous entries, $K_{\mu_A} \in L^1(D)$ be locally bounded and, for each $z_0 \in \partial D$, a neighborhood U_{z_0} of z_0 and $\alpha(z_0) > 0$,*

$$\int\limits_{U_{z_0}} e^{\alpha(z_0) K^T_{\mu_A}(z,z_0)}\, dm(z) < \infty\ . \tag{6.25}$$

Then all the conclusions of Theorem 12 on solutions of the Dirichlet problem (1.3) *with continuous data $\varphi : \partial D \to \mathbb{R}$ to the Poisson type equation* (1.4) *hold.*

Corollary 22. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $g \in L_{p'}(D)$, $p' > 1$, with compact support, and $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function with locally*

Hölder continuous entries, $K_{\mu_A} \in L^1(D)$ be locally bounded and, for a neighborhood U of ∂D,

$$\int\limits_U \Phi\left(K_{\mu_A}(z)\right)\, dm(z) < \infty\ , \tag{6.26}$$

where $\Phi : [0,\infty] \to [0,\infty]$ is a convex non-decreasing function such that, for some $\delta > 0$,

$$\int\limits_{\delta}^{\infty} \log\, \Phi(t)\, \frac{dt}{t^2} \ =\ +\infty\ . \tag{6.27}$$

Then all the conclusions of Theorem 12 on solutions of the Dirichlet problem (1.3) *with continuous data $\varphi : \partial D \to \mathbb{R}$ to the Poisson type equation* (1.4) *hold.*

Remark 19. By Theorems 2.5 and 5.1 in [44], condition (6.27) is not only sufficient but also necessary to have a regular enough weak solution u of the Dirichlet problem (1.3) in D for all the Poisson type equations (1.4), satisfying the integral constraints (6.26), for arbitrary continuous functions $\varphi : \partial D \to \mathbb{R}$ because such solutions have the representation through regular homeomorphic solutions $f = f^{\mu}$ of the homogeneous Beltrami equation (2.1) with $\mu = \mu_A$.

Corollary 23. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $g \in L_{p'}(D)$, $p' > 1$, with compact support, and $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function with locally Hölder continuous entries, $K_{\mu_A} \in L^1(D)$ be locally bounded and, for a neighborhood U of ∂D and $\alpha > 0$,*

$$\int\limits_U e^{\alpha K_{\mu_A}(z)}\, dm(z)\ <\ \infty\ . \tag{6.28}$$

Then all the conclusions of Theorem 12 on solutions of the Dirichlet problem (1.3) *with continuous data $\varphi : \partial D \to \mathbb{R}$ to the Poisson type equation* (1.4) *hold.*

As a result, we have a number of effective integral criteria for the solvability of the classical Dirichlet problem (1.3) in the most general admissible domains to one of the main equations (1.4) of the hydromechanics (fluid mechanics) in anisotropic and inhomogeneous media.

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