Український математичний вісник
Том 20 (**2023**), № 1, 73 – 86
https://doi.org/10.37069/1810-3200-2023-20-1-4

On Grunsky norm of univalent functions

Samuel L. Krushkal

Abstract. We establish an intrinsic lower bound for the Grunsky norm of univalent functions in the disk. This bound sheds light on the intrinsic geometric features of complex analysis and of Teichmüllerv space theory.

2020 MSC. Primary: 30C55, 30C62, 30F45; Secondary 30F60, 32F45, 32G15, 46G20.

Key words and phrases. Univalent function, Grunsky norm, quasiconformal extension, universal Teichmüller space, Teichmüller norm, hyperbolic metric, generalized Gaussian curvature, supporting metric.

1. The Grunsky norm

The classical Grunsky theorem of 1939 implies the necessary and sufficient conditions for univalence of holomorphic functions in a finitely connected domain on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ in terms of an infinite system of the coefficient inequalities. In particular, for the canonical disk $\mathbb{D}^* = \{z \in \widehat{\mathbb{C}} : \ |z| > 1\}$ this theorem yields that a holomorphic function $f(z) = z + \text{const} + O(z^{-1})$ in a neighborhood of $z = \infty$ can be extended to a univalent holomorphic function on $\mathbb{D}^*$ if and only if the Taylor coefficients α_{mn} of the function

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (\mathbb{D}^*)^2, \tag{1}$$

called the **Grunsky coefficients** of $f(z)$, satisfy the inequality

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| \leq 1, \tag{2}$$

Received 19.02.2023

for any sequence $\mathbf{x} = (x_n)$ from the unit sphere $S(l^2)$ of the Hilbert space l^2 with norm $\|\mathbf{x}\| = (\sum\limits_1^\infty |x_n|^2)^{1/2}$; here the principal branch of the logarithmic function is chosen (cf. [7]). The quantity

$$\varkappa(f) = \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| : \ \mathbf{x} = (x_n) \in S(l^2) \Big\} \leq 1$$

is called the **Grunsky norm** of f.

The univalent functions $f(z) = z + b_0 + b_1 z^{-1} + \dots$ in $\mathbb{D}^*$ admitting quasiconformal extensions across the unit circle $\mathbb{S}^1 = \partial \mathbb{D}^*$ onto the disk $\mathbb{D} = \{|z| < 1\}$ form the class Σ_Q. To have their uniqueness for a given Beltrami coefficient $\mu(z) = \partial_{\bar{z}} f / \partial_z f$ in $\mathbb{D}$, compactness in the topology of locally uniform convergence on $\mathbb{C}$, etc., we add the third normalization condition

$$f(0) = 0.$$

All such $f \in \Sigma_Q$ are zero free in $\mathbb{D}^*$, hence their inversions $F_f(z) = 1/f(1/z) = z + a_2 z^2 + \dots$ are holomorphic and univalent in the disk $\mathbb{D}$ with $F_f(\infty) = \infty$. The functions f and F_f have the same Grunsky coefficients, and $\varkappa(F_f) = \varkappa(f)$.

Note also that the norm $\varkappa(f)$ is defined for all $f \in \Sigma_Q$ and does not depend on the additional normalization at 0.

For the functions with k-quasiconformal extensions ($k < 1$), we have instead of (2) a stronger bound

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| \leq k \quad \text{for any} \ \ \mathbf{x} = (x_n) \in S(l^2), \qquad (3)$$

established first in [19] (see also [15]).

Note that the Grunsky matrix operator $\mathcal{G}(f) = (\sqrt{mn}\ \alpha_{mn}(f))_{m,n=1}^{\infty}$ acts as a linear operator $l^2 \to l^2$ contracting the norms of elements $\mathbf{x} \in l^2$; the norm of this operator equals $\varkappa(f)$.

The method of Grunsky inequalities was generalized in several directions, even to bordered Riemann surfaces with a finite number of boundary components (see [6, 7, 16, 23, 24, 26, 30]), replacing the generating function (1) by appropriate bilinear differential; this leads to the generalized Grunsky norm. In this paper, we shall deal only with the canonical case of disk $\mathbb{D}^*$.

The Grunsky norm $\varkappa(f)$ is dominated by the **Teichmüller norm** $k(f)$, which is equal to the infimum of dilatations $k(w^\mu) = \|\mu\|_\infty$ of quasiconformal extensions of f to $\widehat{\mathbb{C}}$. Here w^μ denotes a homeomorphic solution to the Beltrami equation $\partial_{\overline{z}} w = \mu \partial_z w$ on $\mathbb{C}$ extending f; accordingly, μ is called the **Beltrami coefficient** (or complex dilatation) of w.

For most functions f, we have the strong inequality $\varkappa(f) < k(f)$ (moreover, the functions satisfying this inequality form a dense subset of Σ_Q), while the functions with the equal norms play a crucial role in many applications.

On the other hand, the important result of Pommerenke and Zhuravlev states that if *a function* $f \in \Sigma$ *satisfies the inequality* $\varkappa(f) < k$ *with some constant* $k < 1$*, then* f *has a quasiconformal extension to* $\widehat{\mathbb{C}}$ *with a dilatation* $k_1 = k_1(k) \geq k$ [26, 32]; [17, pp. 82–84]. On explicit bounds $k_1(k)$ see, e.g., [14, 18, 22].

Each coefficient $\alpha_{mn}(f)$ in (4) is represented as a polynomial of a finite number of the initial coefficients $b_1, b_2, \dots, b_s$ of f; hence it depends holomorphically on Beltrami coefficients of quasiconformal extensions of f as well as on the **Schwarzian derivatives**

$$S_f(z) = \Big(\frac{f''(z)}{f'(z)}\Big)' - \frac{1}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2, \quad z \in \mathbb{D}^*.$$

These derivatives range over a bounded domain in the complex Banach space $\mathbf{B}(\mathbb{D}^*)$ of hyperbolically bounded holomorphic functions $\varphi \in \mathbb{D}^*$ with norm

$$\|\varphi\|_{\mathbf{B}} = \sup_{\mathbb{D}^*}(|z|^2 - 1)^2 |\varphi(z)|,$$

This domain models the **universal Teichmüller space T** (the space of complex structures on the disk) in holomorphic Bers' embedding of $\mathbf{T}$.

The following two sets of holomorphic functions ψ (equivalently, of holomorphic quadratic differentials ψdz^2)

$$A_1(\mathbb{D}) = \{\psi \in L_1(D) : \ \psi \ \text{ holomorphic in } \ \mathbb{D}\},$$
$$A_1^2(\mathbb{D}) = \{\psi = \omega^2 \in A_1(\mathbb{D}) : \ \omega \ \text{ holomorphic } \ \text{in } \ \mathbb{D}\}$$

are intrinsically connected with the extremal Beltrami coefficients (with minimal norm in their equivalence class) hence, with the Teichmüller norm and Grunsky inequalities.

The Beltrami coefficients of quasiconformal extensions w^μ of functions $f(z) \in \Sigma_Q$ range over the unit ball

$$\mathrm{Belt}(\mathbb{D})_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu(z)|\mathbb{D}^* = 0, \ \ \|\mu\|_\infty < 1\},$$

and the well-known criterion for extremality (the Hamilton–Krushkal–Reich–Strebel theorem) implies that a Betrami coefficient $\mu_0 \in \mathrm{Belt}(\mathbb{D})_1$

is extremal if an only if

$$\|\mu_0\|_\infty = \sup_{\|\psi\|_{A_1(\mathbb{D})}=1} \Big| \iint\limits_{\mathbb{D}} \mu_0(z)\psi(z)dxdy \Big| \quad (z = x + iy). \tag{4}$$

The same condition is necessary and sufficient for the infinitesimal extremality of μ_0 at the origin of the space $\mathbf{T}$ in the direction $t\phi_{\mathbf{T}}(\mu_0)$, where $\phi_{\mathbf{T}}$ is the defining (factorizing) holomorphic projection $\mathrm{Belt}(\mathbb{D})_1 \to \mathbf{T}$; see, e.g., [4, 5, 10].

Due to [11, 16], the elements of $A_1^2(\mathbb{D})$ are represented in the form

$$\psi(z) = \omega(z)^2 = \frac{1}{\pi} \sum_{m+n=2}^{\infty} \sqrt{mn}\, x_m x_n z^{m+n-2},$$

with $\|\mathbf{x}\|_{l^2} = \|\omega\|_{L_2}$; here $\mathbf{x} = (x_n)$.

Using the pairing

$$\langle \mu, \psi \rangle_{\mathbb{D}} = \iint\limits_{\mathbb{D}} \mu(z)\psi(z)dxdy \quad \psi \in L_1(\mathbb{D}), \quad \mu \in \mathrm{Belt}(\mathbb{D})_1,$$

we define the set

$$A_1^2(\mathbb{D})^\perp = \{\mu \in \mathrm{Belt}(\mathbb{D})_1 : \ \langle \mu, \psi \rangle_{\mathbb{D}} = 0 \quad \text{for all} \quad \psi \in \widetilde{A}_1^2(\mathbb{D})\}, \tag{5}$$

where $\widetilde{A}_1^2(\mathbb{D})$ is the span of elements from $A_1^2(\mathbb{D})$ (with A_1-norm). A crucial similar set for the Teichmüller norm is

$$A_1(\mathbb{D})^\perp = \mathrm{Ker}\, \phi_{\mathbf{T}}'(\mathbf{0}) = \{\mu \in \mathrm{Belt}(\mathbb{D})_1 : \langle \mu, \psi \rangle_{\mathbb{D}} = 0 \text{ for all } \psi \in A_1(\mathbb{D})\}.$$

An important fact is that the extremal Teichmüller Beltrami coefficients $\mu_0 = k|\psi|/\psi$ with $\psi \in A_1(\mathbb{D})$ cannot lie in $A_1(\mathbb{D})^\perp$.

2. A lower bound for Grunsky norm

The aim of this paper is to prove the following theorem giving an intrinsic lower bound for the Grunsky norm. This bound sheds light on the intrinsic geometric features of complex analysis and of Teichmüllerv space theory.

Theorem 1. *For any function $f \in \Sigma_Q$, its Grunsky norm satisfies*

$$\varkappa(f) \ge \alpha(f) := \sup_{\psi \in A_1^2(\mathbb{D}), \|\psi\|_{A_1}=1} \Big| \iint\limits_{\mathbb{D}} \mu_0(z)\psi(z)dxdy \Big|, \tag{6}$$

where μ_0 is an extremal Beltrami coefficient among quasiconformal extensions f^μ of f in the disk $\mathbb{D}$.

This theorem naturally relates to the intrinsic features of the Grunsky norm, though the estimate (6) is rough. This estimate is trivial for functions whose extremal coefficients μ_0 lie in the set (5), and therefore (in view of continuity of the Grunsky norm on $\mathbf{T}$), for all functions f^μ with small $\|\mu - \mu_0\|_\infty$, we have in (6) a strong inequality. This is valid, for example, for f^{μ_n} with (extremal) Beltrami coefficients $\mu_n = k|z^n|/z^n, \ n = 2p-1$ with $p \in \mathbb{N}$.

Note also that generically the extremal extension f^{μ_0} is not unique, but for a dense subset of functions f in $\Sigma_Q(D^*)$ their extremal coefficients μ is of Teichmüller type, which means that $\mu = k|\psi|/\psi$, where $k = \text{const} < 1$ and $\psi \in A_1(D)$ (such μ is unique in its equivalence class). It determines the Strebel point of the universal Teichmüller space $\mathbf{T}$; such points are dense in $\mathbf{T}$; see [5, 31]).

3. Two applications

We mention two important consequences of this theorem.

First, Theorem 1 improves the known results characterizing the univalent functions, for which the inequality $\varkappa_{D^*}(f) \le k$ is also sufficient for existence of k-quasiconformal extension to $\widehat{\mathbb{C}}$ established in [11, 15]: *the equality $\varkappa(f) = k(f)$ is valid if and only if the extremal extension of $f(z)$ to $\mathbb{D}$ satisfies*

$$\|\mu_0\|_\infty = \sup_{\psi \in A_1^2(\mathbb{D}), \|\psi\|_{A_1}=1} \Big| \iint\limits_{\mathbb{D}} \mu_0(z)\psi(z)dxdy \Big|.$$

This important fact found various applications. Its part "if" is an infinite dimensional generalisation of Kra's theorem for finite dimensional Teichmüller spaces [9].

Theorem 1 also improves some related results obtained in [14].

Another application concerns the **Fredholm eigenvalues** of Jordan curves. Recall that the Fredholm eigenvalues ρ_n of an oriented smooth closed Jordan curve $L \subset \widehat{\mathbb{C}}$ are the eigenvalues of its double-layer potential, or equivalently, of the integral equation

$$u(z) + \frac{\rho}{\pi} \int\limits_L u(\zeta) \frac{\partial}{\partial n_\zeta} \log \frac{1}{|\zeta - z|} ds_\zeta = h(z),$$

which often appears in applications (here n_ζ is the outer normal and ds_ζ is the length element at $\zeta \in L$).

The least positive eigenvalue $\rho_L = \rho_1$ plays a crucial role in many applications and is naturally connected with conformal and quasiconformal maps related to L. It can be defined for any oriented closed Jordan curve L by

$$\frac{1}{\rho_L} = \sup \frac{|\mathcal{D}_G(u) - \mathcal{D}_{G^*}(u)|}{\mathcal{D}_G(u) + \mathcal{D}_{G^*}(u)},$$

where G and G^* are, respectively, the interior and exterior of L; $\mathcal{D}$ denotes the Dirichlet integral, and the supremum is taken over all functions u continuous on $\widehat{\mathbb{C}}$ and harmonic on $G \cup G^*$. In particular, $\rho_L = \infty$ only for the circle. This quantity remains invariant under the action of the Moebius group $PSL(2, \widehat{\mathbb{C}})$.

The indicated value is intrinsically connected with the Grunsky coefficients of the exterior conformal map $f^* : \mathbb{D}^* \to D^*$; this is qualitatively expressed by the Kühnau–Schiffer theorem on reciprocity of ρ_L to the Grunsky norm $\varkappa(f^*)$ [20, 29].

The above theorem implies that for any quasiconformal curve $L \subset \widehat{\mathbb{C}}$ its Fredholm eigenvalue is estimated from below by

$$\frac{1}{\rho_L} \geq \sup_{\psi \in A_1^2(D), \|\psi\|_{A_1} = 1} |\langle \mu_0, \psi(z) \rangle_{\mathbb{D}}|,$$

where μ_0 is the extremal Beltrami coefficient of the appropriately normalized exterior conformal mapping function f^* on which the Teichmüller norm of f^* is attained.

This gives simultaneously the lower bound for the extremal dilatation of quasiconformal reflections across the curve L (the orientation reversing quasiconformal automorphisms of $\widehat{\mathbb{C}}$ which preserve L pointwise fixed); see, e.g. [13].

4. Preliminary lemmas

The proof of Theorem involves certain known results on conformal metrics $ds = \lambda(t)|dt|$ on the disk $\mathbb{D}$ with $\lambda(t) \geq 0$ (called also semimetrics) of negative generalized Gaussian curvature and of negative integral curvature bounded from above.

Recall that the **generalized Gaussian curvature** κ_λ of an upper semicontinuous Finsler metric $ds = \lambda|dt|$ in a domain $\Omega \subset \mathbb{C}$ is defined by

$$\kappa_\lambda(t) = -\frac{\Delta \log \lambda(t)}{\lambda(t)^2}, \tag{7}$$

where Δ is the **generalized Laplacian**

$$\Delta \lambda(t) = 4 \liminf_{r \to 0} \frac{1}{r^2} \Big\{ \frac{1}{2\pi} \int_0^{2\pi} \lambda(t + re^{i\theta}) d\theta - \lambda(t) \Big\}$$

(provided that $-\infty \le \lambda(t) < \infty$). Similar to C^2 functions, for which Δ coincides with the usual Laplacian, one obtains that λ is subharmonic on Ω if and only if $\Delta\lambda(t) \ge 0$; hence, at the points t_0 of local maximuma of λ with $\lambda(t_0) > -\infty$, we have $\Delta\lambda(t_0) \le 0$.

The sectional **holomorphic curvature** of a Finsler metric on a complex Banach manifold X is defined in a similar way as the supremum of the curvatures (7) over appropriate collections of holomorphic maps from the disk into X for a given tangent direction in the image.

As is well-known [1,12], the holomorphic curvature of the Kobayashi–Teichmüller metric $\mathcal{K}_{\mathbf{T}}(x,v)$ of universal Teichmüller space $\mathbf{T}$ equals -4 at all points (x,v) of the tangent bundle $\mathcal{T}(\mathbf{T})$ over $\mathbf{T}$. Instead, the holomorphic curvature of metric $\lambda_{\varkappa}$ generated on $\mathbb{D}$ by the Grunsky Finsler structure satisfies the inequality $\Delta \log \lambda \ge 4\lambda^2$, where Δ is again the generalized Laplacian (see [14]).

We shall consider a more general inequality

$$\Delta \log \lambda \ge K\lambda^2, \quad K = \text{const} > 0, \tag{8}$$

and use here somewhat different generalizations of curvature. Following [2], we say that a conformal metric $\lambda(t)|dt|$ in a domain G on $\mathbb{C}$ (or on a Riemann surface) has curvature less than or equal to K **in the supporting sense** if for each $K' > K$ and each t_0 with $\lambda(t_0) > 0$, there is a C^2-smooth supporting metric λ_0 for λ at t_0 (i.e., such that $\lambda_0(t_0) = \lambda(t_0)$ and $\lambda_0(t) \le \lambda(t)$ in a neighborhood of t_0) with $\kappa_{\lambda_0} \le K'$ (cf. [8]).

There are also integral generalizations of the inequality (8) (see, e.g. [27,28]). We shall use its generalization in the potential sense due to [28] and say that λ has curvature at most K **in the potential sense** at t_0 if there is a disk U about t_0 in which the function

$$\log \lambda + K \operatorname{Pot}_U(\lambda^2),$$

where Pot_U denotes the logarithmic potential

$$\operatorname{Pot}_U h = \frac{1}{2\pi} \int_U h(\zeta) \log|\zeta - z| d\xi d\eta \quad (\zeta = \xi + i\eta),$$

is subharmonic. One can replace U by any open subset $V \subset U$, because the function $\operatorname{Pot}_U(\lambda^2) - \operatorname{Pot}_V(\lambda^2)$ is harmonic on U. Note that *having curvature at most K in the potential sense is equivalent to λ satisfy (8) in the sense of distributions*.

The following three lemmas are proven in [28].

Lemma 1. *If a conformal metric has curvature at most K in the supporting sense, then it has curvature at most K in the potential sense.*

Lemma 2. *Let $\lambda|dz|$ be a conformal metric on the unit disk which has curvature at most -4 in the potential sense. Then the metric $\widetilde{\lambda} = e^{\mathcal{M}u}$, where $u = \log \lambda$ and $\mathcal{M}u$ is the circular mean*

$$\mathcal{M}u(r) = \frac{1}{2\pi} \int\limits_0^{2\pi} u(re^{i\theta}) d\theta,$$

also has curvature at most -4 in the potential sense.

Lemma 3. *If a circularly symmetric conformal metric $\lambda(|t|)|dt|$ in the unit disk has curvature at most -4 in the potential sense and $\lambda(0) = a > 0$, then*

$$\lambda(r) \geq \lambda_a(r) = \frac{a}{1 - a^2 r^2}. \tag{9}$$

The right hand-side of (9) defines a supporting conformal metric for λ at the origin with constant Gaussian curvature -4 on the whole disk $\mathbb{D}$ (actually, λ_a is the hyperbolic metric of the broader disk $\mathbb{D}_{1/a} = \{|z| < 1/a\}$).

Note also that any circularly symmetric subharmonic function $u(r)$ in a disk $\mathbb{D}_s = \{|t| < s\}$ has one-sided derivatives for each $r < 1$ and $ru'(r)$ is monotone increasing, $u'(0)$ exists and is nonnegative, and $u(r)$ is convex with respect to $\log r$.

The strengthened version of Lemma 2.3 for singular metrics with prescribed singularities given in [15] states:

Lemma 4. *Let $\lambda(|t|)d|t|$ be a circularly symmetric subharmonic metric on $\mathbb{D}$ such that*

$$\lambda(r) = mcr^{m-1} + O(r^m) \quad \textit{as} \;\; r \to 0 \;\; \textit{with} \;\; 0 < c \leq 1 \quad (m = 1, 2, \dots) \tag{10}$$

and this metric has curvature at most -4 in the potential sense. Then

$$\lambda(r) \geq \frac{mcr^{m-1}}{1 - c^2 r^{2m}}. \tag{11}$$

We shall apply Lemmas 3 and 4 to metric $\lambda_\varkappa$ generated on $\mathbb{D}$ by the Grunsky Finsler structure. This metric also is circularly symmetric, because the class Σ_Q contains every its function $f(z)$ with rotations $e^{-i\theta} f(e^{i\theta} z)$, $-\pi \leq \theta \leq \pi$.

5. Proof of Theorem 1

First observe that the Grunsky coefficients $\alpha_{mn}(f^\mu)$ of functions $f^\mu \in \Sigma_Q$ generate for each $\mathbf{x} = (x_n) \in l^2$ with $\|\mathbf{x}\| = 1$ the holomorphic maps

$$h_{\mathbf{x}}(f^\mu) = \sum_{m,n=1}^{\infty} \alpha_{mn}(f^\mu) x_m x_n : \ \mathrm{Belt}(D)_1 \to \mathbb{D}, \tag{12}$$

and

$$\sup_{\mathbf{x}} |h_{\mathbf{x}}(f^\mu)| = \varkappa_{D^*}(f^\mu). \tag{13}$$

The holomorphy of these functions follows from the holomorphy of coefficients α_{mn} with respect to Beltrami coefficients $\mu \in \mathrm{Belt}(D)_1$ mentioned above, by applying the estimate

$$\Big| \sum_{m=j}^{M} \sum_{n=l}^{N} \beta_{mn} x_m x_n \Big|^2 \le \sum_{m=j}^{M} |x_m|^2 \sum_{n=l}^{N} |x_n|^2 \tag{14}$$

which holds for any finite M, N and $1 \le j \le M, \ 1 \le l \le N$ (see [26, p. 61], [24, p. 193]).

Similar arguments imply that the maps (12) regarded as functions of points $\varphi^\mu = S_{f^\mu}$ in the universal Teichmüller space $\mathbf{T}$ are holomorphic on $\mathbf{T}$. This holomorphy provides, together with the equality (13), that the Grunsky norm $\varkappa_{D^*}$ regarded as a function of the Schwarzians S_f is logarithmically plurisubharmonic on the space $\mathbf{T}$. In addition, as it is established in [16], the functions $\varkappa_{D^*}(S_f)$ is continuous; moreover, it is Lipschitz continuous on this space. The Teichmüller norm has the similar properties.

We calculate the differentials of maps $h_{\mathbf{x}}(S_{f^\mu})$ at the origin using the well-known variational formula for $f \in \Sigma_Q$ with extensions to $\mathbb{D}$ satisfying $f^\mu(0) = 0$:

$$f^\mu(z) = z - \frac{1}{\pi} \iint_{\mathbb{D}} \mu(w) \left(\frac{1}{w - z} - \frac{1}{w} \right) du dv + O(\|\mu^2\|_\infty), \quad w = u + iv, \tag{15}$$

where the ratio $O(\|\mu^2\|_\infty^2)/\|\mu^2\|_\infty^2$ is uniformly bounded on compact sets of $\mathbb{C}$ (see, e.g., [10]). Comparing the right-hand side of (15) with the expansion $f^\mu(z) = z + b_0 + b_1 z^{-1} + \dots$, one derives that the coefficients b_j are given by

$$b_n = \frac{1}{\pi} \iint_{\mathbb{D}} \mu(w) w^{n-1} du dv + O(\|\mu^2\|_\infty), \quad n = 1, 2, \dots,$$

and from (1),

$$\alpha_{mn}(\mu) = -\pi^{-1} \iint_{\mathbb{D}} \mu(z) z^{m+n-2} dxdy + O(\|\mu\|_\infty^2), \quad \|\mu\|_\infty \to 0.$$

Hence, the differential at zero of the corresponding map $h_{\mathbf{x}}(t\mu_0/\|\mu_0\|_\infty)$ with $\mathbf{x} = (x_n) \in S(l^2)$ is given by

$$dh_{\mathbf{x}}(0)(\mu_0/\|\mu_0\|_\infty) = -\frac{1}{\pi} \iint_{\mathbb{D}} \mu^*(z) \sum_{m+n=2}^{\infty} \sqrt{mn}\, x_m x_n z^{m+n-2} dxdy. \tag{16}$$

For simplicity of notation, we have preserved above for the lifts of functions (12) to $\mathrm{Belt}(\mathbb{D})_1$ and $\mathbf{T}$ the same symbols.

Using these maps, we pull back the hyperbolic metric $\lambda_{\mathbb{D}}(t)|dt| = |dt|/(1-|t|^2)$ of the disk $\mathbb{D}$ onto this disk , getting conformal metrics $\lambda_{h_{\mathbf{x}}}(t)|dt|$ on $\mathbb{D}$ with

$$\lambda_{h_{\mathbf{x}}}(t) = |h'_{\mathbf{x}}(t)|/(1 - |h_{\mathbf{x}}(t)|^2) \tag{17}$$

of Gaussian curvature -4 at noncrical points. We take the upper envelope of these metrics

$$\widetilde{\lambda}_{\varkappa}(t) = \sup\{\lambda_{h_{\mathbf{x}}}(t) : \ \mathbf{x} \in S(l^2)\}$$

and its upper semicontinuous regularization $\lambda_{\varkappa}(t) = \limsup_{t' \to t} \widetilde{\lambda}_{\varkappa}(t')$, which implies a logarithmically subharmonic metric on $\mathbb{D}$.

On a standard way (see, e.g. [14]), one obtains that $\lambda_{\varkappa}$ has at any its noncritical point t_0 a supporting subharmonic metric λ_0 of Gaussian curvature at most -4, and hence, $\kappa_{\lambda_{\varkappa}} \leq -4$.

On the other hand, (16) yields that if the function $f \in \Sigma_Q$ has in $\mathbb{D}^*$ the expansion

$$f(z) = z + b_m z^m + b_{m+1} z^{-m-1} + \dots, \quad b_m \neq 0, \quad m \geq 1, \tag{18}$$

then each of the corresponding metrics and their envelope $\lambda_{\varkappa}(r)$ have near $r = 0$ the form (10), which implies

$$\lim_{r \to 0} \frac{\lambda_{\varkappa}(r)}{r^m} = \sup_{\mathbf{x} \in S(l^2)} \frac{|dh_{\mathbf{x}}(0)(t\mu_0/\|\mu_0\|_\infty|)}{|t|}$$

$$\geq \sup_{\psi \in A_1^2(\mathbb{D}), \|\psi\|_{A_1} = 1} \Big| \iint_{\mathbb{D}} \frac{\mu_0(z)}{\|\mu_0\|_\infty} \psi(z) dxdy \Big|. \tag{19}$$

Now we apply the following reconstruction lemma for Grunsky norm proven in [13], which provides that this norm is the integrated form of $\lambda_{\varkappa}$ along the extremal Teichmüller disks.

Lemma 5. *On any extremal Teichmüller disk* $\mathbb{D}(\mu_0) = \{\phi_{\mathbf{T}}(t\mu_0/\|\mu_0\|_\infty)\} \subset \mathbf{T}$, *we have the equality*

$$\tanh^{-1}[\varkappa(f^{r\mu_0/\|\mu_0\|_\infty})] = \int\limits_0^r \lambda_{\varkappa}(t)dt.$$

Together with Lemmas 3 and 4 and with the equality (19), this implies the desired inequality (6), completing the proof of the theorem.

6. Additional remarks

1. Geometric picture. It follows from above that the value of Grunsky norm $\varkappa(f^\mu)$ on Σ_Q is located in the interval

$$\tanh^{-1}\varrho(1/\alpha(f)) \le t \le \tanh^{-1}\varrho(1),$$

where $\varrho(r)$ is the hyperbolic distance between 0 and $\|\mu_0\|_\infty$ on the disk $\{|z| < r\}$, taking the extremal μ_0 in the class of μ (i.e., among quasiconformal extensions of $f^\mu|\mathbb{D}^*$ onto $\mathbb{D}$), and $\alpha(f)$ is given by (6).

Generically, the Finsler metric $\lambda_{\varkappa}$ generated by $\varkappa(f)$ on the space $\mathbf{T}$ has the holomorphic curvature less or equal to -4, while the Teichmüller norm is determined by hyperbolic metric of curvature -4.

2. Small dilatations. Generically Theorem 1 provides a rough lower bound $a(f^{t\mu_0})$ for $\varkappa(f^{t\mu_0})$. In the case of small dilatations $\|\mu\|_\infty$, the situation is somewhat different.

Using the variation (14) and holomorphic functions (11), one can prove the following result established in [13] for $f \in \Sigma_Q$ whose Schwarzians belong to sufficiently small neighborhood of the origin of the space $\mathbf{T}$.

Proposition 1. *For* $f \in \Sigma(\mathbb{D}^*)$ *with sufficiently small norm* $\|S_f\|_{\mathbf{B}}$ *of its Schwarzian,*

$$\varkappa(f) = \alpha(f) + O(\|S_f\|^2_{\mathbf{B}}), \tag{20}$$

where the ratio $O(\|S_f\|^2_{\mathbf{B}})/\|S_f\|^2_{\mathbf{B}}$ *remains bounded as* $\|S_f\|_{\mathbf{B}} \to 0$.

Theorem 1 yields that the remainder term in (20) is nonnegative.

3. Grunsky norm of homotopy functions. Consider the complex homotopy

$$f_t(z) = tf(z/t) = z + b_0 t + b_1 t^2 z^{-1} + b_2 t^3 z^{-2} + \cdots :\ \mathbb{D}^* \times \mathbb{D} \to \widehat{\mathbb{C}},$$

of functions $f \in \Sigma_Q$; it plays an important role in many applications.This homotopy has a deep connection with Teichmüller and Grunsky norms.

In particular, due to Kühnau's result [21], for any $f \in \Sigma_Q$ with $b_1 \neq 0$, there exists a sufficiently small $r_o(f)$ such that the extensions of functions f_t with $|t| < r_0(f)$ are defined by nonvanishing holomorphic quadratic differentials in $\mathbb{D}$, and therefore, for such t the equality

$$\varkappa(f_t) = k(f_t).$$

is valid (which yields the equality in (6) for such f_t); generically, $r_0(f) < 1$.

A point is that $f_t(z) = J_t(z) + h(z,t)$ with $J_t(z) = z + b_1 t^2/z$; hence, the function J_t has the affine extension $\widehat{J_t}(z) = z + b_1 t^2 \overline{z}$ with constant dilatation.

It is not hard to establish, using the variation (14) and the properties of bounded functions with sup-norm depending holomorphically on complex parameters, that Teichmüller norms of f_t and J_t differ on a quantity of order t^3 as $t \to 0$ (the proof is given in [18]).

4. Underlying features of (19). For any $f \in \Sigma_Q$ with the expansion (18) the corresponding holomorphic map $\chi_f : \mathbb{D} \to \mathbf{T}$ generated by $t \mapsto S_{f_t}(z)$ has near $t = 0$ the growth $\chi_f(t) = O(t^{m+1})$ in $\mathbf{B}$-norm; hence there exists its quasiconformal extension f^μ onto $\mathbb{D}$ with dilatation $k(f) = \|\mu\|_\infty = O(t^{m+1})$. In representation (16), such μ is orthogonal to $\psi_l(z) = z^l,\ l = 0, 1, \dots, m-1$.

However, under the additional restriction at $z = 0$, the corresponding extremal Beltrami coefficient μ_0 satisfies $\|\mu_0\|_\infty = O(t)$ as $t \to 0$. This is equivalent to applying Teichmüller's Verschiebungssatz in domain $\widehat{\mathbb{C}} \setminus \overline{f^\mu(\mathbb{D}^*)}$ (see [13, 33]).

Note also that $m = 1$ for a dense subset of $f \in \Sigma_Q$.

7. Open question. *Find the functions $f \in \Sigma_Q$ different from the known case $\varkappa(f) = k(f)$, on which the equality in (6) can be realized, i.e., with $\varkappa(f) = \alpha(f)$.*

References

[1] Abate, M., Patrizio, G. (1998). Isometries of the Teichmüller metric. *Ann. Scuola Super. Pisa Cl. Sci., 26*(4), 437–452.

[2] Ahlfors, L. (1938). An extension of Schwarz's lemma. *Trans. Amer. Math. Soc., 43*, 359–364.

[3] Dineen, S. (1989). *The Schwarz Lemma.* Clarendon Press, Oxford.

[4] Earle, C.J., Kra, I., Krushkal, S.L. (1994). Holomorphic motions and Teichmüller spaces. *Trans. Amer. Math. Soc., 944*, 927–948.

[5] Gardiner, F.P., Lakic, N. (2000). *Quasiconformal Teichmüller Theory.* Amer. Math. Soc., Providence, RI.

[6] Goluzin, G. M. (1969). *Geometric Theory of Functions of Complex Variables.* Transl. of Math. Monographs, vol. 26, Amer. Math. Soc., Providence, RI.

[7] Grunsky, H. (1939). Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen. *Math. Z., 45*, 29–61.

[8] Heins, M. (1962). A class of conformal metrics. *Nagoya Math. J., 21*, 1–60.

[9] Kra, I. (1981). The Carathéodory metric on abelian Teichmüller disks. *J. Anal. Math., 40*, 129–143.

[10] Krushkal, S. L. (1979). *Quasiconformal Mappings and Riemann Surfaces.* Wiley, New York.

[11] Krushkal, S. L. (1989). Grunsky coefficient inequalities, Carathéodory metric and extremal quasiconformal mappings. *Comment. Math. Helv., 64*, 650–660.

[12] Krushkal, S. L. (2004). Plurisubharmonic features of the Teichmüller metric. *Publications de l'Institut Mathématique-Beograd, Nouvelle série, 75(89)*, 119–138.

[13] Krushkal, S.L. (2005). Quasiconformal extensions and reflections, Ch 11. In: *Handbook of Complex Analysis: Geometric Function Theory, Vol 2 (R. Kühnau, ed.), Elsevier Science, Amsterdam*, pp. 507–553.

[14] Krushkal, S. L. (2007). Strengthened Moser's conjecture, geometry of Grunsky coefficients and Fredholm eigenvalues. *Central European J. Math., 5(3)*, 551–580.

[15] Krushkal, S. L. (2010). Proof of the Zalcman conjecture for initial coefficients. *Georgian Math. J., 17*, 663–681; (2012). *Corrigendum, 19*, 777.

[16] Krushkal, S.L. (2016). Strengthened Grunsky and Milin inequalities. *Contemp. Mathematics, 667*, 159–179.

[17] Krushkal, S. L., Kühnau, R. (1983). *Quasikonforme Abbildungen - neue Methoden und Anwendungen.* Teubner-Texte zur Math., Bd. 54, Teubner, Leipzig.

[18] Krushkal, S. L., Kühnau, R. (2011). Quasiconformal reflection coefficient of level lines. *Contemp. Math., 553*, 155–172.

[19] Kühnau, R. (1971). Verzerrungssätze und Koeffizientenbedingungen vom Grunskyschen Typ für quasikonforme Abbildungen. *Math. Nachr., 48*, 77–105.

[20] Kühnau, R. (1982). Quasikonforme Fortsetzbarkeit, Fredholmsche Eigenwerte und Grunskysche Koeffizientenbedingungen. *Ann. Acad. Sci. Fenn. Ser. AI. Math., 7*, 383–391.

[21] Kühnau, R. (1986). Wann sind die Grunskyschen Koeffizientenbedingungen hinreichend für Q-quasikonforme Fortsetzbarkeit? *Comment. Math. Helv., 61*, 290–307.

[22] Kühnau, R. (2000). Über die Grunskyschen Koeffizientenbedingungen. *Ann. Univ. Mariae Curie-Sklodowska, sect. A, 54*, 53–60.

[23] Lebedev, N. A. (1975). *The Area Principle in the Theory of Univalent Functions.* Nauka, Moscow [in Russian].

[24] Milin, I. M. (1977). *Univalent Functions and Orthonormal Systems*, Transl. of Mathematical Monographs, vol. 49. Transl. of *Odnolistnye funktcii i normirovannie systemy*, Amer. Math. Soc., Providence, RI.

[25] Minda, D. (1987). The strong form of Ahlfors' lemma. *Rocky Mountain J. Math., 17*, 457–461.

[26] Pommerenke, Chr. (1975). *Univalent Functions*. Vandenhoeck & Ruprecht, Göttingen.

[27] Reshetnyak, Yu. G. (1993). *Two-dimensional manifolds of bounded curvature*. Geometry, IV, Encyclopaedia Math. Sci. **70**, Springer, Berlin, pp. 3–163, 245–250. Engl. transl. from: *Two-dimensional manifolds of bounded curvature*, Geometry, 4, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989, pp. 189, 273–277, 279 [in Russian].

[28] Royden, H. L. (1986). The Ahlfors-Schwarz lemma: the case of equality. *J. Anal. Math., 46*, 261–270.

[29] Schiffer, M. (1981). Fredholm eigenvalues and Grunsky matrices. *Ann. Polon. Math., 39*, 149–164.

[30] Schiffer, M., Spencer, D. (1954). *Functionals of Finite Riemann Surfaces*. Princeton Univ. Press, Princeton.

[31] Strebel, K. (1976). On the existence of extremal Teichmueller mappings. *J. Analyse Math., 30*, 464–480.

[32] Zhuravlev, I. V. (1979). *Univalent functions and Teichmüller spaces*. Inst. of Mathematics, Novosibirsk, preprint, 23 pp. [in Russian].

[33] Teichmüller, O. (1944). Ein Verschiebungssatz der quasikonformen Abbildung. *Deutsche Math., 7*, 336–343.

Contact information

Samuel L. Krushkal
Department of Mathematics,
Bar-Ilan University, Israel,
Department of Mathematics,
University of Virginia,
Charlottesville, USA
E-Mail: `krushkal@math.biu.ac.il`