Український математичний вісник
Том 20 **(2023)**, № 2, 283 – 307
https://doi.org/10.37069/1810-3200-2023-20-2-8

On the best approximation of non-integer constants by polynomials with integer coefficients

Roald M. Trigub

Abstract. In this paper, exact rate of decrease of best approximations of non-integer numbers by polynomials with integer coefficients of growing degrees is found on a disk in the complex plane, on a cube in $\mathbb{R}^d$, and on a ball in $\mathbb{R}^d$. While in the first two cases the sup-norm is used, in the third one that in L_p, $1 \le p < \infty$, is applied.

Detailed comments are also given (two remarks in the end of the paper).

2020 MSC. Primary 41A10; Secondary 30E10, 41A17, 41A25.

Key words and phrases. Transfinite diameter, Chebyshev polynomial, extreme properties of polynomials, best approximation, integer algebraic numbers, q-adic fractions.

Introduction

At the first USSR mathematical congress (1930), S. N. Bernstein raised the question of the best approximation of a non-integer number by polynomials with integer coefficients of increasing degrees (see [1], v. I, p. 468–471 and 519). Shortly after, R. O. Kuz'min and L. V. Kantorovich indicated some estimate of the approximation from above not depending on the nature of a number [2]. The question was completely solved for the interval $[\delta, 1-\delta]$, $\delta \in \big(0, 1/2\big)$ in [3] (see also [4], **5.4.16**). Further, in [5], four cases of exact rates of best approximations of constants were indicated on the intervals $[\alpha, \beta]$, $0 < \alpha < \beta < 1$, depending on the arithmetic nature of both the number and interval. In the same paper, the problems of approximation of both smooth functions and constants by polynomials with positive integer coefficients (on a segment lying on the negative semi-axis of $\mathbb{R}$) were considered for the first time.

Received 09.05.2023

It is clear that if some function different from a polynomial admits uniform approximation by polynomials q_n with integer coefficients ($\mathbb{Z} + i\mathbb{Z}$) on a compactum $K \subset \mathbb{C}$, then there exists a polynomial X with integer coefficients such that

$$0 < \max_{z \in K} |X(z)| < 1.$$

Hence, the transfinite diameter of K is less than one (see, e. g., [6], **2.13**, **13**).

Here and in what follows p_n is a polynomial of degree not greater than n with arbitrary coefficients, while q_n is the same but with integer coefficients.

The Chebyshev polynomial $C_n(K)$ is defined by

$$\|C_n(K)\|_\infty = \min_{p_{n-1}} \max_{z \in K} |z^n + p_{n-1}(z)|.$$

Then the Chebyshev constant always coinciding with the transfinite diameter is equal to

$$d(K) = \lim_{n \to \infty} \|C_n(K)\|_\infty^{\frac{1}{n}}.$$

The existence of the limit follows from the well-known lemma: *if* $0 < x_{n+m} \le x_n \cdot x_m$ *for any* n *and* m*, then there exists the limit* $\lim x_n^{\frac{1}{n}}$. Note that for such sequences, the limit $\lim \frac{x_{n+1}}{x_n}$ may not exist.

For a disk, the Chebyshev constant is equal to the radius, and for an ellipse, it is equal to the half-sum of semi-axes.

If μ is the outer planar Lebesgue measure, then $\mu(K) \le \pi(d(K))^2$ (see [20], Ch. YII, §2]).

For an interval $[a, b] \subset \mathbb{R}$, the Chebyshev polynomial equals

$$C_n(x; a, b) = 2\Big(\frac{b-a}{4}\Big)^n \cdot T_n(x; a, b),$$

where

$$\begin{aligned} T_n(x; a, b) &= \cos n \arccos \frac{2x - a - b}{b - a} \\ &= \frac{1}{2}\Bigg\{ \Bigg(\frac{2x-a-b}{b-a} + \sqrt{\Big(\frac{2x-a-b}{b-a}\Big)^2 - 1} \Bigg)^n \\ &+ \Bigg(\frac{2x-a-b}{b-a} - \sqrt{\Big(\frac{2x-a-b}{b-a}\Big)^2 - 1} \Bigg)^n \Bigg\}. \end{aligned}$$

It has many extreme properties. Let us present one of them that will essentially be used below.

For any polynomial p_n of $x \in \mathbb{R} \setminus (a, b)$, we have

$$|p_n(x)| \le |T_n(x; a, b)| \max_{x \in [a,b]} |p_n(x)|. \tag{1}$$

Note also that if $d(K) < 1$, then there exists a polynomial X with integer coefficients and the leading coefficient 1 for which $\max |X| < 1$ as well (see [12], p. 272). Such a polynomial X is crucial in many problems. For example, if such a polynomial does exist, with the condition $0 < |X(z)| < 1$ for any $z \in K$, then every function that can be approximated by polynomials p_n with arbitrary coefficients admits approximation by polynomials q_n (it suffices to approximate the constant $\lambda = \frac{1}{2}$); see ibid. If there is at least one integer point (i.e., a point with integer coordinates) on the compactum K or, more generally, integer algebraic numbers together with their conjugates, then the function must satisfy certain arithmetic conditions. For example, a real continuous function on $[-1, 1]$ is the limit of polynomials q_n as $n \to \infty$ if and only if $f(0)$ and $\frac{1}{2}\big(f(-1) \pm f(1)\big) \in \mathbb{Z}$ (see, e. g., [7], Ch. 2, §4).

For the exact rate of best approximation in the uniform metrics of individual constants on an interval of the real axis, see [23].

In this paper, we study the best approximations of constants in the cases where the compactum K is a disk in $\mathbb{C}$ (§1, Theorem 1), a cube in $\mathbb{R}^d$ (§2, Theorem 2), and a ball and a cube in $\mathbb{R}^d$ (§3, Theorem 3). In the case of a ball and a cube centered at zero, both the integral metrics and arithmetic conditions are dropped ($d = 1$, see [4], **5.4.16**). The exact rate of approximation is established to an individual constant in Theorem 3 and on the class in Theorems 1 and 2 (the latter means that there exist both a constant and a compactum of the indicated form for which this rate of approximation is exact).

The problem of the rate of decrease of the difference between the best approximations of a continuous function by polynomials with arbitrary coefficients and and that with only integer ones on a compactum without integer points is also considered (see the second part of Theorem 1).

By c we denote absolute positive constants, and by $c(\alpha, P)$ some positive values depending only on α and P.

In §4 detailed comments are given in Remarks 1 and 2, in which certain supplements, historical information and open problems can be found.

1. Approximation of constants on a disk in $\mathbb{C}$

Let $\lambda \in \mathbb{C}$ be non-integer, that is, $\lambda \notin \mathbb{Z} + i\mathbb{Z}$. Suppose that in the disk

$$K_r = K_r(z_0) = \{z \in \mathbb{C} : \ |z - z_0| \le r\}$$

there are no integer points, which implies $0 < r < \dfrac{1}{\sqrt{2}}$. Shifting by an integer, we may assume that $z_0 \in \Pi_{0,1}$, where $\Pi_{0,1}$ is the closed square with the vertices at 0, 1, $1+i$, i. Suppose also that $\Pi_{0,\frac{1}{2}} = \frac{1}{2}\Pi_{0,1}$. By linear transformation $w = w(z)$ with integer coefficients we can map the square $\Pi_{0,\frac{1}{2}} + \frac{1+i}{2}$ (the algebraic sum) into $\Pi_{0,\frac{1}{2}}$ $(w = 1+i-z)$, the square $\Pi_{0,\frac{1}{2}} + \frac{1}{2}-$ into $\overline{\Pi_{0,\frac{1}{2}}}$ (the complex conjugation, $w = 1 - z$), and $\Pi_{0,\frac{1}{2}} + \frac{i}{2}$ into $\overline{\Pi_{0,\frac{1}{2}}}$ $(w = z - i)$. Since the inverse transformation is also with integer coefficients, without loss of generality, let us assume that the center z_0 is contained in $\Pi_{0,\frac{1}{2}} \cup \overline{\Pi_{0,\frac{1}{2}}}$.

Theorem 1. *Under the assumptions made, for any $n \in \mathbb{N}$, we have*

$$E_n^e(\lambda; K_r) = \min_{q_n} \max_{K_r} |\lambda - q_n(z)| \le (n+1)\rho^n,$$

where $\rho = \max\{\rho_1, \rho_2\}$ and $\rho_1 = \dfrac{r}{|z_0|}$, $\rho_2 = |z_0| + r$.

In general, ρ cannot be lessened.

Moreover, if a function f is analytic in the closed disk $K_R(z_0)$ of radius $R > r$, with the same center z_0, then there holds the equality

$$E_n^e(f; K_r) = \min_{q_n} \max_{K_r} \bigl|f(z) - q_n(z)\bigr| = O\Bigl(\Bigl(\frac{r}{R}\Bigr)^n + (n+1)\rho^n\Bigr).$$

Proof. Since

$$\Bigl(\frac{z - z_0}{-z_0}\Bigr)^m = 1 - \sum_{k=1}^{m} a_k z^k,$$

we have

$$\max_{K_r} \Bigl|\lambda - \sum_{k=1}^{n} \lambda a_k z^k\Bigr| \le |\lambda| \Bigl(\frac{r}{|z_0|}\Bigr)^n = |\lambda| \rho_1^n.$$

Replacing λa_1 with the nearest integer c_1 and applying the same inequality with n replaced by $n-1$ to their difference ("the fractional part"), we obtain

$$\max_{K_r} \Bigl|\lambda - c_1 z - \sum_{k=2}^{n} b_k z^k\Bigr| \le |\lambda|\rho_1^n + \Bigl(\frac{r}{|z_0|}\Bigr)^{n-1} \max_{K_r} |z| = |\lambda|\rho_1^n + \rho_1^{n-1}\rho_2.$$

We assume $|\lambda| \le 1$, without loss of generality. Further, we single out the integral part of the coefficient b_1, and so on. Continuing in this manner until all non-integer coefficients are corrected, we arrive at a polynomial q_n such that

$$\max_{K_r} |\lambda - q_n(z)| \le \sum_{k=0}^{n} \rho_1^{n-k} \rho_2^k \le (n+1) \max\left\{\rho_1^n, \rho_2^n\right\} = (n+1)\rho^n,$$

as desired.

Clearly, for $\rho_1 \ne \rho_2$, the factor $(n+1)$ can be replaced by a value bounded in n.

Now, to estimate the approximation from below, we use the following inequality: for any polynomial p_n with $|z - z_0| > r$, there holds (see, e. g., [4]4.7.1]):

$$|p_n(z)| \le \left(\frac{|z - z_0|}{r}\right)^n \max_{K_r} |p_n(z)|.$$

For some polynomial q_n, we have

$$E_n^e(\lambda; K_r) = \max_{K_r} |\lambda - q_n(z)| \ge |\lambda - q_n(0)| \Big(\frac{r}{|z_0|}\Big)^n \ge c(\lambda)\rho_1^n.$$

This estimate of approximation from below is obtained for $\rho_1 \ge \rho_2$.

Let now $\rho_2 > \rho_1$, i. e., $r < \dfrac{|z_0|^2}{1 - |z_0|}$. We choose a number and a disk for which the same estimate of the approximation from below hold true. Put $\lambda = \frac{1}{q+1}$, where $q \in \mathbb{N}$ and $q \ge 2, z_0 = x_0 = \frac{1}{q} - \frac{1}{(q+1)^2}$, $r = \frac{1}{(q+1)^2}$. Then

$$\begin{aligned} E_n^e(\lambda; K_r) &\ge \min_{q_n} \left| \frac{1}{q+1} - q_n(\frac{1}{q}) \right| \ge \min_{s \in \mathbb{Z}} \left| \frac{1}{q+1} - \frac{s}{q^n} \right| \\ &= \frac{1}{(q+1)q^n} \min_{s \in \mathbb{Z}} |q^n - s(q+1)|, \end{aligned}$$

and this is not less than $\frac{1}{(q+1)q^n} = \frac{1}{q+1}(x_0 + r)^n = \frac{1}{q+1}\rho_2^n$. Thus, there exist a number λ and a disk K_r such that ρ_2 cannot be lessened too. This means that the inequality is exact in the general case, i. e., on the class. $\square$

It is important that we can pass from approximation of constants to approximation of arbitrary functions that admit approximation by polynomials p_n.

Let us proceed to the proof of the second part of Theorem 1.

Suppose that f is an analytic function in the closed disk $K_R(z_0)$ with the same center z_0 for some radius $R > r$ and that not all Taylor's coefficients at z_0 are integer. The point is that, as follows from the Cauchy–Hadamard formula, if $f(z) = \sum_{k=0}^{\infty} c_k z^k$, where all c_k are integer, then the radius of convergence of the series is not greater than one if the function is not equal to a polynomial, and it is possible to use partial sums of the series for the approximation.

Due to the Bernstein theorem (see, e. g., [4, 4.7.2]), there exists a sequence $\{p_n\}$ such that for $0 \le \nu \le s$ (s is the smallest number of a non-integer Taylor's coefficient) we get

$$\max_{K_r} \left| f^{(\nu)}(z) - p_n^{(\nu)}(z) \right| \le c(r, R, s) \left(\frac{r}{R} \right)^n.$$

This is the rate of best approximation. So, we can additionally suppose that $p_n^{(\nu)}(z_0) = f^{(\nu)}(z_0)$ for $(0 \le \nu \le s)$.

Without loss of generality, we will also assume that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \qquad \sum_{k=0}^{\infty} |a_k| \cdot R^k < \infty.$$

Then

$$\max_{K_r} \Big| \sum_{k=n+1}^{\infty} a_k (z - z_0)^k \Big| \le \sum_{k=n+1}^{\infty} |a_k| \cdot R^k \left(\frac{r}{R} \right)^k \le \left(\frac{r}{R} \right)^{n+1} \sum_{k=n+1}^{\infty} |a_k| \cdot R^k.$$

It is crucial now that p_n must be approximated by a polynomial q_n. We have

$$a_s (z - z_0)^s = a_s (-z_0)^s \left(\frac{z - z_0}{-z_0} \right)^s = a_s (-z_0)^s \Big(1 - \sum_{k=1}^{s} b_k z^k \Big).$$

As above, we replace $a_s(-z_0)^s$ by the nearest integer, and further increasing the degree of z, we repeat the proof of the first part of the theorem. Theorem 1 is proved. $\square$

Similarly, we can consider the approximation on the ellipse containing no integer points (with one of its axes less than 4 and the diameter less than 1).

2. Approximation of constants on a cube in $\mathbb{R}^d$

Suppose that $x = (x_1, ..., x_d) \in \mathbb{R}^d$, $x^k = \prod_{j=1}^{d} x_j^{k_j}$ for $k_j \in \mathbb{Z}_+$, $|k| = \sum_{j=1}^{d} k_j$, and a polynomial $p_n(x) = \sum_{0 \le k_j \le n_j} a_k x^k$ has the degree $n = \sum_{j=1}^{d} n_j$. Moreover, let

$$K = \Pi_{a,b} = \big\{x \in \mathbb{R}^d : \ x_j \in [a, b], \ 1 \le j \le d\big\} = [a, b]^d.$$

Since the cube must not contain integer points, we may assume, without loss of generality, that $0 < a < b \le 1 - a$ and $O = (0, ..., 0)$ is the integer point nearest to $\Pi_{a,b}$.

For $\lambda \in (0, 1)$, we put

$$E_n^e(\lambda; \Pi_{a,b}) = \min_{q_n} \max_{x \in \Pi_{a,b}} |\lambda - q_n(x)|.$$

Theorem 2. *Under the above assumptions, for any $n \in \mathbb{N}$, we have*

$$E_n^e(\lambda; \Pi_{a,b}) \le c(d) n^d \rho^n, \qquad \rho = \max\left\{\frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}}, b\right\},$$

and ρ cannot be taken smaller in general.

Proof. As is follows from (1) and the previous formula for T_n, if $0 < a$, we have

$$|T_n(0; a, b)| = \theta \left(\frac{\sqrt{b} + \sqrt{a}}{\sqrt{b} - \sqrt{a}}\right)^n, \quad \theta \in \left(0, \frac{1}{2}\right].$$

But

$$T_n(t; a, b) = T_n(0; a, b) - T_n(0; a, b) \sum_{k=1}^{n} a_k t^k,$$

and for $t \in [a, b]$, there holds

$$\Big|1 - \sum_{k=1}^{n} a_k t^k\Big| \le \Big|\frac{T_n(t; a, b)}{T_n(0; a, b)}\Big| \le \frac{1}{|T_n(0; a, b)|}.$$

Multiplying such inequalities with $t = x_j$ and n_j $(1 \le j \le d)$, for $x \in \Pi_{a,b}$ and $n = \sum n_j$, we obtain

$$\Big|1 - \sum_{1 \le |k| \le n} b_k x^k\Big| \le \prod_{j=1}^{d} \frac{1}{|T_{n_j}(0; a, b)|} \le 2^d \left(\frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}}\right)^n. \qquad (2)$$

Let us multiply this inequality by $\lambda \in (0,1)$ and take into account that for $\delta_k \in [0,1)$, the equality $\lambda b_k = [\lambda b_k] + \delta_k$ holds. We need to approximate monomials δx^s for $\delta \in (0,1)$ and $s_j \in [0, n_j], 1 \le j \le d$, by polynomials q_n. To this end, we multiply inequality (2) by δx^s, replacing n by $n-s$. This yields

$$\Big|\delta x^s - \sum_{s+1\le|k|\le n} \delta b_{k-s} x^k\Big| = \Big|\delta x^s - \sum_{1\le k\le n-s} \delta b_k x^{k+s}\Big| \le x^s \cdot 2^d \left(\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}}\right)^{n-s}.$$

Further, we select the integral part $[\delta b_{k-s}]$ and apply the same inequality to the fractional part again, and so on up to $s = n$. We arrive at the following inequality: for $x \in \Pi_{a,b}$, there holds

$$\begin{aligned}|\lambda - q_n(x)| &\le 2^d \sum_{s=0}^{n} x^s \left(\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}}\right)^{n-s} \le 2^d \sum_{s=0}^{n} b^s \left(\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}}\right)^{n-s} \\ &\le 2^d \rho^n \mathrm{card}\big\{x^k : \ |k| \le n\big\}.\end{aligned}$$

But, as is known (see, e. g., [9], Ch. IV, it. 2),

$$\mathrm{card}\big\{x^k : \ |k| = s\big\} = \binom{d+s-1}{d-1} = \frac{s(s+1)...(s+d-1)}{(d-1)!},$$

wherefrom

$$\begin{aligned}\mathrm{card}\big\{x^k : \ |k| \le n\big\} &= \sum_{s=0}^{n} \mathrm{card}\big\{x^k : \ |k| = s\big\} \le (n+1)\binom{d+n-1}{d-1} \\ &\le (n+1)(n+d-1)^{d-1} \le c_1(d) n^d.\end{aligned}$$

The estimate for the approximation from above is proved.

To estimate the approximation from below, we use the following well-known inequality ([1], v. II, pp. 434-436):

$$|p_n(0)| \le \max_{\Pi_{a,b}} |p_n(x)| \prod_{j=1}^{d} |T_{n_j}(0; a, b)| \le \max_{\Pi_{a,b}} |p_n(x)| \left(\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}\right)^n.$$

Substituting the difference $\lambda - q_n$, for p_n, we obtain

$$E_n^e(\lambda; \Pi_{a,b}) \ge \min_{c\in\mathbb{Z}} |\lambda - c| \left(\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}}\right)^n.$$

Hence, for $a \le b\left(\frac{1-b}{1+b}\right)^2$ or, which is the same, for $b \le \frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}}$, we cannot take smaller ρ.

If $a > b\left(\frac{1-b}{1+b}\right)^2$ and $b = \frac{1}{q}$ $(q \in \mathbb{N}, q \geq 2)$, then, for example, for $d = 2$, we have

$$E_n^e(\lambda; \Pi_{a,b}) \geq \min_{q_n} \max_{x_1 \in [a, \frac{1}{q}]} \left|\lambda - q_n\left(x_1, \frac{1}{q}\right)\right|.$$

Choose $x_1 = \frac{1}{q}$ and take into account that $q_n(\frac{1}{q}, \frac{1}{q}) = \sum\limits_{k=0}^{n} c_k \frac{1}{q^k}$. But for $\lambda = \frac{1}{q+1}$ (see the end of the proof of the first part of Theorem 1), we obtain

$$\left|\frac{1}{q+1} - \sum_{k=0}^{n} c_k \frac{1}{q^k}\right| \geq \frac{1}{q+1} \cdot \frac{1}{q^n} = \frac{b^n}{q+1}.$$

Theorem 2 is proved. □

Similarly (see the proof of the 2nd part of Th. 1) we can consider approximations of functions by polynomials with integer coefficients on some parallelepipeds without integer points.

Note that in the case of a cube $\Pi_{a,b}$, $0 < a < b$, of any size, the above argument yields exact estimates of the approximation of constants and functions by polynomials of the form

$$\sum_{|k|=0}^{n} \frac{m_k}{q^{|k|}} x^k \quad (m_k \in \mathbb{Z}, q > b).$$

Using the similarity transformation, it suffices to pass to approximation on $\Pi_{\alpha,\beta}$, $0 < \alpha < \beta < 1$, by polynomials with integer coefficients, and then return to $\Pi_{a,b}(0 < a < b)$. Here the coefficients of polynomials are rational numbers with known denominators.

3. Integral approximations

Let

$$K_r = \left\{x = (x_1, ..., x_d) : \ |x| \leq r\right\}$$

be a Euclidean ball of radius r centered at the origin, and let

$$\Pi_{-r,r} = [-r, r]^d = \left\{x = (x_1, ..., x_d) : \ |x_j| \leq r, 1 \leq d\right\}$$

be a cube.

Theorem 3. *Let $\lambda \in \mathbb{R} \setminus \mathbb{Z}, r \in (0, 2), p \in [1, +\infty)$, and let the degrees of polynomials in all variables n tend to infinity. We have*

$$E_n^e(\lambda; K_r)_p = \min_{q_n} \left(\int\limits_{K_r} |\lambda - q_n(x)|^p dx\right)^{\frac{1}{p}} \asymp n^{-\frac{d}{p}}$$

and

$$E_n^e(\lambda;\Pi_{-r,r})_p = \min_{q_n}\left(\int\limits_{\Pi_{-r,+r}} |\lambda - q_n(x)|^p dx\right)^{\frac{1}{p}} \asymp n^{-\frac{d}{p}},$$

and for $r \in (0,1]$, *we have*

$$E_n^e(\lambda;\Pi_{0,r})_p \asymp n^{-\frac{2d}{p}},$$

(two-sided inequalities with positive constants not depending on n*).*

For $r \geq 2$, *we have*

$$E_n^e(\lambda;\Pi_{-r,r})_p \geq E_n^e(\lambda;K_r)_p \geq c(\lambda,r,p) > 0.$$

Proof. To estimate the approximation from below, we need the following result.

Lemma 1. *For* $\alpha > -1$, $p \in [1,+\infty), r > 0$ *and any polynomial* $\{p_n\}$, *we have*

$$\int\limits_0^r t^\alpha |p_n(t)|^p dt \geq c(\alpha,p) r^{1+\alpha}|p_n(0)|^p \frac{1}{n^{2p+2\alpha p}},$$

and for even polynomials p_n, *we have*

$$\int\limits_0^r t^\alpha |p_n(t)|^p dt \geq c(\alpha,p) r^{1+\alpha}|p_n(0)|^p \frac{1}{n^{p+\alpha p}}.$$

Proof. Proving the first inequality for $r = 1$ will imply the general case.

If $p_n(t) = \sum\limits_{k=0}^n a_k t^k$, where $a_0 = p_n(0)$, then for $p = 1$ and $n \geq 2$, we obtain

$$\int\limits_0^1 t^\alpha |p_n(t)| dt \geq \max_{[0,1]}\Big|\int\limits_0^x t^\alpha p_n(t) dt\Big| =$$

$$= \max_{[0,1]} x^{\alpha+1}\Big|\sum_{k=0}^n a_k \frac{x^k}{k+\alpha+1}\Big| \geq \frac{1}{n^{2\alpha+2}} \max_{\left[\frac{1}{n^2},1\right]}\Big|\sum_{k=0}^n a_k \frac{x^k}{k+\alpha+1}\Big|.$$

In view of the extreme property of Chebyshev's polynomials C_n (the growth of the norm of a polynomial when the interval expands, see (1)), with the absolute constant $c > 0$, it is easy to check that

$$\max_{[0,1]} |p_n(x)| \leq c \max_{\left[\frac{1}{n^2},1\right]} |p_n(x)|. \tag{3}$$

Therefore,

$$\int_0^1 t^\alpha |p_n(t)|dt \geq \frac{1}{c}\cdot\frac{1}{n^{2+2\alpha}}\max_{[0,1]}\Big|\sum_{k=0}^n a_k\frac{x^k}{k+\alpha+1}\Big| \geq \frac{1}{c}\cdot\frac{|a_0|}{n^{2+2\alpha}}.$$

It remains to apply the Hölder inequality:

$$\int_0^1 t^\alpha |p_n(t)|dt \leq \Big(\frac{1}{\alpha+1}\Big)^{\frac{1}{p'}}\Big(\int_0^1 t^\alpha |p_n(t)|^p dt\Big)^{\frac{1}{p}}.$$

In the case of even polynomials, one obtains this after replacing t^2 by t with another α.

Lemma 1 is proved. □

We will now prove the estimate of approximation from below for λ on a ball and a cube, both centered at the origin.

By the symmetry of a ball with respect to the coordinate planes, for some $c_k \in \mathbb{Z}$, we have

$$(E^e_{2n}(\lambda; K_r)_p)^p = \int_{K_r} |\lambda - \sum_{|k|\leq n} c_k x^{2k}|^p dx.$$

Let us pass to the spherical or polar coordinates at $|x| = t \in (0, r]$. For example, if $d = 2$, where $x_1 = t\sin\varphi, x_2 = t\cos\varphi$, we get

$$\int_0^{2\pi} d\varphi \int_0^r t\Big|\lambda - \sum_{|k|\leq n} c_k t^{2|k|}\sin^{2k_1}\varphi\cos^{2k_2}\varphi\Big|^p dt.$$

Applying Lemma 1 for even polynomials, we derive that for any $r > 0$ this value is not less than

$$c(p)r^2|\lambda - c_0|\frac{1}{n^2}\int_0^{2\pi} d\varphi.$$

This holds for $d = 2$, while for $d \geq 3$ the Jacobian equals $t^{d-1}F(\varphi)$, where $F(\varphi) = \prod_{m=1}^{d-2}(\sin\varphi_m)^{d-2}$ and all $\varphi_m \in [0, \frac{\pi}{2}]$.

Consider the case $p = 1$ and then apply Hölder's inequality:

$$(E^e_{2n}(\lambda; K_r)_1 = \int_0^r t^{d-1}dt \int_{[0,\frac{\pi}{2}]^{d-2}} F(\varphi)\Big|\lambda - \sum_{|k|\leq n} c_k t^{2k}(F(\varphi))^{2k}\Big|d\varphi.$$

Since the absolute value of the integral of a function is not greater than the integral of the absolute value of the function, this expression is not less than

$$\int_0^r t^{d-1}dt\Big|\lambda\int_{[0,\frac{\pi}{2}]^{d-2}} F(\varphi)-\sum_{|k|\le n} c_k t^{2k}\int_{[0,\frac{\pi}{2}]^{d-2}}(F(\varphi))^{2k+1}d\varphi\Big|.$$

It remains to apply Lemma 1. The estimate of the approximation from below is proved for $r\in(0,2)$, since we always have $E_n^e(\lambda;\Pi_{-r,r})_p \ge E_n^e(\lambda;K_r)_p$.

Now we consider the case $r\ge 2$.

By the Korkin–Zolotaryov inequality (see, e. g., [6], 2.9.31 or [4], p. 223]), for $m\in\mathbb{Z}_+$, we have

$$\int_a^b t^m\Big|\sum_{k=0}^n a_k t^k\Big|dt \ge 4\Big(\frac{b-a}{4}\Big)^{n+m+1}|a_n|.$$

For $d\ge 2$ and $p\in[1,\infty)$, we choose $\beta\ge -\dfrac{1}{p'}$ such that

$$\frac{1}{p'}\Big(\frac{d}{2}-1\Big)+\beta=m\in\mathbb{Z}_+.$$

Then, due to Hölder's inequality, for any polynomial q_n, there holds

$$4\le\int_0^4 t^m|\lambda-q_n(t)|dt\le\Big(\int_0^4 t^{\frac{d}{2}-1}|\lambda-q_n(t)|^p dt\Big)^{\frac{1}{p}}\Big(\int_0^4 t^{\beta p'}dt\Big)^{\frac{1}{p'}}.$$

This yields the desired estimate from below. □

The case $d=1$ was considered in [4], p. 231].

Lemma 2. *If $\alpha>-1$, $p\in[1,+\infty)$ and $m\in\mathbb{Z}_+$, there exists a sequence of polynomials $\{p_n\}_m^\infty$, $p_n(0)=1$, such that for some constant $c(\alpha,p,m)$, there holds*

$$\int_0^1 t^\alpha|p_n(t)|^p dt\le c(\alpha,p,m)\cdot\frac{1}{n^{2p+2p\alpha}}$$

and

$$(p_n(t)-p_n(0))^{(\nu)}(0)=0\ (0\le\nu\le m),\quad \text{and}\ \max_{[0,1]}|p_n(t)|\le c(\alpha,p,m).$$

Proof. For $p=2$, the corresponding problem was solved long ago. We have (see [10], Ch. I, **14**)

$$\min_{\{a_k\}_m^n}\int\limits_0^1 t^\alpha\Big|1-\sum_{k=m}^n a_k t^k\Big|^2 dt=\min_{\{a_k\}_m^n}\int\limits_0^1\Big|t^{\frac{\alpha}{2}}-\sum_{k=m}^n a_k t^{k+\frac{\alpha}{2}}\Big|^2 dt$$
$$=\frac{1}{\alpha+1}\prod_{k=m}^n\Big(\frac{k}{k+\alpha+1}\Big)^2=\frac{1}{\alpha+1}\prod_{k=m}^n\Big(1+\frac{\alpha+1}{k}\Big)^{-2}.$$

But $\ln(1+x)=x+O(x^2)$ as $x\to 0$. Therefore,

$$\ln\prod_{k=m}^n\Big(1+\frac{\alpha+1}{k}\Big)^2\geq 2\Big(\sum_{k=m}^n\frac{\alpha+1}{k}+O(1)\Big)=(2\alpha+2)\ln\frac{n+1}{m}+O(1),$$

and the desired inequality is proved for $p=2$.

Let us deduce from it the boundedness of the extremal sequence $\{p_n\}$ in the space $C[0,1]$.

Taking into account that $t^\alpha\geq\dfrac{1}{n^{2\alpha}}$ at $t\in\left[\frac{1}{n^2},1\right]$ $(n\geq 2)$, we have

$$\max_{\left[\frac{1}{n^2},1\right]}\int\limits_{\frac{1}{n^2}}^x p_n^2(t)dt\leq c_1(\alpha,m)\cdot\frac{1}{n^2}.$$

Now, from Markov's inequality for the derivative of a polynomial ([6, 4.8(32)], [4, 5.4.6]) it follows that

$$\max_{\left[\frac{1}{n^2},1\right]} p_n^2(x)\leq c_1(\alpha,m)\frac{(2n+1)^2}{n^2}\cdot\frac{2}{1-\frac{1}{n^2}}\leq c_2(\alpha,m).$$

Further, we apply (3).

And this sequence $\{p_n\}$ (more precisely, $\{p_n^2\}$) may be used for any $p\in[1,+\infty)$ since all the conditions at $x=0$ are fulfilled.

Indeed, if a polynomial p_n meets assumptions of Lemma 2 at $p=2$ and $(p_n(x))^2\leq M$, then

$$\int\limits_0^1 t^\alpha(p_n(t))^{2p}dt\leq M^{p-1}\int\limits_0^1 t^\alpha(p_n(t))^2dt\leq M^{p-1}c(\alpha,2,m)\cdot\frac{1}{n^{2+2\alpha}},$$

which completes the proof. $\square$

Lemma 3. *Let $p \in [1, +\infty)$, $m \in \mathbb{Z}_+$ and $\{t_\nu\}_0^s \subset [-r, r]$, where $t_0 = 0$. Then there exists a sequence of polynomials p_n such that*

$$\|1 - p_n\|_{L_p[-r,r]} = O\Big(\frac{1}{n^{\frac{1}{p}}}\Big)$$

and

$$\sup_n \|p_n\|_{C[-r,r]} < \infty, \qquad p_n^{(k)}(t_\nu) = 0 \qquad (0 \le \nu \le s,\ 0 \le k \le m).$$

Proof. By Lemma 2, at $\alpha = s = 0$ there exists a sequence of even polynomials p_n such that

$$\|1 - p_n\|_{L_p[-2r,2r]} = O\left(\frac{1}{n^{\frac{1}{p}}}\right), \quad p_n^{(k)}(0) = 0 \quad (0 \le k \le m).$$

Then for $t_\nu \neq t_0$, we obtain

$$\|1 - p_n(\cdot \pm t_\nu\|_{L_p[-r,r]} \le \|1 - p_n\|_{L_p[-2r,2r]} = O\Big(\frac{1}{n^{\frac{1}{p}}}\Big),$$

and new polynomials $1 - p_n$ are bounded in the aggregate, and their derivatives equal to zero at $\pm t_\nu$ up to the order m. Both the equality

$$1 - fg = g(1 - f) + (1 - g)$$

and the inequality $\|g\|_{C[-r,r]} \le M$ imply that

$$\|1 - fg\|_{L_p[-r,r]} \le M(2r)^{\frac{1}{p}}\|1 - f\|_{L_p[-r,r]} + \|1 - g\|_{L_p[-r,r]}.$$

Thus, for $s \ge 1$, we have

$$\|1 - p_n(\cdot)p_n(\cdot \pm t_1)\|_{L_p[-r,r]} = O\Big(\frac{1}{n^{\frac{1}{p}}}\Big).$$

For $s \ge 2$, adding all other points one by one in the same way, we arrive at the inequality

$$\|1 - \prod_{\nu=0}^{s} p_n(\cdot \pm t_\nu)\|_{L_p[-r,r]} = O\Big(\frac{1}{n^{\frac{1}{p}}}\Big).$$

Lemma 3 is proved. $\square$

In what follows we consider $d = 2$.

Lemma 4. *If $\{t_\nu\}_0^s \subset [-r,r]$ and $m \in \mathbb{Z}_+$, then for any function $f \in C^2[-r,r]^2$, there exists a sequence of polynomials $\{p_n(x_1,x_2)\}$ such that*

$$\|f-p_n\|_{L_p[-r,r]^2} = O\Big(\frac{1}{n^{\frac{2}{p}}}\Big)$$

and

$$\frac{\partial^k p_n(x_1,x_2)}{\partial x_1^k}(t_\nu,t_\mu) = \frac{\partial^k p_n(x_1,x_2)}{\partial x_2^k}(t_\nu,t_\mu) = 0,$$

for $0 \le \nu,\mu \le s$ and $0 \le k \le m$.

Proof. If p_n is a polynomial from Lemma 3, then for

$$p_{2n}(x_1,x_2) = p_n(x_1) + p_n(x_2) - p_n(x_1)\cdot p_n(x_2),$$

we have

$$1 - p_{2n}(x_1,x_2) = (1-p_n(x_1))(1-p_n(x_2)),$$

and hence,

$$\|1-p_{2n}\|_{L_p[-r,r]^2} = \|1-p_n\|^2_{L_p[-r,r]} = O\Big(\frac{1}{n^{\frac{2}{p}}}\Big).$$

We multiply now this inequality by the function $f \in C^2[-r,r]^2$, which is approximated by polynomials due to Jackson's theorem:

$$\|f-\tilde{p}_n\|_{C[-r,r]^2} = O\Big(\frac{1}{n^2}\Big).$$

Taking into account that

$$f - \tilde{p}_n\cdot p_{2n} = f(1-p_{2n}) + f - \tilde{p}_n + (\tilde{p}_n - f)\cdot(1-p_{2n}) =$$
$$= O\Big(\frac{1}{n^{\frac{2}{p}}}\Big) + O\Big(\frac{1}{n^2}\Big) + O\Big(\frac{1}{n^{2+\frac{2}{p}}}\Big) = O\Big(\frac{1}{n^{\frac{2}{p}}}\Big),$$

we conclude that the polynomials $P_{3n} = \tilde{p}_n \cdot p_{2n}$ are desired.

Lemma 4 is proved. □

To pass to approximation by polynomials q_n, we note that for any compactum K in $\mathbb{R}$ with the transfinite diameter less than one, there exists a polynomial

$$P(x) = x^m + \dots, \qquad \max_K |P(x)| < 1.$$

This implies that there exists a polynomial X with integer coefficients and the leading coefficient one such that

$$0 < \max_K |X(x)| < 1.$$

Such a polynomial is sometimes called fundamental for a given compactum.

For example, for $K = [0,1]$ we can take $X(t) = t(1-t)$, and for $[0,2]$ we can take $X(t) = t(t-1)^2(t-2)$ (see also a polynomial X for $[0,3]$ in §4). The number of zeros of such a polynomial increases indefinitely as the length of the segment tends to 4, and there are finitely many integer algebraic numbers of different degrees on the segment $[0,4]$ (zeros of Chebyshev's polynomials). For $K = [-r,r]$, $r \in (0,2)$, the polynomial X in question can be treated as even, and, after its squaring, as positive. Its zeros are integer algebraic numbers, some of which may be found out of $[-r,r]$. So, there exist even polynomials X_1 and X_2 with integer coefficients and the leading coefficient one such that $X_2(t) > 0$ for $t \in [-r,r]$, $X_1(t) \geq 0$ and

$$X(t) = X_1(t) \cdot X_2(t), \quad 0 \leq X(t) \leq \rho < 1 \quad (t \in [-r,r]).$$

In case of the square $[-r,r]^2, r \in (0,2)$, we take the product $X(x_1) \cdot X(x_2)$ as $X(x)$.

Lemma 5. *If*

$$p_n(x) = X^m(x) \cdot p_{0,n_1}(x),$$

then there exists a sequence of polynomials q_n with integer coefficients satisfying the inequality

$$\|p_n - q_n\|_{C[-r,r]^2} = O\Big(\frac{1}{n^{2m}}\Big).$$

Proof. It is clear that any polynomial of $t \in \mathbb{R}$ can be represented as

$$p_n = \sum_{k=0}^{N} a_k X^k (1-X)^{N-k},$$

where the degrees of polynomials a_k less than the degree of X, and N is the maximal positive integer such that the degree of the whole polynomial is not greater than n.

A polynomial in two variables x_1 and x_2 is a polynomial of x_1 with coefficients that are polynomials of x_2. By doing the same transformation of polynomials of x_2 we arrive at the following representation:

$$p_n(x) = \sum_{k_1=0}^{N} \sum_{k_2=0}^{N} a_{k_1,k_2} X^{k_1}(x_1)(1-X^{k_1}(x_1))^{N-k_1} X^{k_2}(x_2)(1-X^{k_2}(x_2))^{N-k_2}.$$

Now let

$$p_n(x) = X^m(x) \cdot p_{0,n_1}(x).$$

Then we get

$$p_n(x_1,x_2)=\sum_{k_1=m}^{N}\sum_{k_2=m}^{N}a_{k_1,k_2}X^{k_1}(x_1)(1-X(x_1))^{N-k_1}X^{k_2}(x_2)(1-X(x_2))^{N-k_2}.$$

Replacing coefficients of polynomials a_{k_1,k_2} of a fixed degree by the nearest integers (in this case, the polynomials themselves change by a value bounded with respect to n), we obtain a polynomial q_n. For $x\in[-r,r]^2$, we have

$$\begin{aligned}&|p_n(x)-q_n(x)|\le\\ &\le c(X)\sum_{k_1=m}^{N}X^{k_1}(x_1)(1-X(x_1))^{N-k_1}\sum_{k_2=m}^{N}X^{k_2}(x_2)(1-X(x_2))^{N-k_2}.\end{aligned}$$

For $0\le X\le\rho<1$, we obtain

$$\begin{aligned}\sum_{k=m}^{N}X^k(1-X)^{N-k}&=\sum_{k=m}^{N-m}X^k(1-X)^{N-k}\\ &\quad+\sum_{k=N-m+1}^{N}X^k(1-X)^{N-k}\\ &\le\frac{1}{\binom{N}{m}}\sum_{k=m}^{N-m}\binom{N}{k}X^k(1-X)^{N-k}+m\rho^{N-m+1}\\ &\le\frac{1}{\binom{N}{m}}+m\rho^{N-m+1}=O\Big(\frac{1}{n^m}\Big).\end{aligned}$$

This implies the relation

$$|p_n(x)-q_n(x)|=O\Big(\frac{1}{n^{2m}}\Big).$$

Lemma 5 is proved. □

We will now prove the estimate of approximation of λ from above in Theorem 3 in the case of a cube centered at the origin, and consequently for a ball.

Let $\{t_\nu\}_0^s$ be all zeros of an even polynomial X on $[0,r]$, and let $X=X_1\cdot X_2$, where X_2 does not vanish on $[-r,r]$. Applying Lemma 4 for $f(x)=\frac{1}{X_2^m(x_1)X_2^m(x_2)}$, we get

$$\|f-p_n\|_{L_p[-r,r]}=O\Big(\frac{1}{n^{\frac{2}{p}}}\Big).$$

Due to the conditions on partial derivatives of p_n at the points $\{t_\nu\}_0^s$, this polynomial is divided by $X_1^m(x_1)X_1^m(x_2)$. Multiplying by $X_2^m(x_1)X_2^m(x_2)$, we get a good approximation of the unit by polynomials that are divided by X^m. For $d=2$, when one can take $m=1$, it remains to multiply this ratio by λ and then apply Lemma 5.

We now proceed to the proof of Theorem 3 in the case of a cube $[0,r]^d$, $r\in(0,1]$ where there are no integer points inside it, for $d=2$. We have

$$E^e_{2n}(;\Pi_{0,r})_p=\min_{\{c_k\}}\Big(\int_0^r dx_1\int_0^r\Big|\lambda-\sum_{0\le k_1,k_2\le n}c_k x_1^{k_1}x_2^{k_2}\Big|^p dx_2\Big)^{\frac{1}{p}}.$$

To estimate approximation from below, we first apply Lemma 1 at $\alpha=0$ with respect to x_2 and then with respect to x_1. As in the proof of Lemma 4, the estimate of approximation from above is implied by the case $d=1$ [4], **5.4.15**.

Theorem 3 is completely proved. □

4. Comments

Remark 1.

In the above proofs (see Sections 2 and 3), theorems on the growth of the norm of polynomials p_n outside the given compactum in $\mathbb{C}$ and $\mathbb{R}^d$ play an essential role in estimating approximation of λ from below. What is the maximal growth of $|p_n(x_0)|$ with respect to n at $x_0\notin K$ if

$$\max_{x\in K}|p_n(x)|=1?$$

In the complex plane $\mathbb{C}$ and a “good" compactum K, a desired value is determined by the level line of a function that conformably and unilaterally maps, under a special normalization, the exterior of K to the exterior of the unit circle. The point x_0 is on this line (see, e. g., [8], Ch. IX). Based on this, the following result was obtained for a part of the circle [15].

Let $r\in(0,1)$, $\alpha\in\big(0,\frac{\pi}{2}\big)$, and let

$$K_{r,\alpha}=\big\{x\in\mathbb{C}:\ |z|\le r,\ \operatorname{Re} z\le -r\cos\alpha\big\}.$$

Then for any $\lambda\in\mathbb{R}\setminus\mathbb{Z}$ and $r\le 2\sin\dfrac{\pi\alpha}{2(2\pi-\alpha)}$ with positive integer coefficients, there holds

$$\lim_{n\to\infty}\min_{\{c_k\}\in\mathbb{Z}_+}\max_{K_{r,\alpha}}\Big|\lambda-\sum_{k=0}^n c_k z^k\Big|^{\frac{1}{n}}=2\sin\frac{\pi\alpha}{2(2\pi-\alpha)}.$$

If $r > 2\sin\dfrac{\pi\alpha}{2(2\pi-\alpha)}$ (in particular, $r = \frac{1}{2}$) and λ is not a dyadic rational number, then the same limit equals $\dfrac{1}{2}$.

It is noteworthy that on the line $\mathbb{R}$ the problem of the growth of norms was solved for sets of the positive Lebesgue measure long ago.

E. Ya. Remez (1936) proved that for $h \in (0,2)$, there holds

$$\operatorname{meas}\big\{x \in [-1,1] : \ |p_n(x)| \le 1\big\} \le 2-h \ \Rightarrow \ \max_{[-1,1]} |p_n(x)| \le T_n\Big(\frac{2+h}{2-h}\Big),$$

where T_n is Chebyshev's polynomial for the interval $[-1,1]$, and the inequality is sharp (see the proof of the theorem in [7], Ch. 2, it. 7]).

Note also that at approximately the same time G. Pólya proved a similar result for the maximum of absolute value of $p_n^{(n)}$, i. e., for the leading coefficient of p_n (see [6], 2.9.13). Recently a sharp inequality for the growth of norms for sets of positive measure on the circle has been obtained [11]. A list of earlier papers on Remez-type inequalities for polynomials in several real variables is given there.

Remark 2 (on polynomials with integer coefficients in analysis).

First problem.

Let K be a compactum in $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{R}^d$, with the norm in C or L_p. What is the "integer" transfinite diameter

$$q(K) = \lim_{n\to\infty} \min_{q_n \not\equiv 0} \|q_n\|^{\frac{1}{n}} ?$$

In view of Nikol'skii's inequality of different metrics for polynomials [6], 4.9(36)], for instance, $q(K)$ in $L_p[a,b]$ is independent of $p > 0$.

This problem was successfully dealt with by L. Kronecker, H. Minkowski, D. Hilbert, I. Schur, M. Fekete, A. O. Gelfond–L. G. Shnirelman, D. S. Gorshkov, E. Aparisio, the author (see a survey paper [12] in which theorems by the mentioned mathematician are given with proofs, and [13] with references therein), and also by F. Amoroso, B. S. Kashin, G. V. Chudnovskii, P. Borwein, T. Erdelyi, C. G. Pinner, I. E. Pritsker; see also [16, Ch. 10].

Both this list of authors as well as the following one are chronological regarding the time of publication.

The idea of Gelfond–Shnirelman to use the information about the smallest non-zero norms $\max\limits_{[0,1]} |q_n(x)|$ has failed to give the proof of the asymptotic law of distribution of prime numbers. The reason was that it turned out that $q([0,1]) > \dfrac{1}{e}$ (Gorshkov, see, e.g., [13]). However, there is a chance that this can be done after passing to polynomials in many variables as K. Roth (1955) did in the problem of approximation of algebraic numbers by rational ones (see [21], Ch. YI). The latter idea appeared in the author's survey [12]; it was also discussed in [19], Ch. II] with the reference to this survey.

Note also that Chebyshev's polynomial of degree n in $L_p[a,b]$ (with the leading coefficient one and the smallest norm) is only known if $p=\infty$ (Chebyshev's polynomial), $p=2$ (Legendre's polynomial), and $p=1$ (Chebyshev's polynomial of second kind).

Due to the inequality of different metrics (see also [18]) Chebyshev's constant for the interval $[a,b]$ is the same for any $p>0$: $\frac{b-a}{4}$. In addition, $q([a,b])$ for polynomials with integer coefficients in L_p does not depend on $p>0$, depends continuously on the interval but is not known for any interval. It follows from the Gilbert-Fekete theorem that for $b-a<4$, there holds $q([a,b]) \le \left(\frac{b-a}{4}\right)^{\frac{1}{2}}$, and this inequality cannot be strengthened for small $b-a$ [12], p. 317], see also [13].

Second problem (on approximation of functions by polynomials q_n).

It is about the possibility of approximation and its rate depending on both a function and degree of polynomials. Among the people who contributed to this problem were I. Pál, S. Kakeya, I. Okada, I. N. Khlodovskii, R. O. Kuz'min, L. V. Kantorovich, M. Fekete, G. Szegö, I. N. Sanov, E. Aparisio, A. O. Gelfond, H. Matts, E. Hewitt-H. Zuckerman, the author, S. Ya. Al'per, L. B. O. Ferquson, and M. von Golitschek (see the survey in [12]).

If a function admits approximation by polynomials p_n on a compactum $K\subset\mathbb{C}$, then for the approximation by polynomials q_n, it suffices to approximate the constant $\lambda=\frac{1}{2}$. For this, it is necessary and sufficient a polynomial X to exist with integer coefficients and satisfying the inequality

$$0<|X(z)|<1 \quad (z\in K).$$

In the general case, if at least one function different from a polynomial admits approximation by polynomials q_n, then there exists a polynomial X with the condition

$$0<\max_K |X(z)|<1.$$

One may assume that its leading coefficient is one (Kakeya).

Thus, we get the first necessary condition: the transfinite diameter of K is less than one. If such polynomials X have compulsory zeros on K (integer algebraic numbers along with their conjugates), then necessary arithmetic conditions on a function appear (in contrast to the integral metrics, seen Theorem 3 above). The criterion for the approximation of a continuous function is known, i.e., the necessary and sufficient condition simultaneously (see, e. g., [7], Ch. 2, §4). A strengthening of the well-known Münz criterion for $[0,1]$ is also found (see [7], **6.5**).

In [3], a scheme of a proof for direct theorems for smooth functions on an interval $[a,b]\subset\mathbb{R}$ has been elaborated. If, for example, in the particular case where X has zeros only on $K\subset\mathbb{R}$, $0\le X(x)<1$ on K, then we approximate a function by polynomials p_n that are divided by X (using arithmetic conditions on the function), and then by polynomials divided by X^m (see Lemma 5 above or [4], **5.4.14**).

Direct theorems on approximation by polynomials p_n on compacta in $\mathbb{C}$ were obtained long ago (see [8], Ch. IX). On the other hand, the passage to approximation by polynomials q_n is established only for the square $[0,1]^2$ [22]. Note that in the case of functions of several real variables, where the question of divisibility of polynomials becomes much more complicated (see [14]), direct theorems were obtained only for Cartesian products of one-dimensional compacta (see [12], §1.4]).

We now present one of the direct theorems in an asymptotically exact form on the class [5]: *Let for some $r\in\mathbb{N}$, the derivative $f^{(r-1)}$ be absolutely continuous and $|f^{(r)}(x)|\le 1$ a. e. on $[0,1]$, and let $\frac{f^{(\nu)}(0)}{\nu!}$ and $\frac{f^{(\nu)}(1)}{\nu!}\in\mathbb{Z}$ for $0\le\nu\le r-1$ (these arithmetic conditions are also necessary). Then for any $n\ge 4r+2$, there exists a polynomial q_n such that for any $x\in[0,1]$*

$$|f(x)-q_n(x)|\le K_r\Big(\frac{\sqrt{x(1-x)}}{n}\Big)^r+c(r)\frac{\Big(\sqrt{x(1-x)}\Big)^{r-1}}{n^{r+1}},$$

where the constant cannot be taken smaller than the known Eiler–Bernulli constant

$$K_r=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k(r+1)}}{(2k+1)^{r+1}},$$

as it was called by Bernstein ([1], Ch. II, **61** (1935)) before Favard's paper and the following ones on this topic.

A direct theorem on approximation by polynomials with positive integer coefficients on the interval $[-2,0]$ with sharp arithmetic conditions on a function was also proved in [5]. It is not known how a similar theorem looks like for $[-3,0]$, for example, when the polynomial X is known:

$$\max_{[-3,0]}\big|x(x+1)(x+2)(x+3)\big(x^2+3x+1\big)\big|=\frac{45}{64}.$$

This also implies the form of the polynomial X for the interval $[-\sqrt{3},+\sqrt{3}]$ (see [13]).

Let us add one more direct theorem along with interpolation.

If $f \in C[0,1], f(0)$ and $f(1) \in \mathbb{Z}$, and $\psi \in C[0,1]$ and strictly increases from zero to one, then for any $n \in \mathbb{N}$, there exists a polynomial q_n such that for $x \in [0,1]$ we have

$$|f(x)-q_n(\psi(x))| \leq c\Big(\omega(fo\psi^{-1};\frac{\sqrt{\psi(x)(1-\psi(x))}}{n})+\frac{\sqrt{\psi(x)(1-\psi(x))}}{n}\Big),$$

where ω is the modulus of continuity.

For $\psi(x) = x$, this theorem is in [17], and it immediately implies the presented result. If $\psi(x) = x^\gamma, \quad \gamma > 0$, then we arrive at the first strengthening of Münz's theorem, in a particular case but with the rate of convergence indicated.

For approximation of functions analytic in a neighborhood of on a compactum $K \subset \mathbb{C}$, the Bernshtein–Walsh theorem ([4], **4.7.2**) is known: $E_n(f;K) = O(\rho^n)$ *for some* $\rho < 1$. A similar theorem holds in the case of polynomials q_n [12].

Let $X(z)$ be a polynomial with integer coefficients and the leading coefficient 1, and let z_ν be all its zeros. Let a function f be analytic within the lemniscate $|X(z)| = 1$, and the Hermitian interpolation polynomial defined by

$$p^{(s)}(z_\nu) = f^{(s)}(z_\nu) \quad (s \in [0,r])$$

be a polynomial with integer coefficients for any $r \in \mathbb{Z}_+$ (this is also necessary). Then the relation $E^e_n(f;K) = 0(\rho^n)$ holds for any compactum within the lemniscate,where $\rho \in (0,1)$ and depends on K (see [12], p. 300).

Exact theorems on approximation of functions by polynomials q_n in the L_p metrics, $p \in (0,1)$, can be proved in a similar way.

Note also that in contrast to approximation by polynomials q_n, in the question of approximation by integral-valued polynomials which, by definition, take integer values at all integer points, the size of a compactum K is not essential. Thus, for a compactum lying on $(0,m)$, $m \in \mathbb{N}$, one can take

$$X(x) = \frac{(x-1)(x-2)...(x-m+1)}{(m-1)!} \quad (|X(x)| < 1).$$

On the other hand, for example, in the case of an interval $[a,b]$, $0 < a < b$, it is possible to get rid of arithmetic conditions on a function as not lose the rate of approximation if instead of q_n polynomials of the form $\sum\limits_{k=0}^{n} \frac{m_k}{q^k}x^k \quad (m_k \in \mathbb{Z}, q \in \mathbb{N}, q > b)$ be taken, since for $\alpha = \frac{a}{q}$, $\beta = \frac{b}{q}$ and for the known $\rho = \rho(\alpha,\beta) < 1$, there holds

$$E^e_n(f;[\alpha,\beta]) = O(\rho^n).$$

One example of polynomials of best approximation is in order. If

$$E_n^e(f;[a,b]) = \|f - q_n^*(f)\|_C,$$

then for $f(x) = \max\{1-2x,\ 0\}$ and $n \in [0,2]$, there holds $E_n^e(f;[0,1]) = 1$, with $q_0^* \equiv 0$, $q_0^* \equiv 1$; for $n = 1$, there additionally holds $q_1^*(x) = 1 - 2x$ and $q_1^*(x) = x$, and for $n = 2$ there additionally holds $q_2^*(x) = (2x-1)x$.

It is clear that for any n, a number of polynomials $q_n^*(f)$ is finite for any function.

In [23], it is proved that for any $n \in N$ and natural $q \geq 2$, there holds

$$E_n^e\left(\frac{1}{q};\left[\frac{1}{q+1},\frac{q+2}{q(q+1)}\right]\right) = \frac{1}{q(q+1)^n}.$$

Conjecture: For $n \in N$ and $q \geq 3$, the polynomial q_n^* is unique.

In conclusion, let us mention two simple facts on the relation between $E_n^e(f;[a,b])$ and $q([a,b])$.

If there exists an $m \in \mathbb{N}$ such that $E_{n+m}^e(f;[a,b]) < E_n^e(f;[a,b])$ for infinitely many n, then

$$0 < \|q_n^* - q_{n+m}^*\| \leq E_n^e(f;[a,b]) + E_{n+m}^e(f;[a,b]) \leq 2E_n^e(f;[a,b]).$$

Therefore,

$$q([a,b]) \leq \overline{\lim_{n\to\infty}} \left(E_n^e(f;[a,b])\right)^{\frac{1}{n}}.$$

Here the known asymptotics of best approximation to functions analytic on the segment may be useful (see [6], 7.5). To prove the above inequality, one can also use polynomials $q_n(f)$ with approximation of the best rate instead of $q_n^*(f)$, and there are many such polynomials in C and L_p (see Sections 1-2 above). If the transfinite diameter of a compactum K is less than one and for any small $\epsilon > 0$ and large n, there holds $\|q_n\| < (q(K)+\epsilon)^n$, then for the analytic function

$$f(x) = \sum_{k=0}^{\infty} q_k(x)$$

we obtain

$$\overline{\lim_{n\to\infty}} \left(E_n^e(f;K)\right)^{\frac{1}{n}} \leq q(K) + \varepsilon.$$

References

[1] Bernstein, S.N. (1952, 1954). *Collected works. Vol. I, Vol. II.* Acad. Nauk SSSR, Moscow.

[2] Kantorovic, L. (1931). Quelques observations sur l'approximation de fonctions au moyen de polynomes a coefficients entiers. *Bulletin de l'Academie des Sciences de l'URSS. Classe des sciences mathematiques et na, 9*, 1163–1168.

[3] Trigub, R.M. (1962). Approximation of functions by polynomials with integer coefficients. *Izv. Acad. Nauk SSSR, Ser., Mat., 26*, 261–280.

[4] Trigub, R., Belinsky, E. (2004). *Fourier Analysis and Approximation of Functions.* Kluwer-Springer.

[5] Trigub, R.M. (2001). On Approximation of Smooth Functions and Constants by Polynomials with Integer and Natural Coefficients. *Mat. Zametki, 70*, 123–136; English transl.: *Math. Notes, 70*, 110–122.

[6] Timan, A.F. (1960). *Theory of approximation of functions of a real variable.* Fizmatgiz, Moscow (in Russian); English transl.: (1963). Pergamon, Press, MacMillan, N.Y.

[7] Lorentz, G.G., Golitschek, M.V., Makovoz, Yu. (1996). *Constructive Approximation.* Advanced Problems, Springer.

[8] Dzyadyk, V.K. (1977). *Introduction to the theory of uniform approximation of functions by polynomials.* Nauka, Moscow (in Russian).

[9] Stein, E.M., Weiss, G. (1971). *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton Univ. Press, Princeton.

[10] Akhiezer, N.I. (1965). *Lectures on Approximation Theory,* 2nd ed. Nauka, Moscow (in Russian); English transl. of the 1st ed. (1947). Theory of approximation. Ungar, New York, 1956.

[11] Tikhonov, S., Yuditskii, P. (2020). Sharp Remez inequality. *Constr. Approx., 52*(2), 233–246.

[12] Trigub, R.M. (1971). Approximation of functions with Diophantine conditions by polynomials with integer coefficients. *Metric Questions of the Theory of Functions and Mappings, Naukova Dumka, Kiev, 2*, 267–333.

[13] Trigub, R.M. (2019). Chebyshev Polynomials and Integer Coefficients. *Mat. Zametki, 105*(2), 302–312.

[14] Prasolov, V.V. (2004). *Polynomials.* Translated from the 2001 Russian second edition by Dimitry Leites. Algorithms and Computation in Mathematics, 11. Springer–Verlag, Berlin.

[15] Trigub, R.M. (2009). Approximation of Functions by Polynomials with Various Constraints. *Izvestiya NAN Armenii. Matematika, 4*, 32–44; English transl.: (2009). *J. of Contemporary Math. Analysis, 44*(4), 173–185.

[16] Montgomery, H.I. (1994). *Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis.* Amer. Math. Soc. Providence. RI.

[17] Trigub, R.M. (2003). Approximation of functions by polynomials with Hermitian interpolation and restrictions on the coefficients. *Izv. Ross. Akad. Nauk, Ser. Mat., 67*, 1–23 ; English transl.: *Izv. Russ. Acad. Sci., Math., 67*, 199–221.

[18] Ganzburg, M.I. (2018). Polynomial inequalities on sets with km – concave weighted measures. *Journal D' Analyse Math, 135*, 389–411.

[19] Chudnovsky, G.V. (1983). Number theoretic applications of polynomials with rational coefficients fined by extremality conditions. *Arithmetic and Geometry. Vol. I.* In: Progress in Math. Birkhauser. Boston, vol. 35, pp. 61–105.

[20] Goluzin, G.M. (1969). *Geometric theory of functions of a complex variable.* Translations of Mathematical Monographs, Vol. 26. American Mathematical Society, Providence, R. I.

[21] Cassels, J.W.S. (1967). *An introduction to Diophantine approximation,* Vol. 45. Cambridge Tracts in Mathematics and Mathematical Physics, Cambridge University Press.

[22] Volchkov, V.V. (1996). Approximation of analytic functions by polynomials with integer coefficients. *Math. Notes, 59*(2), 128–132.

[23] Trigub, R.M. (2023). On approximation of constants by polynomials with integer coefficients, *J. Math. Sciences, 266*(6), 959–966.

Contact information

Roald Mikhailovich Trigub

Sumy State University,
Sumy, Ukraine
E-Mail: roald.trigub@gmail.com