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On boundary extension of mappings on Riemannian surfaces in terms of prime ends

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Abstract. We investigate non-homeomorphic mappings of Riemannian surfaces of Sobolev class. There are obtained some estimates of distortion of moduli of families of paths. We have proved that, under some conditions, these mappings have a continuous extension to a boundary of a domain in terms of prime ends.

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1. Introduction

Some important results concerning the boundary behavior of Sobolev homeomorphisms between Riemannian surfaces were obtained in [15] and [16]. In particular, in [15] the authors considered the case when the domains under consideration are locally connected at their boundary, while the paper [16] refers to domains of a more complex structure. In the latter case, mappings, as a rule, do not have a pointwise continuous boundary extension. However, the construction of prime ends, introduced by Caratheodory, allows us to interpret this extension in another (more successful) sense.

In this article, we intend to abandon the condition of the injectivity of mappings, which significantly distinguishes it from [15] and [16]. We will show that similar classes of open-closed discrete maps also have a continuous boundary extension. Definitions and notions used below and not mentioned in the text, may be found in [15, 16] or [18].

In what follows, unless otherwise specified, the Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$ have hyperbolic type. In the following, $ds_{\widetilde{h}}$ and $d\widetilde{v}$, $ds_{\widetilde{h_*}}$ and

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$d\widetilde{v_*}$ denote the elements of length and area on the Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, respectively. We also use the notation $\widetilde{h}$ for the metric on the surface $\mathbb{S}$, in particular,

$$\widetilde{B}(p_0,r):=\{p\in\mathbb{S}:\widetilde{h}(p,p_0)<r\},\quad \widetilde{S}(p_0,r):=\{p\in\mathbb{S}:\widetilde{h}(p,p_0)=r\}$$

are a disk and a circle on the surface $\mathbb{S}$ centered at a point p_0 and a radius $r>0$, respectively.

The following definitions refer to Caratheodory [2], see also [4, 8] and earlier papers related to prime ends ([11, 21]). Recall that, a continuous mapping $\sigma:\mathbb{I}\to\mathbb{S}$, $\mathbb{I}=(0,1)$, is called the *Jordan arc* in $\mathbb{S}$, if $\sigma(t_1)\neq\sigma(t_2)$ for $t_1\neq t_2$. Next, we will sometimes use σ for $\sigma(\mathbb{I})$, $\overline{\sigma}$ for $\overline{\sigma(\mathbb{I})}$ and $\partial\sigma$ for $\overline{\sigma(\mathbb{I})}\setminus\sigma(\mathbb{I})$. A *cut* of a domain D is called either the Jordan arc $\sigma:\mathbb{I}\to D$, ends of which belongs to ∂D, or a closed Jordan path in D. A sequence $\sigma_1,\sigma_2,\ldots,\sigma_m,\ldots$ of cuts of the domain D is called a *chain* if:

(i) $\overline{\sigma_i}\cap\overline{\sigma_j}=\varnothing$ for any $i\neq j$, $i,j=1,2,\ldots$;

(ii) σ_m *splits* D, i.e. $D\setminus\sigma_m$ consists from two components, one of which contains σ_{m-1}, and another contains σ_{m+1},

(iii) $\widetilde{h}(\sigma_m)\to 0$ as $m\to\infty$, $\widetilde{h}(\sigma_m)=\sup\limits_{p_1,p_2\in\sigma_m}\widetilde{h}(p_1,p_2)$.

By the definition, a chain $\{\sigma_m\}$ defines the sequence of domains $d_m\subset D$ such that $\partial\, d_m\cap D\subset\sigma_m$ and $d_1\supset d_2\supset\ldots\supset d_m\supset\ldots$. Two chains $\{\sigma_m\}$ and $\{\sigma_k^{\prime}\}$ are called *equivalent*, if for any $m=1,2,\ldots$ the domain d_m contains all $d_k^{\prime}$ excepting a finite number and, on the other hand, for any $k=1,2,\ldots$ the domain $d_k^{\prime}$ contains all d_m excepting a finite number, as well. A *prime end* of D is a class of equivalent chains of cuts of D.

Let K be a prime end in $D\subset\mathbb{R}^n$, and let $\{\sigma_m\}$ and $\{\sigma_m^{\prime}\}$ are chains in K. In addition, let d_m and $d_m^{\prime}$ are corresponding domains with a respect to σ_m and $\sigma_m^{\prime}$. Then

$$\bigcap\limits_{m=1}^{\infty}\overline{d_m}\subset\bigcap\limits_{m=1}^{\infty}\overline{d_m^{\prime}}\subset\bigcap\limits_{m=1}^{\infty}\overline{d_m}\ ,$$

and, thus,

$$\bigcap\limits_{m=1}^{\infty}\overline{d_m}=\bigcap\limits_{m=1}^{\infty}\overline{d_m^{\prime}}\ ,$$

i.e. the set

$$I(K)=\bigcap\limits_{m=1}^{\infty}\overline{d_m}$$

depends only on K and does not depend on the chain $\{\sigma_m\}$. A set $I(K)$ is said to be an *impression of a prime end* K. In what follows, by E_D we denote the set of all prime ends in D, and $\overline{D}_P:=D\cup E_D$ denotes the completion of D by its prime ends. Now, let us consider $\overline{D}_P$ as a topological space in the following way. First of all, we consider that open sets in D are also open in $\overline{D}_P$. Next, we defined a based neighborhood of $P\subset E_D$ as a union of any domain d, which contains in some chain of P, with the rest prime ends in d. In particular, in the topology mentioned above, a sequence $x_n\in D$ converges to $P\in E_D$ if and only if, for any domain d_m, belonging to a sequence of domains $d_1, d_2, d_3,\ldots$, containing in P, there is $n_0=n_0(m)$ such that $x_n\in d_m$ for $n\geqslant n_0$.

A (maximal) dilatation of f at z is defined in local coordinates by the relation

$$K_f(z)=\frac{|f_z|+|f_{\overline{z}}|}{|f_z|-|f_{\overline{z}}|} \tag{1.1}$$

for $J_f(z)\ne 0$, $K_f(z)=1$ for $\|f^{\,\prime}(z)\|=0$ and $K_f(z)=\infty$ otherwise. It is not difficult to see that, K_f does not depend on local coordinates because the transition mappings between two charts are conformal by the definition of the Riemannian surface.

A mapping $f:D\rightarrow D_*$ is called a *mapping with finite distortion,* if $f\in W_{\rm loc}^{1,1}(D)$ and, in addition, there is almost everywhere a finite function $K(z)$ such that $\|f^{\,\prime}(z)\|^2\leqslant K(z)\cdot J_f(z)$ for almost all $z\in D$. A mapping $f:D\rightarrow D_*$ is called *discrete* if the preimage $f^{-1}(y)$ of any point $y\in D_*$ consists of isolated points only. A mapping $f:D\rightarrow D_*$ is called *open* if the image of any open set $U\subset D$ is an open set in D_*. A mapping $f:D\rightarrow D_*$ is called *closed* if the image of any closed set $U\subset D$ is an closed set in D_*.

Let $p_0\in {\Bbb S}$ and let $\varphi:{\Bbb S}\rightarrow{\Bbb R}$ be a function integrable in some neighborhood U of the point p_0 with respect to the area $\widetilde{v}$ on ${\Bbb S}$. Following [10, Section 6.1, Ch. 6], we say that a function $\varphi:{\Bbb S}\rightarrow{\Bbb R}$ has a *finite mean oscillation* at the point $p_0\in D$, we write $\varphi\in FMO(p_0)$, if

$$\limsup\limits_{\varepsilon\rightarrow 0}\frac{1}{\widetilde{v}(\widetilde{B}(p_0, \varepsilon))}\int\limits_{\widetilde{B}(p_0, \varepsilon)} |\varphi(p)-\overline{\varphi}_{\varepsilon}|\ d\widetilde{v}(p)<\infty\,,$$

where $\overline{\varphi}_{\varepsilon}=\frac{1}{\widetilde{v}(\widetilde{B}(p_0,\varepsilon))}\int\limits_{\widetilde{B}(p_0,\varepsilon)}\varphi(p)\,d\widetilde{v}(p)$.

The main result of the paper is the following, cf. [17, Theorem 1].

Theorem 1.1. *Let D and D_* are domains in ${\Bbb S}$ and ${\Bbb S}_*$, correspondingly, which have compact closures $\overline{D}\subset {\Bbb S}$ and $\overline{D_*}\subset {\Bbb S}_*$, while ∂D and ∂D_* has a finite number of components, where all components of ∂D_* are non-degenerate. Assume that, $Q:{\Bbb S}\rightarrow (0, \infty)$ is a given function which is measurable with a respect to the measure $\widetilde{v}$ on ${\Bbb S}$, $Q(p)\equiv 0$ in ${\Bbb S}\setminus D$. Let $f:D\rightarrow D_*$ be an open, discrete and closed mapping with a finite* *distortion of D onto D_*, such that $K_f(p) \leqslant Q(p)$ for almost all $p \in D$. Then f has a continuous extension $f : \overline{D}_P \to \overline{D_*}_P$, $f(\overline{D}_P) = \overline{D_*}_P$, if one of the following conditions hold:*

1) the relations

$$\int\limits_{\varepsilon}^{\varepsilon_0} \frac{dt}{\|Q\|(t)} < \infty\,, \qquad \int\limits_{0}^{\varepsilon_0} \frac{dt}{\|Q\|(t)} = \infty\,, \tag{1.2}$$

hold at any $p_0 \in \partial D$, for some $\varepsilon_0 = \varepsilon_0(p_0) > 0$ and any $0 < \varepsilon < \varepsilon_0$, where $\|Q\|(t) := \int\limits_{\widetilde{S}(p_0,t)} Q(p)\, ds_{\widetilde{h}}(p)$ denotes the L_1-norm of the function Q over the circle $\widetilde{S}(p_0, t)$,

2) the condition $Q \in FMO(\partial D)$ holds.

2. Preliminaries

In what follows, we need the following statement the proof of which may be found in [18, Proposition 4.5], cf. [10, Lemma 7.4, Ch. 7].

Proposition 2.1. *Let $p_0 \in \mathbb{S}$, let U be a normal neighborhood of p_0, $0 < r_1 < r_2 < \mathrm{dist}\,(p_0, \partial U)$, let $Q(p)$ be a measurable function with a respect to the measure $\widetilde{v}$, $Q : \mathbb{S} \to [0, \infty]$, $Q \in L^1(U)$. Set $\widetilde{A} = \widetilde{A}(p_0, r_1, r_2) = \{p \in \mathbb{S} : r_1 < \widetilde{h}(p, p_0) < r_2\}$, $\|Q\|(r) = \int\limits_{\widetilde{S}(p_0,r)} Q(p)\; ds_{\widetilde{h}}(p)$,*

$$\eta_0(r) := \frac{1}{I \cdot \|Q\|(r)}\,, \tag{2.3}$$

where

$$I = I(p_0, r_1, r_2) := \int\limits_{r_1}^{r_2} \frac{dr}{\|Q\|(r)}\,.$$

Then

$$\begin{aligned} \frac{1}{I} &= \int\limits_{\widetilde{A}(p_0,r_1,r_2)} Q(p) \cdot \eta_0^2(\widetilde{h}(p, p_0))\; d\widetilde{v}(p) \\ &\leqslant \int\limits_{\widetilde{A}(p_0,r_1,r_2)} Q(p) \cdot \eta^2(\widetilde{h}(p, p_0))\; d\widetilde{v}(p) \end{aligned} \tag{2.4}$$

for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)dr = 1\,. \tag{2.5}$$

Given a mapping $f : D \to \mathbb{S}_*$ and a set $E \subset \overline{D} \subset \mathbb{S}$, we put

$$C(f, E) = \{y \in \mathbb{S}_* : \exists\, x \in E, x_k \in D : x_k \to x, f(x_k) \to y, k \to \infty\}\,.$$

The following statement holds.

Proposition 2.2. *Assume that, a domain $D \subset \mathbb{S}$ has a finite number of boundary components $\Gamma_1, \Gamma_2, \ldots, \Gamma_n \subset \partial D$. Then:*

1) for any Γ_i, $i = 1, 2, \ldots, n$ there is a neighborhood $U_i \subset \mathbb{S}$ and a conformal mapping H of $U_i^* := U_i \cap D$ onto $R = \{z \in \mathbb{C} : 0 \leqslant r_i < |z| < 1\}$ such that $\gamma_i := \partial U_i^* \cap D$ is a closed Jordan path

$$C(H, \gamma_i) = \{z \in \mathbb{C} : |z| = 1\}; \quad C(H, \Gamma_i) = \{z \in \mathbb{C} : |z| = r_i\},$$

while $r_i = 0$ if and only if Γ degenerates into a point. Moreover, H extends to a homeomorphism of $\overline{U_i^*}_P$ onto $\overline{R}$, see [16, Lemma 2];

2) a space $\overline{D}_P$ is metrizable with some metric $\rho : \overline{D}_P \times \overline{D}_P \to \mathbb{R}$ such that, the convergence of any sequence $x_n \in D$, $n = 1, 2, \ldots,$ to some prime end $P \in E_D$ is equivalent to the convergence x_n in one of spaces $\overline{U_i^*}_P$, see [16, Remark 2];

3) any prime end $P \in E_D$ contains a chain of cuts σ_m, $m = 1, 2, \ldots,$ which belong to spheres $\widetilde{S}(z_0, r_m)$, $r_m \to 0$ as $m \to \infty$, see [16, Remark 1];

4) for any $P \subset E_D$ its impression $I(P)$ is a continuum in ∂D, while there is some unique $1 \leqslant i \leqslant n$ such that $I(P) \subset \Gamma_i$, see [16, Proposition 1, Remark 1].

The technique for proving the main result is based on using modulus of families of paths. Proceeding from this, we consider some (wider) class of mappings for which the required distortion of modulus is satisfied. Everywhere below, $M(\cdot)$ is the modulus of families of paths on $\mathbb{S}$ (see, for example, [15–18]). Let $\rho\colon \mathbb{S} \to [0, \infty]$ is a function measurable with respect to the area $\widetilde{v}$. We say that, ρ is *extensively admissible* for Γ, abbr. $\rho \in \operatorname{ext}\operatorname{adm}\Gamma$, if the ratio

$$\int\limits_{\gamma} \rho\, ds_{\widetilde{h}}(p) \geqslant 1$$

holds for all locally rectifiable paths $\gamma \in \Gamma \setminus \Gamma_0$, while $M(\Gamma_0) = 0$. The following class is a generalization of quasiconformal mappings in Gehring sense (see, e.g., [10, Chapter 9]). Let D and D_* be domains in $\mathbb{S}$ and $\mathbb{S}_*$, respectively, and let $Q\colon D \to (0, \infty)$ be a measurable function with a respect to $\widetilde{v}$ on $\mathbb{S}$. We say that, $f\colon D \to D_*$ is a *lower Q-mapping at* *a point* $p_0\in \overline{D}$, if there is $\varepsilon_0=\varepsilon_0(p_0)>0$, $\varepsilon_0<d_0=\sup\limits_{p\in D}\widetilde{h}(p,p_0)$, such that

$$M(f(\Sigma_{\varepsilon}))\geqslant \inf\limits_{\rho\in \mathrm{ext\,adm}\,\Sigma_{\varepsilon}}\int\limits_{D\cap \widetilde{A}(p_0,\varepsilon,\varepsilon_0)}\frac{\rho^2(p)}{Q(p)}\,d\widetilde{v}(p) \qquad (2.6)$$

for any ring $\widetilde{A}(p_0,\varepsilon,\varepsilon_0)=\{p\in {\mathbb S}: \varepsilon<\widetilde{h}(p,p_0)<\varepsilon_0\}$, where Σ_{ε} denotes the family of all intersections of circles $\widetilde{S}(p_0,r)=\{p\in {\mathbb S}: \widetilde{h}(p,p_0)=r\}$ with D, $r\in (\varepsilon,\varepsilon_0)$.

In many cases, we need to verify the property (2.6) without of a verification of infinitely many inequalities. Such a possibility follows by the following statement (cf. [10, Theorem 9.2] and [7, Lemma 4.2]), the proof of which may be found in [18, Lemma 2.3].

Lemma 2.1. *Let D and D_* be domains in ${\mathbb S}$ and ${\mathbb S}_*$, respectively, $p_0\in \overline{D}$ and $Q\colon D\rightarrow (0,\infty)$ be a given function. Then $f\colon D\rightarrow D_*$ is a lower Q-mapping at a point p_0 if and only if there is $0<d_0<\sup\limits_{p\in D}\widetilde{h}(p,p_0)$ such that*

$$M(f(\Sigma_{\varepsilon}))\geqslant \int\limits_{\varepsilon}^{\varepsilon_0}\frac{dr}{\Vert\, Q\Vert(r)}\quad \forall\ \varepsilon\in(0,\varepsilon_0)\,,\ \varepsilon_0\in(0,d_0)\,, \qquad (2.7)$$

where Σ_{ε} denotes the family of all intersections $\widetilde{S}(p_0,r)$ with a domain D, $r\in (\varepsilon,\varepsilon_0)$, in addition,

$$\Vert Q\Vert(r)=\int\limits_{D(p_0,r)}Q(p)\,ds_{\widetilde{h}}(p)$$

denotes L_1-norm of the function Q over $D\cap \widetilde{S}(p_0,r)=D(p_0,r)=\{p\in D\,:\,\widetilde{h}(p,p_0)=r\}$.

The following lemma holds.

Lemma 2.2. *Let D and D_* be domains in ${\mathbb S}$ and ${\mathbb S}_*$, correspondingly, which have compact closures $\overline{D}\subset {\mathbb S}$ and $\overline{D_*}\subset {\mathbb S}_*$, while ∂D and ∂D_* consist of finite boundary components, and all components of ∂D_* are non-degenerate. Assume that, $f:D\rightarrow D_*$, $f(D)=D_*$, be a closed open discrete mapping. Then:*

1) $C(f, P)$ is a continuum in ∂D_*, where

$$C(f, P)=\{y\in {\mathbb S}_*: \exists\, x_k\in D: x_k\rightarrow P, f(x_k)\rightarrow y, k\rightarrow\infty\}\,.$$

In particular, there is a unique component $\Gamma\subset \partial D_*$ such that $C(f, P)\subset \Gamma$;

2) if $P \subset E_D$ and d_k, $k = 1, 2, \ldots$, be a sequence of domains corresponding to P and $U \subset \mathbb{S}_*$ be a neighborhood of Γ from item 1) of Proposition 2.2, then there is $s_0 \in \mathbb{N}$ such that

$$f(d_k) \subset U^* \quad \forall\, k \geqslant s_0\,, \tag{2.8}$$

where $U^* := U \cap D_*$.

Proof. Let us to prove that $C(f, P)$ is a continuum in ∂D_*, where

$$C(f, P) = \{y \in \mathbb{S}_* : \exists\, x_k \in D : x_k \to P, f(x_k) \to y, k \to \infty\}\,.$$

For this goal, let us to show that

$$C(f, P) = \bigcap_{k=1}^{\infty} \overline{f(d_k)}\,, \tag{2.9}$$

where d_k, $k = 1, 2, \ldots$, is a sequence of domains of cuts corresponding to a prime end P. Indeed, let $y \in C(f, P)$, then $y = \lim\limits_{k\to\infty} y_k$, $y_k \to P$ as $k \to \infty$. We may consider that $y_k = f(x_k)$, $x_k \in d_k$. Now, for any $m \in \mathbb{N}$ there is $k_0 = k_0(m)$ such that $x_k \in d_m$ for $k \geqslant k_0$, because the sequence d_m is decreasing. It follows from this that, $y \in \overline{f(d_k)}$ for any $k = 1, 2, \ldots$. Thus $C(f, P) \subset \bigcap\limits_{k=1}^{\infty} \overline{f(d_k)}$. On the other hand, let $y \in \bigcap\limits_{k=1}^{\infty} \overline{f(d_k)}$. Then, for a given $k \in \mathbb{N}$, we obtain that $y = \lim\limits_{m\to\infty} y_m^{(k)}$, where $y_m^{(k)} \in f(d_k)$, $m = 1, 2, \ldots$. Then there are $x_m^{(k)} \in d_k$, $m = 1, 2, \ldots$, such that $f(x_m^{(k)}) \to y$ as $m \to \infty$. Then, for a number $1/2^k$, there is $m = m_k \in \mathbb{N}$ such that $\widetilde{h}_*(f(x_{m_k}^{(k)}), y) < 1/2^k$. By the definition, a sequence $x_{m_k}^{(k)}$ converges to P as $k \to \infty$ and $f(x_{m_k}^{(k)}) \to y$ as $k \to \infty$, i.e., $y \in C(f, P)$. Thus, $C(f, P) \subset \bigcap\limits_{k=1}^{\infty} \overline{f(d_k)}$, $\bigcap\limits_{k=1}^{\infty} \overline{f(d_k)} \subset C(f, P)$ and, consequently, the relation (2.9) is established. Then, by [9, Theorem 5.II.5] $C(f, P)$ is a continuum.

It remains to show that $C(f, P) \subset \partial D_*$. Observe that, $C(f, P) \neq \varnothing$ because $\overline{D_*}$ is a compactum by the assumption. Let $y \in C(f, P)$, then $y = \lim\limits_{k\to\infty} y_k$, $y_k \to P$ as $k \to \infty$ and $y_k = f(x_k)$, $x_k \in d_k$. Without loss of generality, by the compactness of $\overline{D}$, we may consider that x_k converges to x_0 as $k \to \infty$. Then, by item 4) of Proposition 2.2, since $x_0 \in \bigcap\limits_{k=1}^{\infty} \overline{d_k}$, we have that $x_0 \in I(P) \subset \partial D$. Since f is an open, discrete and closed mapping, it is boundary preserving. Now, the sequence $f(x_k) = y_k$ may converges only to a boundary point as $k \to \infty$, i.e., $y \in \partial D_*$. The item 1) of Lemma 2.2 is established.

Let us to prove item 2). Let $U \subset \mathbb{S}_*$ be a neighborhood of Γ which corresponds to item 1) of Proposition 2.2. In other words, there is a conformal mapping H of $U^* := U \cap D_*$ onto the ring $R = \{z \in \mathbb{C} : 0 < r < |z| < 1\}$ such that $\gamma := \partial U^* \cap D$ is a closed Jordan path,

$$C(H, \gamma) = \{z \in \mathbb{C} : |z| = 1\}; \quad C(H, \Gamma) = \{z \in \mathbb{C} : |z| = r\}\,.$$

Let us to prove (2.8). Assume the contrary. Then there is an increasing sequence of numbers k_l, $l = 1, 2, \ldots$, and a sequence $y_{k_l} \in f(d_{k_l})$ such that $y_{k_l} \in D_* \setminus U^*$ for any $l \in \mathbb{N}$. By the compactness of $\overline{D_*}$ we may assume that y_{k_l} converges to some point y_0 as $l \to \infty$. Then $y_0 \in \Gamma$ by the inclusion $C(f, P) \subset \Gamma$, where Γ is some boundary component of D_* (see item 1)). Let $\varepsilon_1 > 0$ be such that $B(y_0, \varepsilon_1) \subset U$; this ε_1 exists because U is a neighborhood of Γ. Then $y_{k_l} \in B(y_0, \varepsilon_1) \cap D_* \subset U^*$ for large $l \in \mathbb{N}$, that contradicts with $y_{k_l} \in D_* \setminus U^*$ for $l \in \mathbb{N}$. The contradiction obtained above proves the relation (2.8). □

An analog of the following statement is proved for homeomorphisms in [16, Lemma 4], cf. [8, Lemma 3] and [5, Lemma 5.1].

Theorem 2.1. *Let D and D_* domains in $\mathbb{S}$ and $\mathbb{S}_*$, correspondingly, which have compact closures $\overline{D} \subset \mathbb{S}$ and $\overline{D_*} \subset \mathbb{S}_*$, while ∂D and ∂D_* consist of a finite number of components, and all components of ∂D_* are non-degenerate. Assume that, $Q : \mathbb{S} \to (0, \infty)$ is a given function which is measurable with a respect to $\widetilde{v}$ on $\mathbb{S}$, $Q(p) \equiv 0$ in $\mathbb{S} \setminus D$. Let $f : D \to D_*$, $D_* = f(D)$, be a lower Q-mapping at any point $p_0 \in \partial D$, and let f be an open, discrete and closed. Then f has a continuous extension $f : \overline{D}_P \to \overline{D_*}_P$, $f(\overline{D}_P) = \overline{D_*}_P$, whenever one of the following conditions hold:*

1) either the relation

$$\int\limits_{\varepsilon}^{\varepsilon_0} \frac{dt}{\|Q\|(t)} < \infty\,, \qquad \int\limits_{0}^{\varepsilon_0} \frac{dt}{\|Q\|(t)} = \infty \tag{2.10}$$

holds for any $p_0 \in \partial D$, for some $\varepsilon_0 = \varepsilon_0(p_0) > 0$ and all $0 < \varepsilon < \varepsilon_0$, where $\|Q\|(t) := \int\limits_{\widetilde{S}(p_0,t)} Q(p)\, ds_{\widetilde{h}}(p)$ denotes the L_1-norm of the function Q over the circle $\widetilde{S}(p_0, t)$,

2) or $Q \in FMO(\partial D)$.

Proof. Let us firstly prove that f has a continuous extension $f : \overline{D}_P \to \overline{D_*}_P$. Let us consider the case 1), i.e., when the relations (2.10) hold. Put $P \in E_D$.

1) By the item 1) of Lemma 2.2, the set $C(f,P)$ is a continuum in ∂D_*. Then there is a component $\Gamma\subset\partial D_*$ which contains $C(f,P)$. Let $U\subset\mathbb{S}_*$ be a neighborhood Γ which corresponds to Proposition 2.2, and let H be a corresponding conformal mapping of a domain $U^*:=U\cap D_*$ onto the ring $R=\{z\in\mathbb{C}:0<r<|z|<1\}$ such that $\gamma:=\partial U^*\cap D$ is a closed Jordan path,

$$C(H,\gamma)=\{z\in\mathbb{C}:|z|=1\};\quad C(H,\Gamma)=\{z\in\mathbb{C}:|z|=r\}\,.$$

By Proposition 2.2 there is a chain of cuts σ_n, corresponding to a prime end P, which belongs to spheres $\widetilde{S}(p_0,r_n)$, $p_0\in\partial D$, $r_n\rightarrow 0$ as $n\rightarrow\infty$. Let d_n, $n=1,2,\ldots$, be a sequence of domains corresponding to cuts σ_n. By the inclusion (2.8) we may consider that $f(d_1)\subset U^*$. Now, we set $\widetilde{f}:=f|_{d_1}$, $g:=H\circ\widetilde{f}$, $g:d_1\rightarrow R$, $g(d_1)\subset R$. Observe that, R is a domain, any point of which has a sufficiently small neighborhood the intersection of which with R is quasiconformally equivalent to the unit disk (besides the direct arguing, this statement may be obtained by the corresponding Väisälä's result [23, Theorem 17.12], since R is a union of two circles which are C^1-manifolds. In this context, we also mention [14, sect. 2.2] and [12, Remark 1.5]). Then, by [14, Theorem 4.1] and due to [16, Remark 2], we may consider that $\overline{R}_P=\overline{R}$. In this case, for the proof of Theorem, it is sufficient to establish the continuous extension $\overline{g}:d_1\cup\{P\}\rightarrow\overline{R}$.

2) Moreover, by the compactness of $\overline{R}$, it is sufficiently to prove that the set

$$L=C(g,P):=\left\{y\in\partial R: y=\lim\limits_{m\rightarrow\infty}g(p_m), p_m\rightarrow P, p_m\in d_1\right\}$$

consists from a unique point $y_0\in\partial R$. The mapping g, as usual, is open and discrete in d_1, but is not necessary closed. Let us to show that, g satisfies the relation

$$M(g(\Sigma_{\varepsilon}^1))\geqslant\int\limits_{\varepsilon}^{\varepsilon_1}\frac{dr}{\|\,Q\|(r)}\quad\forall\,\varepsilon\in(0,\varepsilon_1)\,,\ \varepsilon_1\in(0,r_1)\,,\tag{2.11}$$

where Σ_{ε}^1 denotes the family of all intersections of circles $\widetilde{S}(p_0,r)=\{p\in\mathbb{S}:\widetilde{h}(p,p_0)=r\}$ with d_1, $r\in(\varepsilon,\varepsilon_1)$. To proof this fact, let us to show that

$$\widetilde{S}(p_0,r)\cap D>\widetilde{S}(p_0,r)\cap d_1,\quad r<r_1\,.\tag{2.12}$$

(Here and below the notation $\Gamma_1>\Gamma_2$ denotes that, for any dished line $\alpha\in\Gamma_1$, $\alpha:\bigcup\limits_{i=1}^{\infty}(a_i,b_i)\rightarrow\mathbb{S}$, there is a dashed line $\beta\in\Gamma_2$, where

$\beta : \bigcup\limits_{i,k=1}^{\infty}(a_{ik}, b_{ik}) \to \mathbb{S}$, $\bigcup\limits_{k=1}^{\infty}(a_{ik}, b_{ik}) \subset (a_i, b_i)$, $\alpha|_{(a_{ik},b_{ik})} = \beta$ for any $i = 1, 2, \ldots$, $k = 1, 2, \ldots$ and, besides that, at least one interval (a_{ik}, b_{ik}) is not empty).

Indeed, we put $0 < r < r_1$. Then there is $i \in \mathbb{N}$ such that $r_i < r$. Let $\sigma_i \subset \widetilde{S}(p_0, r_i) \cap d_1$ be a cut corresponding to a domain d_i. Join any point $\omega \in \sigma_i$ with a point $w \in \sigma_1$ in D with a path α_i, $|\alpha_i| \subset D$. Without loss of generality, we may consider that α_i belongs to d_1 instead of its endpoint, because $\partial d_1 \cap D \subset \sigma_1$. Observe that $|\alpha_i| \cap \widetilde{B}(p_0, r) \neq \varnothing \neq |\alpha_i| \cap (\mathbb{S} \setminus \widetilde{B}(p_0, r))$, therefore, by [9, Theorem 1.I.5, § 46] $|\alpha_i| \cap \widetilde{S}(p_0, r) \neq \varnothing$. It follows from this that, $\widetilde{S}(p_0, r) \cap d_1 \neq \varnothing$.

Let now $\alpha := \widetilde{S}(p_0, r) \cap D$ be a dished line $\alpha : \bigcup\limits_{i=1}^{\infty}(a_i, b_i) \to D$, where there at least one non-empty interval in its system (a_i, b_i). By the proved above, $\beta := \widetilde{S}(p_0, r) \cap d_1 \neq \varnothing$, therefore there is at most countable of intervals (c_k, d_k), $k = 1, 2, \ldots$, such that $\beta : \bigcup\limits_{k=1}^{\infty}(c_k, d_k) \to d_1$ and the interval (c_k, d_k) is not empty at least for some $k \in \mathbb{N}$. By the definition, for any $k \in \mathbb{N}$ there is $i \in \mathbb{N}$ such that $(c_k, d_k) \subset (a_i, b_i)$. Denote $(a_{ik}, b_{ik}) := (a_i, b_i) \cap (c_k, d_k)$, and observe that the interval (a_{ik}, b_{ik}) is not empty at least for some $i \in \mathbb{N}$ and $k \in \mathbb{N}$. Observe also that, $\bigcup\limits_{k=1}^{\infty}(a_{ik}, b_{ik}) \subset (a_i, b_i)$ and $\alpha|_{(a_{ik},b_{ik})} = \beta$, that proves (2.12).

It follows from (2.12) that $f(\widetilde{S}(p_0, r) \cap D) > f(\widetilde{S}(p_0, r) \cap d_1) = \widetilde{f}(\widetilde{S}(p_0, r) \cap d_1)$, where $\widetilde{f} := f|_{d_1}$. Let Σ_ε be a family of all intersections of circles $\widetilde{S}(p_0, r) = \{p \in \mathbb{S} : \widetilde{h}(p, p_0) = r\}$ with D. Then by [3, Theorem 1(c)] $M(\widetilde{f}(\Sigma_\varepsilon^1)) \geqslant M(f(\Sigma_\varepsilon))$ and, consequently, by Lemma 2.1

$$M(\widetilde{f}(\Sigma_\varepsilon^1)) \geqslant \int\limits_{\varepsilon}^{\varepsilon_1} \frac{dr}{\| Q\|(r)} \quad \forall\, \varepsilon \in (0, \varepsilon_1)\,,\ \varepsilon_1 \in (0, r_1)\,. \tag{2.13}$$

In this case, (2.11) follows by (2.13), because $g = H \circ \widetilde{f}$ and H is a conformal mapping preserving the family of paths with a respect to Lebesgue measure on the plane (see, e.g., [23, Theorem 8.1], see also the corresponding result about equality of the moduli of families of paths in the hyperbolic and Euclidean metrics and measures [19, Remark 5.2]. On this occasion we also mention on [24, Remark 1], where the notion of the modulus of families of paths is given in some another (equivalent) way.

Put $\delta \in (0, r_1)$ and set $\Gamma_n^\delta := \bigcup\limits_{r \in (r_n, \delta)} g(\widetilde{S}(p_0, r) \cap d_1)$, where the union must be understood not in the theoretical-set sense, but namely as a family of paths "from r_n to δ". By (2.13) and due to (2.10) it follows that

$$M(\Gamma_n^{\delta})\rightarrow\infty\,,\quad n\rightarrow \infty. \tag{2.14}$$

3) Let us prove by contradiction, i.e., assume that g has no a limit as $p\rightarrow P$. Then we may find at least two sequences $p_n, p_n^{\prime}\in d_n$, $n=1,2,\ldots,$ and two points $y\ne y_*$, $y, y_*\in R$ such that $g(p_n)\rightarrow y$ and $g(p_n^{\prime})\rightarrow y_*$ as $n\rightarrow\infty$. Join the points p_n and $p_n^{\prime}$ by a path γ_n in a domain d_n. Let $r_0:=|y-y_*|$ and $U_0:=B(y, r_0/2)$. Observe that, a boundary of R is strongly accessible since R has a finite number of components and is a finitely connected on the boundary (see, e.g., [13, Theorem 6.2 and Corollary 6.8]). Therefore, for a neighborhood U_0 of y there is an another neighborhood $V\subset U_0$ of this point, a compactum $K\subset R$ and a number $\delta>0$ such that the relation

$$M(\Gamma(E, K, R))\geqslant \delta \tag{2.15}$$

holds for any continuum $E\subset R$, $E\cap \partial U\ne\varnothing\ne E\cap \partial V.$

Let $C_n:=g(|\gamma_n|)$. Observe that, $\partial U\cap |C_n|\ne\varnothing$ for sufficiently large $n\in {\Bbb N}$, see [9, Theorem 1.I.5, § 46] (as usually, $|C_n|$ denotes the locus of a path C_n). Then, by the condition (2.15) we obtain that

$$M(\Gamma(|C_n|, K, R))\geqslant \delta \tag{2.16}$$

for sufficiently large n.

Let us to show that

$$\Gamma(|C_n|, K, R)>\Gamma(g(\sigma_n), K, R) \tag{2.17}$$

for sufficiently large $n\in {\Bbb N}$, where σ_n denotes the cut of D, which corresponds to a domain d_n. Due to the relation (2.9) and item 1) of Lemma 2.2, $C(\widetilde{f}, P)=C(f, P)=\bigcap\limits_{k=1}^{\infty}\overline{f(d_k)}\subset \partial D_*$, therefore, $f(d_n)\cap K^*=\varnothing$ for sufficiently large $n\in {\Bbb N}$ and any compactum $K^*\subset D_*$. In this case, under some $n_0\in {\Bbb N}$ and all $n>n_0$, we obtain that $f(d_n)\cap H^{-1}(K)=\varnothing$. Since H is a homeomorphism and $\widetilde{f}(d_n)=f(d_n)$ for such $n\in {\Bbb N}$, it follows that $g(d_n)\cap K=\varnothing$, $g=H\circ\widetilde{f}$. Let now $\gamma\in \Gamma(|C_n|, K, R)$. Since $|C_n|\subset g(d_n)$, by the proven above $|\gamma|\cap g(d_n)\ne\varnothing\ne |\gamma|\cap ({\Bbb C}\setminus g(d_n))$. In this case, by [9, Theorem 1.I.5, § 46]

$$|\gamma|\cap (\partial g(d_n)\cap R)\ne\varnothing\,. \tag{2.18}$$

Let us now establish that

$$\partial g(d_n)\cap R\subset g(\sigma_n)\,. \tag{2.19}$$

First of all, by (2.18) it follows that $\partial g(d_n)\cap R\neq\varnothing$. Let $\zeta_0\in\partial g(d_n)\cap R$. Then we may find a sequence $\zeta_k\in g(d_n)$ such that $\zeta_k\to\zeta_0$ as $k\to\infty$. Since $\zeta_k\in g(d_n)$, we may find $\xi_k\in d_n$ such that $g(\xi_k)=\zeta_k$. Since by the assumption $\overline{D}$ is a compactum in $\mathbb{S}$, we may consider that ξ_k is a convergent sequence $\xi_k\to\xi_0\in\overline{d_n}$, $k\to\infty$. If $\xi_0\in d_n$, then ζ_0 is an inner point of $g(d_n)$ by the openness of g, that contradicts the choice of ζ_0. Thus $\xi_0\in\partial d_n$. Observe also that, $\xi_0\in D$. Indeed, if $\xi_0\in\partial D$, then $f(\xi_k)=\widetilde{f}(\xi_k)$ may converge to the boundary point of D_* by the closeness of f, however, $f(\xi_k)=H^{-1}(g(\xi_k))=H^{-1}(\zeta_k)$ converges to an inner point $H^{-1}(\zeta_0)\in D_*$, because $\zeta_k\to\zeta_0\in R$ as $k\to\infty$ and H is a homeomorphism of U^* onto R. The contradiction obtained above shows that $\xi_0\in\partial d_n\cap D$, i.e., $\xi_0\in\sigma_n$. Then $g(\xi_0)=\zeta_0\in g(\sigma_n)$. Then the inclusion (2.19) is established.

Then, by (2.18) it follows that $|\gamma|\cap g(\sigma_n)\neq\varnothing$. Therefore, the relation (2.17) is also proved. By (2.17) and due to [3, Theorem 1(c)] it follows that $M(\Gamma(|C_n|,K,R))\leqslant M(\Gamma(g(\sigma_n),K,R))$. But now, by (2.16) is also follows that

$$M(\Gamma(g(\sigma_n),K,R))\geqslant\delta\,,\quad n\geqslant n_0\,,\tag{2.20}$$

see Figure 1 on this occasion.

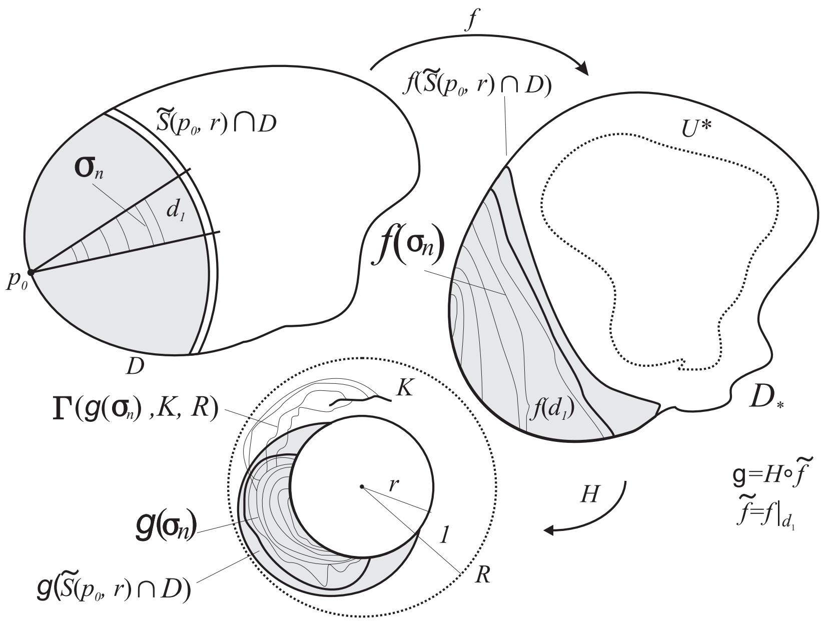

Figure 1: To the proof of Theorem 2.1

4) Let us to show that the condition (2.20) contradicts with (2.14). For this goal, let us to estimate $M(\Gamma_i^{\delta})$ in (2.14) from above, using the Ziemer equality and the connection between joining and separating paths [25]. Let us show that, there exists $\varepsilon_1 > 0$ such that

$$K \cap \overline{g(\widetilde{B}(p_0, r) \cap d_1)} = \varnothing \quad \forall\, r \in (0, \varepsilon_1)\,. \tag{2.21}$$

Assume the contrary. Then, for any $k \in \mathbb{N}$ there is some $q_k \in K \cap \overline{g(\widetilde{B}(p_0, 1/k) \cap d_1)}$. Since $q_k \in \overline{g(\widetilde{B}(p_0, 1/k) \cap d_1)}$, there is a sequence $q_{kl} \in g(\widetilde{B}(p_0, 1/k) \cap d_1)$ such that $q_{kl} \to q_k$ as $l \to \infty$. Since $q_{kl} \in g(\widetilde{B}(p_0, 1/k) \cap d_1)$, there is a sequence $\zeta_{kl} \in \widetilde{B}(p_0, 1/k) \cap d_1$ such that $g(\zeta_{kl}) = q_{kl}$. Put $k \in \mathbb{N}$ and choose $l_k > 0$ such that $|q_{kl_k} - q_k| < 1/2^k$. Without loss of generality, we may assume that q_k converges to z_0 as $k \to \infty$. Then by the triangle inequality

$$|g(\zeta_{kl_k}) - z_0| \leqslant |g(\zeta_{kl_k}) - q_k| + |q_k - z_0| =$$

$$= |q_{kl_k} - q_k| + |q_k - z_0| < 1/2^k + |q_k - z_0| \to 0\,, \quad k \to \infty\,. \tag{2.22}$$

It follows from (2.22) that $z_0 \in \partial R$. Indeed, if $z_0 \in R$, by the continuously of H^{-1} we obtain that

$$\widetilde{h}_*(H^{-1}(g(\zeta_{kl_k})), H^{-1}(z_0)) = \widetilde{h}_*(f(\zeta_{kl_k}), H^{-1}(z_0)) \to 0\,, \quad k \to \infty\,,$$

where $\widetilde{h}_*$ is a corresponding metric on $\mathbb{S}_*$. The last contradicts with the closeness of f in D, because $H^{-1}(z_0) \in C(f, p_0) \subset \partial D_*$ (see [18, Proposition 4.3]). At the same time, $H^{-1}(z_0)$ is an inner point of D_* by the assumption $z_0 \in R$. Thus, $z_0 \in \partial R$, that contradicts with the condition $q_k \to z_0 \in \partial R$ as $k \to \infty$ and because $q_k \in K$, where K is a compactum in R. The obtained contradiction proves (2.21). Then

$$K \subset R \setminus \overline{g(\widetilde{B}(p_0, r) \cap d_1)}\,, \quad r \in (0, \varepsilon_1)\,. \tag{2.23}$$

In particular, (2.23) implies that K and $g(\sigma_n)$ are disjoint for $n > n_1 > n_0$, where $n_1 \in \mathbb{N}$ is such that $r_{n_1} < \varepsilon_1$.

5) Let $n > n_1$. Observe that, for any $r \in (r_n, \varepsilon_1)$, the set $A_r := \partial(g(\widetilde{B}(p_0, r) \cap d_1)) \cap R$ separates K from $g(\sigma_n)$ in R. Indeed,

$$R = B_r \cup A_r \cup C_r \qquad \forall \quad r \in (r_n, \varepsilon_1)\,,$$

where $B_r := g(\widetilde{B}(p_0, r) \cap d_1)$ and $C_r := R \setminus \overline{g(\widetilde{B}(p_0, r) \cap d_1)}$ are open sets in R, $g(\sigma_n) \subset B_r$, $K \subset C_r$ and A_r is closed in R.

Let Σ_n be a family of all sets separating $g(\sigma_n)$ from K in R. Let us establish that

$$(\partial g(\widetilde{B}(p_0, r) \cap d_1)) \cap R \subset g(\widetilde{S}(p_0, r) \cap d_1), \quad 0 < r < r_1\,. \tag{2.24}$$

Indeed, let $\zeta_0 \in (\partial g(\widetilde{B}(p_0, r) \cap d_1)) \cap R$. Then there is a sequence $\zeta_k \in g(\widetilde{B}(p_0, r) \cap d_1)$ such that $\zeta_k \to \zeta_0$ as $k \to \infty$, where $\zeta_k = g(\xi_k)$, $\xi_k \in \widetilde{B}(p_0, r) \cap d_1$. Without loss of generality, we may assume that $\xi_k \to \xi_0$ as $k \to \infty$. If $\xi_0 \in \partial D$, then by the closeness of f in D the sequence $f(\xi_k)$ may converge only to some boundary point $z_1 \in D_*$, however, $H^{-1}(g(\xi_k)) = f(\xi_k)$ converges to some inner point of D_* because H is a homeomorphism, $\zeta_k = g(\xi_k)$ and $\zeta_k \to \zeta_0 \in R$ as $k \to \infty$. Therefore, $\xi_0 \in D$. If $\xi_0 \in \partial d_1$, then $\xi_0 \in \sigma_1 \subset \widetilde{S}(p_0, r_1)$, that is impossible because $\xi_k \in \widetilde{B}(p_0, r)$ by the assumption, $\xi_k \to \xi_0$ as $k \to \infty$ and $r < r_1$. Thus, $\xi_0 \in d_1$.

Now two situations are possible: 1) $\xi_0 \in \widetilde{B}(p_0, r) \cap d_1$ and 2) $\xi_0 \in \widetilde{S}(p_0, r) \cap d_1$. Observe that, the case 1) is impossible because, in this case, $g(\xi_0) = \zeta_0$ and ζ_0 is an inner point of the set $g(\widetilde{B}(p_0, r) \cap d_1)$, that contradicts with the choice of ζ_0. Thus, the inclusion (2.24) is established.

Here and below the unions of the form $\bigcup\limits_{r \in (r_1, r_2)} \partial g(\widetilde{B}(p_0, r) \cap d_1) \cap R$ are understood as families of sets. Denote by Σ_n the family of all sets which separate K from $g(\sigma_n)$ in R (see [25, section 2.3]). Then, by (2.24), we obtain that

$$M(\Sigma_n) \geqslant M\left(\bigcup_{r \in (r_n, \varepsilon_1)} \partial g(\widetilde{B}(p_0, r) \cap d_1) \cap R\right) \geqslant$$

$$\geqslant M\left(\bigcup_{r \in (r_n, \varepsilon_1)} g(\widetilde{S}(p_0, r) \cap d_1)\right) \qquad (2.25)$$

for $n > n_1$, where n_1 is defined in item **4)**.

By (2.25) and (2.14), putting $\delta = \varepsilon_1$, we obtain that

$$M(\Sigma_n) \to \infty\,, \qquad n \to \infty\,. \qquad (2.26)$$

On the other hand, by the Ziemer and Hesse equalities (see [25, Theorem 3.10] and [6, Theorem 5.5]), we obtain that

$$M(\Sigma_n) = \frac{1}{M(\Gamma(g(\sigma_n), K, R))}\,. \qquad (2.27)$$

Now, the relations (2.27) and (2.26) imply that

$$M(\Gamma(g(\sigma_n), K, R)) \to 0\,, \qquad n \to \infty\,,$$

that contradicts with (2.20). The contradiction obtained above proves the statement of the theorem in the case, when the relations (2.10) hold.

Let us consider the case 2), namely, assume that $Q \in FMO(\partial D)$. Let $\varphi : \mathbb{S} \to \mathbb{R}$, $\varphi(x) = 0$ for $x \notin D$, be a nonnegative function which has a finite mean oscillation at a point $p_0 \in \overline{D} \subset \mathbb{S}$. By [1, Theorem 7.2.2] a surface $\mathbb{S}$ is locally 2-regular by Alhfors, so that by [20, Lemma 3]

$$\int\limits_{\varepsilon<\widetilde{h}(p,p_0)<\widetilde{\varepsilon_0}} \frac{\varphi(p)\, d\widetilde{v}(p)}{\left(\widetilde{h}(p,p_0)\log\frac{1}{\widetilde{h}(p,p_0)}\right)^2} = O\left(\log\log\frac{1}{\varepsilon}\right) \tag{2.28}$$

as $\varepsilon \to 0$ for some $0 < \widetilde{\varepsilon_0} < {\rm dist}(p_0, \partial U)$, where U is some normal neighborhood of p_0. Set $0 < \psi(t) = \frac{1}{\left(t\,\log\frac{1}{t}\right)}$. Observe that $\psi(t) \geqslant \frac{1}{t\,\log\frac{1}{t}}$ for sufficiently small $\varepsilon > 0$, therefore $I(\varepsilon, \widetilde{\varepsilon_0}) := \int\limits_{\varepsilon}^{\widetilde{\varepsilon_0}} \psi(t)\,dt \geqslant \log\frac{\log\frac{1}{\varepsilon}}{\log\frac{1}{\widetilde{\varepsilon_0}}}$. Set $\eta(t) := \psi(t)/I(\varepsilon, \widetilde{\varepsilon_0})$. Then, due to the relation (2.28), we may find a constant $C > 0$ such that

$$\int\limits_{\widetilde{A}(p_0,\varepsilon,\widetilde{\varepsilon_0})} Q(p)\cdot\eta^2(\widetilde{h}(p,p_0))\ d\widetilde{v}(p) =$$

$$= \frac{1}{I^2(\varepsilon,\widetilde{\varepsilon_0})}\int\limits_{\varepsilon<\widetilde{h}(p,p_0)<\widetilde{\varepsilon_0}} \frac{Q(p)\, d\widetilde{v}(p)}{\left(\widetilde{h}(p,p_0)\log\frac{1}{\widetilde{h}(p,p_0)}\right)^2} \leqslant$$

$$\leqslant C\cdot\left(\log\frac{\log\frac{1}{\varepsilon}}{\log\frac{1}{\widetilde{\varepsilon_0}}}\right)^{-1} \to 0\,, \quad \varepsilon \to 0\,. \tag{2.29}$$

Then the relations (2.4) and (2.29) imply the conditions (1.2), so that the desired conclusion follows directly by Theorem 2.1.

To complete the proof we may to establish the equality $f(\overline{D}_P) = \overline{R}$. Obviously, $f(\overline{D}_P) \subset \overline{R}$. Let us to show the inverse inclusion. Let $\zeta_0 \in \overline{R}$. If ζ_0 is an inner point of R, then obviously there is $\xi_0 \in D$ such that $f(\xi_0) = \zeta_0$ and, consequently, $\zeta_0 \in f(D)$. Let now $\zeta_0 \in \partial R$. Then there is a sequence $\zeta_n \in R$, $\zeta_n = f(\xi_n)$, $\xi_n \in D$, such that $\zeta_n \to \zeta_0$ as $n \to \infty$. Since $\overline{D}_P$ is a compactum, see item 2) of Proposition 2.2, we may consider that $\xi_n \to P_0$, where P_0 is some prime end in $\overline{D}_P$. Then also $\zeta_0 \in f(\overline{D}_P)$. The inclusion $\overline{R} \subset f(\overline{D}_P)$ is proved and, therefore, $f(\overline{D}_P) = \overline{R}$. Theorem is proved. $\square$

Proof of Theorem 1.1. Observe that, $N(f, D) < \infty$ (see [22, Theorem 5.5]), because f is an open, discrete and closed mapping. Note also that, an open discrete mapping $f : D \to D_*$ with a finite distortion for which $N(f, D) < \infty$,

$$N(f, D) = \sup\limits_{y\in\mathbb{S}_*} N(y, f, D)\,,$$

$$N(y, f, D) = \text{card}\,\{p \in E : f(p) = y\}\ ,$$

is a lower Q-mapping at any point $p_0 \in \overline{D}$ for $Q(p) = c \cdot N(f, D) \cdot K_f(p)$, where $c > 0$ is some constant depending only on p_0 and D_*, and K_f is defined in (1.1). In this case, the desired conclusion follows from Theorem 2.1. □

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