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Conformable fractional derivative in commutative algebras

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Abstract. In this paper, some analog of the conformable fractional derivative is defined in an arbitrary finite-dimensional commutative associative algebra. Functions taking values in the indicated algebras and having derivatives in the sense of a conformable fractional derivative are called φ-monogenic. It is established a relation between the concepts of φ-monogenic and monogenic function in such algebras. We also propose two new definitions of the fractional derivative of functions with values in finite-dimensional commutative associative algebras.

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1. Introduction

The idea of fractional derivative was first raised by L'Hospital in 1695. After introducing this idea, many new definitions have been formulated. The most well-known ones are Riemann–Liouville, Caputo, Hadamard, Riesz, Grünwald–Letnikov, Marchaud, etc. (see e. g., [1,2] and references therein).

Recently, Khalil et al. introduced a new definition of fractional derivative called the conformable fractional derivative [3]. Unlike other definitions, this new definition satisfies the formulas of derivative of product and quotient of two functions and has a simpler chain rule than other definitions. In addition to the conformable fractional derivative definition, the conformable integral definition, Rolle theorem, and Mean value theorem for conformable fractional differentiable functions were given in literature. In [4], Abdeljawad improves this new theory. For instance, definitions of left and right conformable fractional derivatives and fractional

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integrals of higher order (i.e. of order $\alpha > 1$), fractional power series expansion, fractional Laplace transform definition, fractional integration by parts formulas, chain rule and Gronwall inequality are provided by Abdeljawad.

In the paper [5] the conformable partial derivative of the order $\alpha \in (0, 1]$ of several real variables and conformable gradient vector are defined.

In [6], two new results on homogeneous functions involving their conformable partial derivatives are introduced, specifically, the homogeneity of the conformable partial derivatives of a homogeneous function and the conformable version of Euler's Theorem.

In the paper [7] it is present a new general definition of local fractional derivative, that depends on an unknown kernel. It is establish a relation between this new concept and ordinary differentiation. Using such a formula, most of the fundamental properties of the fractional derivative can be derived directly.

In [8–12] a theory of fractional analytic functions in the conformable sense is developed. Namely, in [8] a fractional Cauchy like theorem and a fractional Cauchy like formula for fractional analytic functions are established.

In the paper [11], some interesting results of real fractional Calculus are extended to the context of the complex-valued functions of a real variable. Finally, using all obtained results, the complex conformable integral is defined, and some of its most important properties are established. In [12], the concept of fractional contour integral has also been developed. There is propose and prove some new results on complex fractional integration, and it is establish necessary and sufficient conditions for a continuous function to have antiderivative in the conformable sense. Finally, in [12], some of the well-known Cauchy's integral theorems will also be the subject of the extension that we do in this paper.

Independently of previous authors, in other papers the conformable fractional derivative of order α is defined in complex plane. It is proposed analog of Cauchy–Riemann conditions for α-differentiable functions. Moreover, a discuss about two complex conformable differential equations and solutions with their Riemann surfaces are given.

In short time, many studies about the theory and applications of the fractional differential equations which are based on conformable fractional derivative were conducted in many papers. See, for example, [13–20].

The next natural step is to generalize the concept of a conformable fractional derivative to the case of any multidimensional algebra, and first of all, to commutative and associative algebras.

2. Conformable fractional derivative and α-analytic functions

Definition 2.1. [3] *For a given a function $f : [0,\infty) \to \mathbb{R}$, the conformable fractional derivative of f of order α is defined by*

$$(T_\alpha f)(t) := \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{2.1}$$

for all $t > 0$, $0 < \alpha \leq 1$. If f is α-differentiable in some $(0,b)$, $b > 0$, and $\lim\limits_{t\to 0+0}(T_\alpha f)(t)$ exist, then it is defined as

$$(T_\alpha f)(0) := \lim_{t\to 0+0}(T_\alpha f)(t).$$

See papers [3,4,11,15] for derivative properties.

Now consider the definition of α-differentiation in the complex plane.

Definition 2.2. [8] *A complex function f is called conformable fractional differentiable (or α-analytic) at a point $z \in \mathbb{C}$ if there exists the following limit*

$$(T_\alpha f)(z) := \lim_{\varepsilon\to 0} \frac{f(z+\varepsilon z^{1-\alpha}) - f(z)}{\varepsilon} \tag{2.2}$$

for all z, and $0 < \alpha < 1$. The value $(T_\alpha f)(z)$ is called α-derivative. If f is α-analytic in an open set U, and $\lim\limits_{z\to 0}(T_\alpha f)(z)$ exists, then define $(T_\alpha f)(0) := \lim\limits_{z\to 0}(T_\alpha f)(z)$.

Example 2.1. Let $f(z) = z^2$ and $\alpha = \frac{1}{2}$. Then

$$T_{1/2}(z^2) = \lim_{\varepsilon\to 0} \frac{\left(z+\varepsilon z^{1-1/2}\right)^2 - z^2}{\varepsilon} = 2z^{3/2}.$$

It it obvious that $T_{1/2}(z^2)$ is holomorphic outside some cut connecting the point 0 and ∞.

Remark 2.1. If the function $f(z)$ is holomorphic on $\mathbb{C}$ then conformable fractional derivative $T_\alpha f(z)$, generally speaking, is not holomorphic function on $\mathbb{C}$ (but holomorphic outside some cut of the complex plane).

The following theorem can be found in [8].

Theorem 2.1. *Let $\alpha \in (0,1]$, and f, g be α-analytic at a point z_0. Then*
1. $T_\alpha\left(c_1 f(z) + c_2 g(z)\right) = c_1 T_\alpha f(z) + c_2 T_\alpha g(z)$ *for all* $c_1, c_2 \in \mathbb{C}$*;*
2. $T_\alpha(z^c) = cz^{c-\alpha}$ *for all* $c \in \mathbb{C}$*;*
3. $T_\alpha(\mu) = 0$ *for all constant functions* $f(z) = \mu$*;*
4. $T_\alpha\left(f(z)g(z)\right) = f(z)T_\alpha g(z) + g(z)T_\alpha f(z)$*;*
5. $T_\alpha\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)T_\alpha f(z) - f(z)T_\alpha g(z)}{g^2(z)}$.
6. If, in additional, f is analytic, then $T_\alpha f(z)|_{z=z_0} = z_0^{1-\alpha} f'(z_0)$.

Complex conformable fractional derivative of certain complex functions are as follows:

$$T_\alpha\left(e^{cz}\right) = cz^{1-\alpha}e^{cz}, \quad c \in \mathbb{C};$$

$$T_\alpha(\sin cz) = cz^{1-\alpha}\cos cz, \quad c \in \mathbb{C};$$

$$T_\alpha(\cos cz) = -cz^{1-\alpha}\sin cz, \quad c \in \mathbb{C};$$

$$T_\alpha\left(\frac{1}{\alpha}z^\alpha\right) = 1.$$

For more results of α-analytic functions in the sense of conformable fractional derivative see [8, 9, 11, 12].

3. Monogenic functions in commutative associative algebras

Let $\mathbb{A}$ be an arbitrary n-dimensional $(1 \le n < \infty)$ commutative associative algebra with unit over the field of complex number $\mathbb{C}$. E. Cartan [21, p. 33] proved that in $\mathbb{A}$ there exist a basis $\{I_k\}_{k=1}^n$ such that the first m basis vectors $I_1, I_2, \dots, I_m$ are idempotents and another vectors $I_{m+1}, I_{m+2}, \dots, I_n$ are nilpotents. The element $1 = I_1 + I_2 + \cdots + I_m$ is the unit of $\mathbb{A}$.

In the algebra $\mathbb{A}$ we consider the vectors $e_1, e_2, \dots, e_d$, $2 \le d \le 2n$. Let these vectors have the following decomposition in the basis of the algebra:

$$e_j = \sum_{r=1}^{n} a_{jr}\, I_r\,, \quad a_{jr} \in \mathbb{C}, \quad j = 1, 2, \dots, d. \tag{3.1}$$

Throughout this paper, we will assume that at least one of the vectors $e_1, e_2, \dots, e_d$ is invertible.

For the element $\zeta = x_1e_1 + x_2e_2 + \cdots + x_de_d$, where $x_1, x_2, \dots, x_d \in \mathbb{R}$, the complex numbers

$$\xi_u := x_1a_{1u} + x_2a_{2u} + \cdots + x_da_{du}, \qquad u = 1, 2, \dots, m$$

forms the spectrum of the point ζ.

Consider in the algebra $\mathbb{A}$ a linear span

$$E_d := \{\zeta = x_1e_1 + x_2e_2 + \cdots + x_de_d : \; x_1, x_2, \dots, x_d \in \mathbb{R}\}$$

generated by the vectors $e_1, e_2, \dots, e_d$ of $\mathbb{A}$.

Next, the assumption is essential: for each fixed $u \in \{1, 2, \dots, m\}$ at least one of the numbers $a_{1u}, a_{2u}, \dots, a_{du}$ belongs to $\mathbb{C} \setminus \mathbb{R}$.

We identify a domain S in the space $\mathbb{R}^d$ with the domain

$$S := \{\zeta = x_1e_1 + x_2e_2 + \cdots + x_de_d : (x_1, x_2, \dots, x_d) \in S\} \text{ in } E_d \subset \mathbb{A}.$$

Definition 3.1. [22] *We will call a continuous function* $\Phi : \Omega \to \mathbb{A}$ *monogenic in a domain* $\Omega \subset E_d$ *if* Φ *is differentiable in the sense of Gâteaux at every point of this domain, that is, if for each* $\zeta \in \Omega$ *there exists an element* $\Phi'_G(\zeta) \in \mathbb{A}$ *such that the equality*

$$\lim_{\varepsilon \to 0+0} \frac{\Phi(\zeta + \varepsilon h) - \Phi(\zeta)}{\varepsilon} = h\Phi'_G(\zeta) \quad \forall\, h \in E_d \tag{3.2}$$

holds. The element $\Phi'_G(\zeta)$ *is called the Gâteaux derivative of the function* Φ *at the point* ζ*.*

Consider the decomposition of the function $\Phi : \Omega \to \mathbb{A}$ in the basis $\{I_k\}_{k=1}^n$:

$$\Phi(\zeta) = \sum_{k=1}^{n} U_k(x_1, x_2, \ldots, x_d)\, I_k\,. \tag{3.3}$$

In the case where the functions $U_k : \Omega \to \mathbb{C}$ are $\mathbb{R}$-differentiable in the domain Ω, that is, for an arbitrary $(x_1, x_2, \ldots, x_d) \in \Omega$,

$$U_k\,(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_d + \Delta x_d) - U_k(x_1, x_2, \ldots, x_d)$$

$$= \sum_{j=1}^{d} \frac{\partial U_k}{\partial x_j}\,\Delta x_j + o\left(\sqrt{\sum_{j=1}^{d} (\Delta x_j)^2}\right), \qquad \sum_{j=1}^{d} (\Delta x_j)^2 \to 0\,,$$

the function Φ is monogenic in the domain Ω if and only if the following analogues of the Cauchy–Riemann conditions are fulfilled at each point of the domain Ω:

$$\frac{\partial \Phi}{\partial x_j}\, e_1 = \frac{\partial \Phi}{\partial x_1}\, e_j \qquad \text{for all} \quad j = 2, 3, \ldots, d.$$

Note that the decomposition of the resolvent has the form [23]:

$$(te_1 - \zeta)^{-1} = \sum_{u=1}^{m} \frac{1}{t - \xi_u}\, I_u + \sum_{s=m+1}^{n} \sum_{r=2}^{s-m+1} \frac{Q_{r,s}}{(t - \xi_{u_s})^r}\, I_s\,, \tag{3.4}$$

$$\forall\, t \in \mathbb{C} :\, t \neq \xi_u\,, \quad u = 1, 2, \ldots, m,$$

where the coefficients $Q_{r,s}$ are determined by the following recurrence relations:

$$Q_{2,s} = \xi_s\,, \quad Q_{r,s} = \sum_{q=r+m-2}^{s-1} Q_{r-1,q}\, B_{q,\,s}\,, \quad r = 3, 4, \ldots, s - m + 1,$$

$$B_{q,s} := \sum_{p=m+1}^{s-1} \xi_p \Upsilon_{q,s}^p\,, \quad p = m+2, m+3, \ldots, n,$$

with the structure constants $\Upsilon_{r,p}^s \in \mathbb{C}$ that defined by the equality $I_r I_s = \sum\limits_p \Upsilon_{r,p}^s I_p$ and the natural numbers u_s are defined by following rule:

for any natural $m+1 \leq s \leq n$ there exist a unique natural $1 \leq u_s \leq m$ such that for all natural $1 \leq r \leq m$:

$$I_r I_s = \begin{cases} 0 \;\text{ if }\; r \neq u_s\,, \\ I_s \;\text{ if }\; r = u_s\,. \end{cases}$$

It follows from relations (3.4) that the points $(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ corresponding to the noninvertible elements $\zeta = \sum\limits_{j=1}^{d} x_j\, e_j$ form the set

$$L_u: \quad \begin{cases} x_1 \operatorname{Re} a_{1u} + x_2 \operatorname{Re} a_{2u} + \cdots + x_d \operatorname{Re} a_{du} = 0, \\ x_1 \operatorname{Im} a_{1u} + x_2 \operatorname{Im} a_{2u} + \cdots + x_d \operatorname{Im} a_{du} = 0, \end{cases} \qquad u = 1, 2, \ldots, m$$

in the d-dimensional space $\mathbb{R}^d$.

We say that a domain $\Omega \subset E_d$ is *convex with respect to the set of directions* L_u if Ω contains the segment $\{\zeta_1 + \alpha(\zeta_2 - \zeta_1) : \alpha \in [0,1]\}$ for all $\zeta_1, \zeta_2 \in \Omega$ such that $\zeta_2 - \zeta_1 \in L_u$.

Denote

$$D_u := \{\xi_u = x_1 a_{1u} + x_2 a_{2u} + \cdots + x_d a_{du} \in \mathbb{C} : \, \zeta \in \Omega\}, \quad u = 1, 2, \ldots, m.$$

In the next theorem we present a constructive description of monogenic functions with values in the algebra $\mathbb{A}$ via holomorphic functions of a complex variable.

Theorem 3.1. [23, 24] *Let a domain $\Omega \subset E_d$ be convex with respect to the set of directions L_u, $u = 1, 2, \ldots, m$, and let for all $u = 1, 2, \ldots, m$ at least one of the numbers $a_{1u}, a_{2u}, \ldots, a_{du}$ belong to $\mathbb{C} \setminus \mathbb{R}$. Then every monogenic function $\Phi : \Omega \to \mathbb{A}$ can be represented in the form*

$$\Phi(\zeta) = \sum_{u=1}^{m} I_u \frac{1}{2\pi i} \int\limits_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1}\, dt +$$

$$+ \sum_{s=m+1}^{n} I_s \frac{1}{2\pi i} \int\limits_{\Gamma_{u_s}} G_s(t)(te_1 - \zeta)^{-1}\, dt, \tag{3.5}$$

where F_u and G_s are certain holomorphic functions in the domains D_u and D_{u_s}, respectively, and Γ_q is a closed Jordan rectifiable curve in D_q which surrounds the point ξ_q and does not contain points ξ_ℓ, $\ell, q = 1, 2, \ldots, m$, $\ell \neq q$.

From representation (3.5) it follows that under the conditions of Theorem 3.1 each monogenic in the domain Ω function Φ is differentiable in a strong sense, in particular, in the sense of Lorch [25].

Definition 3.2. [25] *A function $\Phi\colon \Omega \to \mathbb{A}$ given in a domain $\Omega \subset E_d$ is called differentiable in the sense of Lorch at a point $\zeta \in \Omega$ if there exists an element $\Phi'_L(\zeta) \in \mathbb{A}$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $h \in E_d$ with $\|h\| < \delta$ the following inequality is fulfilled:*

$$\big\|\Phi(\zeta + h) - \Phi(\zeta) - h\Phi'_L(\zeta)\big\| \leq \|h\|\, \varepsilon\,. \tag{3.6}$$

The element $\Phi'_L(\zeta)$ is called the Lorch derivative of the function Φ at the point ζ.

The representation of the monogenic function Φ in form (3.5) is unique. It is proved in [23] (in $\mathbb{R}^3$ see [24]) that for every monogenic function $\Phi : \Omega \to \mathbb{A}$ in an arbitrary domain Ω, the Gâteaux r-th derivatives Φ^r_G are monogenic functions in Ω for all r.

Remark 3.1. Under the conditions of Theorem 3.1, a monogenic function $\Phi : \Omega \to \mathbb{A}$ is differentiable in the sense of Lorch in Ω.

Consider examples of representation (3.5) in some low-dimensional commutative algebras.

Example 3.1. In n-dimensional semi-simple algebra $\mathbb{A}_n$ with multiplication table

$\cdot$	I_1	I_2	$\ldots$	I_n
I_1	I_1	0	$\ldots$	0
I_2	0	I_2	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
I_n	0	0	$\ldots$	I_n

representation (3.5) of monogenic function has the form [26]:

$$\Phi(\zeta) = F_1(\xi_1)I_1 + F_2(\xi_2)I_2 + \ldots + F_n(\xi_n)I_n\,,$$

where $\zeta = \xi_1 I_1 + \xi_2 I_2 + \cdots \xi_n I_n$. In particular, in the algebra of bicomplex numbers (or commutative Segre's quaternions) $\mathbb{BC} = \{\zeta = \xi_1 I_1 + \xi_2 I_2 : \xi_1, \xi_2 \in \mathbb{C}\}$ monogenic function has the form [27]

$$\Phi(\zeta) = F_1(\xi_1)I_1 + F_2(\xi_2)I_2. \tag{3.7}$$

Example 3.2. In the biharmonic algebra $\mathbb{B}$ with the basis $\{1, \rho\}$, $\rho^2 = 0$, representation (3.5) of monogenic function has the form [28]:

$$\Phi(\zeta) = F(\xi_1) + \Big[\xi_2 F'(\xi_1) + F_0(\xi_1)\Big]\rho, \tag{3.8}$$

where $\zeta = \xi_1 + \xi_2\,\rho, \quad \xi_1, \xi_2 \in \mathbb{C}$.

Example 3.3. In 3-dimensional algebra $\mathbb{A}_3$ with two-dimensional radical and multiplication table

$\cdot$	1	ρ	ρ^2
1	1	ρ	ρ^2
ρ	ρ	ρ^2	0
ρ^2	ρ^2	0	0

representation (3.5) of monogenic function has the form [29]:

$$\Phi(\zeta) = F(\xi_1) + \Big[\xi_2 F'(\xi) + F_1(\xi_1)\Big]\rho +$$

$$+\Big[\xi_3 F'(\xi_1) + \frac{\xi_1^2}{2} F''(\xi_1) + \xi_2 F_1'(\xi_1) + F_2(\xi_1)\Big]\rho^2, \tag{3.9}$$

where $\zeta = \xi_1 + \xi_2\,\rho + \xi_3\,\rho^2, \quad \xi_1, \xi_2, \xi_3 \in \mathbb{C}$.

Example 3.4. In 3-dimensional algebra $\mathbb{A}_2$ with one-dimensional radical and multiplication table

$\cdot$	I_1	I_2	ρ
I_1	I_1	0	0
I_2	0	I_2	ρ
ρ	0	ρ	0

representation (3.5) of monogenic function has the form [29]:

$$\Phi(\zeta) = F_1(\xi_1) I_1 + F_2(\xi_2) I_2 + \Big[\xi_3 F_2'(\xi_2) + F_0(\xi_2)\Big]\rho,$$

where $\zeta = \xi_1\, I_1 + \xi_2\, I_2 + \xi_3\,\rho, \quad \xi_1, \xi_2, \xi_3 \in \mathbb{C}$.

In the paper [30] for monogenic function given in a domain of a special real subspace E_d, $2 \le d \le 2n$, of an arbitrary finite-dimensional commutative associative algebra, $\mathbb{A}$, it is obtain analogues of the Cauchy integral theorem, the Cauchy integral formula and the Morera theorem for a curvilinear integral. This result in a subspace E_3 is proved in [31]. In [32] we prove an analogue of the Cauchy integral theorem for a surface integral of hyperholomorphic functions given in a domain of three-dimensional space and taking values in the algebra $\mathbb{A}$. In the paper [33] the correspondence between a monogenic function in the algebra $\mathbb{A}$ and a finite set of monogenic functions in a special commutative associative algebra is obtained. In the work [34] it is proposed a relation between monogenic functions taking values in n-dimensional commutative associative algebra and monogenic functions taking values in a special $(n+1)$-dimensional algebra. Finally, in the work [35], the previous results are applied to the solution of the linear PDEs. Using monogenic functions given in certain sequences of commutative associative algebras with increasing dimension of these algebras, we substantiate a recurrence procedure for constructing infinite-dimensional families of solutions of any partial differential equation with constant coefficients in the form of components of the mentioned monogenic functions.

4. φ-monogenic functions in finite-dimensional commutative associative algebras

Let us consider the definition of φ-monogenic functions in an arbitrary n-dimensional $(1 \le n < \infty)$ commutative associative algebra $\mathbb{A}$ with unit over the field of complex number $\mathbb{C}$.

Definition 4.1. *Let fix a continuous function $\varphi : \Omega \to \mathbb{A}$ such that all values of which are invertible in $\Omega \subseteq \mathbb{A}$.*

We will call a continuous function $\Phi : \Omega \to \mathbb{A}$ φ-monogenic in a domain $\Omega \subseteq \mathbb{A}$ if there exists an element $\Phi'_\varphi(\zeta) \in \mathbb{A}$ such that for all $h \in \mathbb{A}$ the equality

$$\lim_{\varepsilon \to 0+0} \frac{\Phi\left(\zeta + \varepsilon h \varphi(\zeta)\right) - \Phi(\zeta)}{\varepsilon} = h\, \Phi'_\varphi(\zeta) \tag{4.1}$$

holds. The element $\Phi'_\varphi(\zeta)$ is called the φ-derivative of the function Φ at a point ζ.

Remark 4.1. If $\varphi(\zeta)=\zeta^{1-\alpha}$, then φ-derivative coincides with α-derivative.

Example 4.1. For the function $\Phi(\zeta) = \zeta^2$ we have

$$\lim_{\varepsilon \to 0+0} \frac{\Big(\zeta + \varepsilon h \varphi(\zeta)\Big)^2 - \zeta^2}{\varepsilon} = \lim_{\varepsilon \to 0+0} \left(2h\zeta\varphi(\zeta) + \varepsilon h^2 \varphi^2(\zeta)\right) = h \cdot 2\zeta\varphi(\zeta).$$

Thus, $\left(\zeta^2\right)'_\varphi = 2\zeta\varphi(\zeta)$.

Real-valued analog of the next theorem was proved in paper [7].

Theorem 4.1. *A function $\Phi : \Omega \to \mathbb{A}$ is φ-monogenic at a point $\zeta \in \Omega$ if and only if Φ is monogenic at ζ. In that case, we have the relation*

$$\Phi'_\varphi(\zeta) = \varphi(\zeta)\Phi'_G(\zeta). \tag{4.2}$$

Proof. Sufficiency. We fix a point ζ. Let a function $\Phi : \Omega \to \mathbb{A}$ is monogenic at ζ. It means that there exists an element $\Phi'_G(\zeta)$ of the algebra $\mathbb{A}$ such that for each $h \in \mathbb{A}$ the equality (3.2) holds. Since ζ is fixed, then $\varphi(\zeta)$ is an element of $\mathbb{A}$. Since equality (3.2) is true for each vector $h \in \mathbb{A}$, then it is true for the vector $h \cdot \varphi(\zeta) \in \mathbb{A}$, i.e., from (3.2) we have

$$\lim_{\varepsilon \to 0+0} \frac{\Phi(\zeta + \varepsilon h\varphi(\zeta)) - \Phi(\zeta)}{\varepsilon} = h\varphi(\zeta)\Phi'_G(\zeta). \tag{4.3}$$

Thus, by virtue of relation (4.1) a function $\Phi : \Omega \to \mathbb{A}$ is φ-monogenic at the point ζ and equality (4.2) fulfilled.

Necessity. Since a function $\Phi : \Omega \to \mathbb{A}$ is φ-monogenic at a point $\zeta \in \Omega$, then equality (4.1) is true for every direction $h \in \mathbb{A}$. Taking into account the invertibility of φ, we conclude that equality (4.1) is also true for the direction $h \cdot \big(\varphi(\zeta)\big)^{-1} \in \mathbb{A}$. Therefore, from (3.2) we have

$$\lim_{\varepsilon \to 0+0} \frac{\Phi(\zeta + \varepsilon h) - \Phi(\zeta)}{\varepsilon} = h\big(\varphi(\zeta)\big)^{-1}\Phi'_\varphi(\zeta). \tag{4.4}$$

Thus, a function $\Phi : \Omega \to \mathbb{A}$ is monogenic at the point ζ and $\Phi'_G(\zeta) = \big(\varphi(\zeta)\big)^{-1}\Phi'_\varphi(\zeta)$. □

Thus, we have, for example, $(e^\zeta)'_\varphi = \varphi(\zeta)\, e^\zeta$, $(\sin\zeta)'_\varphi = \varphi(\zeta)\cos\zeta$ etc.

In view of Remark 3.1, we have the following statement.

Corollary 4.1. *Under the conditions of Theorem 3.1, a function $\Phi : \Omega \to \mathbb{A}$ is φ-monogenic at a point $\zeta \in \Omega$ if and only if Φ is differentiable in the sense of Lorch at ζ. In that case, we have the relation*

$$\Phi'_\varphi(\zeta) = \varphi(\zeta)\Phi'_L(\zeta) = \varphi(\zeta)\Phi'_G(\zeta).$$

Remark 4.2. From equality (4.2) follows the relation

$$e_j\Phi'_\varphi(\zeta) = \varphi(\zeta)\frac{\partial \Phi}{\partial x_j}\,, \qquad j = 1, 2, \ldots, d.$$

In additional, when e_s is an invertible for some $s \in \{1, 2, \ldots, d\}$, then

$$\Phi'_\varphi(\zeta) = \varphi(\zeta)e_s^{-1}\frac{\partial \Phi}{\partial x_s}\,.$$

From Remark 4.2 follows the next properties.

Proposition 4.1. *If a function Φ is φ-monogenic and ψ-monogenic, and at least one of the vectors e_s , $s \in \{1, 2, \ldots, d\}$, is an invertible, then the following equalities are true:*

1. $\Phi'_\varphi + \Phi'_\psi = \Phi'_{\varphi+\psi}$;
2. $\Phi'_{\varphi\cdot\psi} = \varphi\Phi'_\psi = \psi\Phi'_\varphi$.

5. Alternative approaches to defining fractional differentiations in commutative associative algebras

5.1.

Suppose that e_1 is invertible, and a function Φ of a variable $\zeta = x_1e_1 + x_2e_2 + \cdots + x_de_d$, where $x_1, x_2, \ldots, x_d \in \mathbb{R}$, is monogenic. For any $\alpha \in \mathbb{R}$, we define the power function ζ^α in the algebra $\mathbb{A}$ as follows

$$\zeta^\alpha := \exp(\alpha \ln \zeta),$$

where $\ln \zeta$ are defined in the paper [25, p. 422].

Then for natural n we have the equalities

$$\Phi'_G(\zeta) = \frac{\partial \Phi}{\partial x_1}\, e_1^{-1}, \qquad \Phi''_G(\zeta) = \frac{\partial^2 \Phi}{\partial x_1^2}\, e_1^{-2}, \ldots$$

$$\Phi^{(n)}_G(\zeta) = \frac{\partial^n \Phi}{\partial x_1^n}\, e_1^{-n}, \quad \text{where} \quad e_1^{-n} := \left(e_1^{-1}\right)^n.$$

The following definition is natural.

Remark 5.1. Let $\alpha \in \mathbb{R}$. The derivative of order α of the function Φ at a point ζ is called the product

$$\Phi^{(\alpha)}(\zeta) := \frac{\partial^\alpha \Phi}{\partial x_1^\alpha} \cdot e_1^{-\alpha}, \tag{5.1}$$

where the real fractional partial derivative $\frac{\partial^\alpha \Phi}{\partial x_1^\alpha}$ defined in some sense exists at the point x_1.

We note that in relation (5.1) a real fractional partial derivative $\frac{\partial^\alpha \Phi}{\partial x_1^\alpha}$ is not defined. Considering different meanings of a real derivative $\frac{\partial^\alpha \Phi}{\partial x_1^\alpha}$, we will get different meanings for the derivative $\Phi^{(\alpha)}$.

5.2.

The following definition is based on Cauchy's idea in using the integral representation. We will use integral representation (3.5).

Remark 5.2. Let $\alpha \in \mathbb{R}$. The derivative of order α of the function Φ at a point ζ is called the product

$$\Phi^{\alpha}(\zeta) = \sum_{u=1}^{m} I_u \frac{\Gamma(\alpha+1)}{2\pi i} \int_{\Gamma_u} F_u(t) \left((te_1 - \zeta)^{-1}\right)^{\alpha+1} dt +$$

$$+ \sum_{s=m+1}^{n} I_s \frac{\Gamma(\alpha+1)}{2\pi i} \int_{\Gamma_{u_s}} G_s(t) \left((te_1 - \zeta)^{-1}\right)^{\alpha+1} dt, \tag{5.2}$$

where $\Gamma(\alpha+1)$ is the Euler's function. In this case, the integrand must be correctly defined.

Definitions (4.1), (5.1) and (5.2) are of different nature. Therefore, the question of the relations between these definitions is not simple and requires further research.

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