GENERALIZED DEFORMED OSCILLATORS IN THE FRAMEWORK OF UNIFIED $(q;\alpha,\beta,\gamma;\nu)$-DEFORMATION AND THEIR OSCILLATOR ALGEBRAS

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The aim of this paper is to review our results on the description of multiparameter deformed oscillators and their oscillator algebras. We define generalized $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebras and study their irreducible representations. The Arik–Coon oscillator with the main relation $aa^+ - qa^+a = 1$, where $q > 1$, is embedded in this framework. We have found the connection of this oscillator with the Askey q^{-1}-Hermite polynomials. We construct a family of generalized coherent states associated with these polynomials and give their explicit expression in terms of standard special functions. By means of the solution of the appropriate classical Stieltjes moment problem, we prove the property of (over)completeness of these states.

1. Introduction

The oscillator algebra plays the central role in the investigation of many physical systems. It is also useful in the theory of Lie algebra representations. The physical motivation of the study of deformed boson and fermion quanta is connected with the hope for that the deformed oscillators in nonlinear systems will play the same role as a usual oscillator in the standard quantum mechanics.

The investigation of the one-parameter deformed oscillator algebras in theoretical physics originated from the study of the dual resonance models of strong interactions [1]. The q-deformed analog of a harmonic oscillator was introduced in [2, 3].

In parallel with the one-parameter deformed commutation relations, the two-parameter (p,q)-deformation of these relations has been introduced [4, 5]. The connection of the (p,q)-deformed oscillator algebra with (p,q)-hypergeometric functions has been established in [6]. The two-parameter deformed boson algebra invariant under the quantum group SU_{q_1/q_2} ("Fibonacci" oscillator) was studied in [7].

A wide class of generalized deformed oscillator algebras studied in the literature is connected with the generalized deformed oscillators. The description of the systems of particles with continuous interpolating (Bose and Einstein) statistics, the theory of the fractional quantum Hall effect, and high-temperature superconductivity require one to deform the canonical commutation relations. The q-deformed oscillators are used widely in the molecular and nuclear spectroscopies. A considerable attention was paid to nonlinear vector coherent states (NVCSs) of f-deformed spin-orbit Hamiltonians [8]. This class includes a multiparameter generalization of the one- and two-parameter deformed oscillator algebras [8–14]. Some of them have found applications to the investigation of various physical systems.

The multiparameter deformed quantum algebras were used in work [15] to construct integrable multiparameter deformed quantum spin chains. It is natural that the increase of the number of deformation parameters makes the method of deformations more flexible. Although a multiparameter deformed quantum algebra can be mapped in some cases onto a standard one-parameter deformed algebra [16, 17], the physical results in both cases are not the same. The Hamiltonian of the electromagnetic monochromatic field in the Kerr medium [18] is embedded in the framework of a four-parameter deformed oscillator algebra [12]. This gives the complete description of the energy spectrum of this system. The most general famous examples of the multiparameter deformed oscillator algebras are the $(q;\alpha,\beta,\gamma)$- and $(q,p;\alpha,\beta,l)$-deformations of the one- and two-parameter deformed oscillator algebras [9–11].

The modified oscillator algebra [19] has found applications to the study of the integrability of the two-particle Calogero model [20]. This algebra has been generalized to the C_λ-extended oscillator algebra [21] with the hope to apply its to the construction of new integrable models. The generalized C_λ-extended oscillator algebra, namely the S_N-extended oscillator algebra supplemented with a certain projector, underlies an operator solution of the N-particle Calogero model [22]. For the same purpose, a "hybrid" model of the q-deformed and modified oscillator algebras has been proposed in [23].

To complete the cycle of these ideas, we have proposed the generalized $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra as "the synthesis" of the $(q;\alpha,\beta,\gamma)$-deformed [2, 9] and ν-modified oscillator algebras [19]. The unified form of the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra is useful, because it gives a unified approach to the well-known deformed oscillator algebras and presents new partial examples of deformed oscillators with useful properties. By means of the selection of special values of deformation parameters, we have separated a generalized deformed oscillator connected with generalized discrete Hermite II polynomials [13]. In this way, we have constructed the Barut–Girardello-type coherent states of this oscillator and have found the conditions for the $(q;\alpha,\beta,\gamma;\nu)$-deformation parameters, at which the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator approximates the usual anharmonic oscillator in the homogeneous Kerr medium. The Arik–Coon oscillator with the main relation $aa^+ - qa^+a = 1$, where $q > 1$, is embedded in this framework. We find connection of this oscillator with the Askey q^{-1}-Hermite polynomials. We construct a family of generalized coherent states associated with these polynomials and give their explicit expression in terms of standard special functions. By means of the solution of the appropriate classical Stieltjes moment problem, we prove the (over)completeness relation of these states.

2. Oscillator Algebra and Its Generalized Deformations

The oscillator algebra of a quantum harmonic oscillator is defined by the canonical commutation relations

$$[a,a^+] = 1, \quad [N,a] = -a, \quad [N,a^+] = a^+. \tag{1}$$

It allows the different types of deformations. Some of them have been called *generalized deformed oscillator algebras* [9, 24–26]. Each of them defines an algebra generated by the elements (generators) $\{\mathbf{1}, a, a^+, N\}$ and the relations

$$a^+a = f(N), \quad aa^+ = f(N+1),$$

$$[N,a] = -a, \quad [N,a^+] = a^+, \tag{2}$$

where f is called *the structure function of a deformation.* Among them, we mention the multiparameter generalization of one-parameter deformations [9–13, 21, 23, 26].

Let us recall some of them.

1. The Arik–Coon q-deformed oscillator algebra [1]

$$aa^+ - qa^+a = 1, \;\; [N,a] = -a, \;\; [N,a^+] = a^+, \;\; q \in \mathbb{R}_+,$$

$$f(n) = \frac{1-q^n}{1-q}. \tag{3}$$

2. The Biedenharn–Macfarlane q-deformed oscillator algebra [2, 3]

$$aa^+ - qa^+a = q^{-N}, \quad aa^+ - q^{-1}a^+a = q^N,$$

$$[N,a] = -a, \quad [N,a^+] = a^+,$$

$$f(n) = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad q \in \mathbb{R}_+. \tag{4}$$

3. The Chung–Chung–Nam–Um generalized $(q;\alpha,\beta)$-deformed oscillator algebra [9]

$$aa^+ - qa^+a = q^{\alpha N+\beta}, \; [N,a] = -a, \; [N,a^+] = a^+, \; q \in \mathbb{R}_+,$$

$$f(n) = \begin{cases} q^\beta \frac{q^{\alpha n}-q}{q^\alpha - q}, & \text{if } \alpha \neq 1; \\ nq^{n-1+\beta}, & \text{if } \alpha = 1, \end{cases} \tag{5}$$

where $\alpha, \beta \in \mathbb{R}$.

4. The generalized $(q;\alpha,\beta,\gamma)$-deformed oscillator algebra [10]

$$aa^+ - q^\gamma a^+a = q^{\alpha N+\beta}, \quad [N,a] = -a, \quad [N,a^+] = a^+,$$

$$f(n) = \begin{cases} q^\beta \frac{q^{\alpha n}-q^{\gamma n}}{q^\alpha - q^{\gamma n}}, & \text{if } \alpha \neq \gamma; \\ nq^{n-1+\beta}, & \text{if } \alpha = \gamma, \end{cases} \tag{6}$$

where $q \in \mathbb{R}_+, \alpha, \beta, \gamma \in \mathbb{R}$.

5. The ν-modified oscillator algebra [19, 20]

$$[a,a^+] = 1 + 2\nu K, \quad [N,a] = -a, \quad [N,a^+] = a^+,$$

$$aK = -Ka, \quad a^+K = -Ka^+, \quad K^2 = 1, \quad \nu \in \mathbb{R},$$

$$f(n) = \begin{cases} 2k+1+2\nu, & \text{if } n = 2k; \\ 2k+2, & \text{if } n = 2k+1. \end{cases} \tag{7}$$

This oscillator, as was shown in [20], is linked to the two-particle Calogero model [27].

6. The deformed C_λ-extended oscillator algebra [21] is defined by the relations

$$[a, a^+]_q \equiv aa^+ - qa^+a = H(N) + K(N)\sum_{k=0}^{\lambda-1} \nu_k P_k,$$

$$[N, a] = -a, \quad [N, a^+] = a^+,$$

$$aK = -Ka, \quad a^+K = -Ka^+, \quad K^2 = 1, \quad \nu_k \in \mathbb{R}, \tag{8}$$

where $\nu_k \in \mathbb{R}$ and $H(K)$, $K(N)$ are real analytic functions. This algebra involves the two Casimir operators $C_1 = e^{2\pi N}$ and $C_2 = \sum_{k=0}^{\lambda-1} e^{-2\pi i(N-k)/\lambda} P_k$.

7. The new $(q;\nu)$-deformed oscillator algebra [23] with the relations

$$aa^+ - qa^+a = (1+2\nu K)q^{-N}, \quad [N, a] = -a,$$

$$[N, a^+] = a^+, \quad Ka = -aK, \quad Ka^+ = -a^+K, \quad K^2 = 1,$$

$$f(n) = \Big(\frac{q^n - q^{-n}}{q - q^{-1}} + 2\nu\frac{q^n - (-1)^n q^{-n}}{q + q^{-1}}\Big) \tag{9}$$

has been defined by the combination of the Biedenharn–Macfarlane idea [2,3] of a q-deformation with the Brink–Hanson–Vasiliev idea [20] of a ν-modification of the oscillator algebra.

8. In order to complete this cycle of ideas, we consider a $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra, which is a "hybrid" of the $(q;\alpha,\beta,\gamma)$-deformed (6) and ν-modified (7) oscillator algebras or, more exactly, an oscillator algebra defined by the generators $\{I, a, a^+, N, K\}$ and relations

$$aa^+ - q^\gamma a^+a = (1+2\nu K)q^{\alpha N+\beta},$$

$$[N, a] = -a, \quad [N, a^+] = a^+, \quad Ka = -a,$$

$$Ka^+ = -a^+K, \quad [N, K] = 0, \quad N^+ = N, \quad K^+ = K, \tag{10}$$

where $q \in \mathbb{R}_+, \alpha, \beta \in \mathbb{R}, \nu \in \mathbb{R} - \{0\}$. This model unifies all deformations 1–7 of the oscillator algebra (1).

3. Generalized $(q;\alpha,\beta,\gamma;\nu)$-Deformed Oscillator Algebra and Its Simplest Properties

(a) $(q;\alpha,\beta,\gamma;\nu)$-deformed structure function. Description of a deformed oscillator algebra requires the determination of the deformation structure function $f(n)$.

Equations (2) and (10) imply the recurrence relation

$$f(n+1) - q^\gamma f(n) = \Big(1 + 2\nu(-1)^n\Big)q^{\alpha n+\beta}. \tag{11}$$

Its solution is obtained by the mathematical induction method [28]. The solution of Eq. (11) with the initial value $f(0) = 0$ is given by the formula

$$f(n) =$$

$$= \begin{cases} q^\beta\left(\frac{q^{\gamma n} - q^{\alpha n}}{q^\gamma - q^\alpha} + 2\nu\frac{q^{\gamma n} - (-1)^n q^{\alpha n}}{q^\gamma + q^\alpha}\right), & \text{if } \alpha \neq \gamma; \\ nq^{\gamma(n-1)+\beta} + 2\nu q^{\gamma(n-1)+\beta}\left(\frac{1-(-1)^n}{2}\right), & \text{if } \alpha = \gamma. \end{cases} \tag{12}$$

(b) Useful formulas. The following formulas will be useful for the study of this algebra. One of them is

$$a(a^+)^n - q^{\gamma n}(a^+)^n a = [n;\alpha,\gamma;\nu K](a^+)^{n-1}q^{\alpha N+\beta}, \tag{13}$$

where $n \geq 1$, and the other one,

$$[n;\alpha,\gamma;\nu K] =$$

$$= \begin{cases} \left(\frac{q^{\gamma n} - q^{\alpha n}}{q^\gamma - q^\alpha} + 2\nu K\frac{q^{\gamma n} - (-1)^n q^{\alpha n}}{q^\gamma + q^\alpha}\right), & \text{if } \alpha \neq \gamma; \\ nq^{\alpha(n-1)} + 2\nu K q^{\alpha(n-1)}\left(\frac{1-(-1)^n}{2}\right), & \text{if } \alpha = \gamma \end{cases}, \tag{14}$$

is deduced by the method of induction. The direct calculation leads to (13). For $[n;\alpha,\gamma;\nu K]$, the second formula gives the generating function

$$\sum_{n=0}^{\infty}[n;\alpha,\gamma;\nu K]z^n =$$

$$= \begin{cases} \frac{z}{1-q^\gamma z}\left(\frac{1}{1-q^\alpha z} + 2\nu K\frac{1}{1+q^\alpha z}\right), & \text{if } \alpha \neq \gamma; \\ \frac{z}{(1-q^\gamma z)^2} + 2\nu K\frac{z}{1-q^{2\gamma}z^2}, & \text{if } \alpha = \gamma. \end{cases} \tag{15}$$

(d) Deformed C_2-extended and $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebras.

The defining relations for a deformed C_2-extended oscillator are given by

$$aa^+ - q^\gamma a^+ a = H(N) + \nu\Big(E(N+1) + q^\gamma E(N)\Big)(P_0 - P_1),$$

$$[N, a^+] = a^+, \quad [N, P_k] = 0, \quad a^+ P_k = P_{k+1} a^+,$$

$$P_1 + P_2 = I, \quad P_k P_l = \delta_{k,l} P_l, \tag{16}$$

where $q, \nu \in \mathbb{R}$, $k, l = 1, 2$, and $E(N)$, $H(N)$ are real analytic functions. As we saw above, the deformed extended oscillator algebra C_λ involves the two Casimir operators C_1 and C_2. For the C_2-extended oscillator algebra, they have the form

$$C_1 = e^{2\pi i N}, \quad C_2 = e^{i\pi N} K. \tag{17}$$

Let us define the operator

$$\tilde{C}_3 = q^{-\gamma N}\Big(D(N) + \nu E(N) K - a^+ a\Big), \tag{18}$$

where $D(N)$, $E(N)$ are some analytic functions of N. The operator $\tilde{C}_3$ will be the Casimir operator of the oscillator algebra (16) if only one condition $[\tilde{C}_3, a] = 0$ holds. This is equivalent to the solution of the equations

$$K(N)\nu_k = \quad E(N+1)\beta_{k+1} - q^\gamma E(N)\beta_k,$$

$$H(N) = D(N+1) - q^\gamma D(N), \tag{19}$$

where $\nu_0 = -\nu_1 = \nu, \beta_0 = 0, \beta_2 = 0, \beta_1 = \nu, \quad k = 0, 1$. Substituting the solution $E(N) = 2q^{\alpha N + \beta}/(q^\gamma + q^\alpha)$ of Eq. (19) and $H(N) = q^{\alpha N + \beta}$ in (16), we obtain the commutation relations of the $(q; \alpha, \beta, \gamma; \nu)$-deformed oscillator algebra (10). Moreover, the solution

$$D(N) = \begin{cases} q^\beta \left(\frac{q^{\gamma N} - q^{\alpha N}}{q^\gamma - q^\alpha} + 2\nu \frac{q^{\gamma N}}{q^\gamma + q^\alpha}\right), & \text{if } \gamma \neq \alpha; \\ q^\beta (q^{\gamma(N-1)} N + \nu q^{-\gamma}), & \text{if } \gamma = \alpha \end{cases}$$

of the second equation in (19) gives the explicit form of the Casimir operator

$$\tilde{C}_3 =$$

$$= \begin{cases} q^{-\gamma N}\Big(\big(\frac{q^{\gamma N} - q^{\alpha N}}{q^\gamma - q^\alpha} + 2\nu \frac{q^{\gamma N} - (-1)^N q^{\alpha N}}{q^\gamma + q^\alpha}\big) q^\beta - a^+ a\Big), \\ \text{if } \alpha \neq \gamma; \\ q^{-\gamma N}\Big(N + \nu(1 + (-1)^N) q^{\gamma N + \beta} - a^+ a\Big), \\ \text{if } \alpha = \gamma. \end{cases}$$

4. Classification of Representations of Unified $(q; \alpha, \beta, \gamma; \nu)$-Deformed Oscillator Algebra

As has been shown in the previous section, the $(q; \alpha, \beta, \gamma; \nu)$-deformed oscillator algebra allows for a nontrivial center, which means that it has irreducible non-equivalent representations [29, 30]. We give a classification of these representations by a method similar to one in [31, 32].

Due to relations (10) and (17), there exists a vector $|0\rangle$ such that

$$a^+ a|0\rangle = \lambda_0 |0\rangle, \quad aa^+|0\rangle = \mu_0 |0\rangle,$$

$$N|0\rangle = \varkappa_0 |0\rangle, \quad K|0\rangle = \omega e^{-i\pi\varkappa_0}|0\rangle, \tag{20}$$

where $\langle 0|0\rangle = 1$, and ω is the value of Casimir operator C_2 in the given irreducible representation. By means of (13), we find that the vectors

$$|n\rangle' = \begin{cases} (a^+)^n |0\rangle, & \text{if } n \geq 0; \\ (a)^{-n} |0\rangle, & \text{if } n < 0 \end{cases} \tag{21}$$

are the eigenvectors of the operators $a^+ a$ and aa^+ :

$$a^+ a|n\rangle' = \lambda_n |n\rangle', \quad aa^+|n\rangle' = \mu_n |n\rangle'. \tag{22}$$

Let us define a new system of orthonormal vectors $\{|n\rangle\}_{n=-\infty}^{n=\infty}$, by

$$|n\rangle = \begin{cases} \left(\prod_{k=1}^n \lambda_k\right)^{-1/2} (a^+)^n |0\rangle, & \text{if } n \geq 0; \\ \left(\prod_{k=1}^{-n} \lambda_{n+k}\right)^{-1/2} (a)^{-n} |0\rangle, & \text{if } n < 0. \end{cases} \tag{23}$$

Then relations (10) are represented by the operators

$$a^+|n\rangle = \sqrt{\lambda_{n+1}}|n+1\rangle, \quad a|n\rangle = \sqrt{\lambda_n}|n-1\rangle,$$

$$N|n\rangle = (\varkappa_0 + n)|n\rangle, \quad K|n\rangle = \frac{(-1)^n}{2\nu} B|n\rangle, \tag{24}$$

where $B = 2\nu\omega e^{-i\pi\varkappa_0} \in \mathbb{R}$. Due to the non-negativity of the operators $a^+ a$ and aa^+, we have $\lambda_n \geq 0$ and $\mu_n \geq 0$.

From the identity $a(a^+ a)|n\rangle = (aa^+) a|n\rangle$, we find

$$\lambda_n = \mu_{n-1}. \tag{25}$$

Moreover, formula (24) yields the recurrence relation

$$\lambda_{n+1} - q^\gamma \lambda_n = \Big(1 + (-1)^n B\Big) q^{\alpha(n+\varkappa_0)+\beta}. \tag{26}$$

In view of relation (10), the solution of Eq. (26) can be represented as

$$\lambda_n =$$

$$= \begin{cases} \lambda_0 q^{\gamma n} + q^{\alpha\varkappa_0+\beta}\left(\frac{q^{\gamma n}-q^{\alpha n}}{q^\gamma - q^\alpha} + B\frac{q^{\gamma n}-(-1)^n q^{\alpha n}}{q^\gamma+q^\alpha}\right), \\ \text{if } \alpha \neq \gamma; \\ \lambda_0 q^{\gamma n} + q^{\gamma\varkappa_0+\beta}\left(nq^{\gamma(n-1)} + B\frac{1-(-1)^n}{2}\right), \\ \text{ if } \alpha = \gamma. \end{cases} \tag{27}$$

For $n = 2k$ and $n = 2k+1$, the nonnegativity of λ_n ($\gamma - \alpha \neq 0$) yields, respectively,

$$\left(\lambda_0 q^{-(\alpha\varkappa_0+\beta)} + \frac{1}{q^\gamma - q^\alpha} + \right.$$

$$\left. + \frac{B}{q^\gamma + q^\alpha}\right) \geq q^{-2(\gamma-\alpha)k}\left(\frac{1}{q^\gamma - q^\alpha} + \frac{B}{q^\gamma+q^\alpha}\right), \tag{28}$$

$$\left(\lambda_0 q^{-(\alpha\varkappa_0+\beta)} + \frac{1}{q^\gamma - q^\alpha} + \right.$$

$$\left. + \frac{B}{q^\gamma + q^\alpha}\right) \geq q^{-(\gamma-\alpha)(2k+1)}\left(\frac{1}{q^\gamma - q^\alpha} - \frac{B}{q^\gamma+q^\alpha}\right). \tag{29}$$

The representations of the generalized oscillator algebra are reduced to four classes of unitary representations:

(i) Assume $q < 1$, $\alpha = \gamma > 0$, or $q > 1$, $\alpha = \gamma > 0$, $B < 0$. The nonnegativity λ_n implies

$$\lambda_0 q^{-\gamma(\kappa_0+1)-\beta}(n + q^{-\gamma(n-1)})B\frac{1-(-1)^n}{2}) \geq 0.$$

Therefore, there exists n_0 such that $\lambda_n < 0$ for all $n < n_0$. After the possible renumbering, we assume that

$$a|0\rangle = 0, \quad \lambda_0 = 0.$$

Therefore, the representation of relations (10) is given by formula (24) with

$$\lambda_n = q^{\gamma\varkappa_0+\beta}\left(nq^{\gamma(n-1)} + B\frac{1-(-1)^n}{2}\right), \quad \forall n \geq 0.$$

(ii) Assume $\gamma - \alpha > 0$, $q > 1$ ($\gamma - \alpha < 0$, $0 < q < 1$). This implies that at least one of the numbers $\frac{1}{q^\gamma - q^\alpha} \pm \frac{B}{q^\gamma+q^\alpha}$ is positive. Due to (28) and (29), there exists n_0 such that, for all even or odd $n < n_0$, $\lambda_n < 0$. After the possible renumbering, we may assume

$$a|0\rangle = 0, \quad \lambda_0 = 0.$$

The nonnegativity condition for λ_n and $B \geq -1$ lead to the cases:

– Let $B > -1$. The representation relations (10) are given by formulae (24) with

$$\lambda_n = q^{-\alpha\varkappa_0+\beta}\left(\frac{q^{\gamma n} - q^{\alpha n}}{q^\gamma - q^\alpha} - \frac{q^{\gamma n} - (-1)^n q^{\alpha n}}{q^\gamma + q^\alpha}\right). \tag{30}$$

The arbitrary values of the parameter $\varkappa_0$, and $B > -1$ define nonequivalent infinite-dimensional representations (24) of the relation (10).

– Let $B = -1$. In this case, due to (25), $\lambda_1 = \mu_0$, and representations (24) are defined by

$$a = a^+ = 0, \quad N = \varkappa_0, \quad K = -\frac{1}{2\nu}. \tag{31}$$

(iii) Assume $q < 1$, $\gamma - \alpha > 0$, ($q > 1$, $\gamma - \alpha < 0$). From this, it follows that at least one and only one of the numbers $\frac{1}{q^\gamma - q^\alpha} \pm \frac{B}{q^\gamma + q^\alpha}$ is positive. Due to (28) and (29), there exists n_0 such that, for $n > n_0$, λ_n is negative for even and odd values of n. This implies $a^+|n\rangle = 0$ for some $n \geq n_0$. After the possible renumering, we have

$$a^+|0\rangle = 0. \tag{32}$$

This condition implies $\lambda_1 = 0$, or $\lambda_0 = -q^{\alpha\varkappa_0+\beta-\gamma}(1 + B)$. The condition $\lambda_0 \geq 0$ is equivalent to $B \leq -1$.

If $B = -1$, we obtain representation (31).

If $B < -1$, we have

$$\lambda_n = q^{\alpha\varkappa_0+\beta+\gamma n}\left(\frac{1-q^{(\alpha-\gamma)n}}{q^\gamma - q^\alpha} - q^{-\gamma}\right) +$$

$$+ B\left(\frac{1-q^{(\alpha-\gamma)n}(-1)^n}{q^\gamma + q^\alpha} - q^{-\gamma}\right) \geq 0. \tag{33}$$

The nonnegativity condition for λ_n gives a restriction for possible values of B:

- For values $B < \frac{q^\gamma + q^\alpha}{q^\gamma + q^\alpha}$, we have $\lambda_n > 0$. Therefore, the representation of (10) with λ_n given by (33) yields representations (10). The arbitrary values of parameter $\varkappa_0$ and $B < \frac{q^\gamma+q^\alpha}{q^\gamma+q^\alpha}$ distinguish an irreducible representation.
- For $B = \frac{q^\gamma+q^\alpha}{q^\gamma+q^\alpha}$, we have $\lambda_n > 0$. The vector space of this representation is a span of two-dimensional vectors

$$\begin{pmatrix} \psi_{-1} \\ \psi_0 \end{pmatrix}.$$

Due to (24), the representations are defined by

$$a=\begin{pmatrix}0 & \sqrt{\frac{2q^{\alpha\varkappa_0+\beta}}{q^\alpha-q^\gamma}}\\ 0 & 0\end{pmatrix},\ a^+=\begin{pmatrix}0 & 0\\ \sqrt{\frac{2q^{\alpha\varkappa_0+\beta}}{q^\alpha-q^\gamma}} & 0\end{pmatrix},$$

$$N=\begin{pmatrix}\chi_0-1 & 0\\ 0 & \chi_0\end{pmatrix},\ K=\frac{1}{2\nu}\frac{q^\gamma+q^\alpha}{q^\alpha-q^\gamma}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \quad (34)$$

These representations are distinguished by the arbitrary values of $\varkappa_0$ and $B=\pm\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}$, $\lambda_0=\frac{2q^{\alpha(\varkappa_0)+\beta}}{q^\alpha-q^\gamma}$.

(iv) Let $q<1, \gamma-\alpha>0$, ($q>1, \gamma-\alpha<0$), and let λ_n be defined by (27). Then the conditions that both values $\frac{1}{q^\gamma-q^\alpha}\pm\frac{B}{q^\gamma+q^\alpha}$ are nonpositive (then at least one of them must be strictly negative) lead to the cases considered in (28), (29)). There is the following possibility:

$$a)\quad \Big(\lambda_0 q^{-(\alpha\varkappa_0+\beta)}+\frac{1}{q^\gamma-q^\alpha}+\frac{B}{q^\gamma+q^\alpha}\Big)<0. \quad (35)$$

According to (28) and (29), there exists n_0 such that, for $n>n_0$, λ_n is negative for even and odd values of n. As in (ii), this yields

- $B<-\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}$. These representations of relations (10) are given by formulae (24) with λ_n (30).
- $-1<B$. These representations of relations (10) are given by formulae (31).

We have

$$b)\quad \Big(\lambda_0 q^{-(\alpha\varkappa_0+\beta)}+\frac{1}{q^\gamma-q^\alpha}+\frac{B}{q^\gamma+q^\alpha}\Big)>0. \quad (36)$$

This condition implies $\lambda_n>0, \forall n\in\mathbb{Z}$. This representation is given by formulae (24) with λ_n as (27) for $\alpha\neq\beta$ and $n\in\mathbb{Z}$;

$$c)\quad \Big(\lambda_0 q^{-(\alpha\varkappa_0+\beta)}+\frac{1}{q^\gamma-q^\alpha}+\frac{B}{q^\gamma+q^\alpha}\Big)=0; \quad (37)$$

- $|B|<-\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}$ implies $\lambda_n>0\forall n\in\mathbb{Z}$. The representations are the same as in b).
- $|B|=-\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}$ implies $\lambda_n>0\forall n\in\mathbb{Z}$. Moreover, $\lambda_n=0, \forall n=2k$. The vector space of this representation is spanned by the two-dimensional vectors

$$\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}.$$

Therefore, the representation is two-dimensional and given by the formula

$$a=\begin{pmatrix}0 & 0\\ \sqrt{\frac{q^{\alpha(\varkappa_0+1)+\beta}}{q^\gamma-q^\alpha}} & 0\end{pmatrix},\ a^+=\begin{pmatrix}0 & \sqrt{\frac{q^{\alpha(\varkappa_0+1)+\beta}}{q^\alpha-q^\gamma}}\\ 0 & 0\end{pmatrix},$$

$$N=\begin{pmatrix}\chi_0 & 0\\ 0 & \chi_0+1\end{pmatrix},\ K=\frac{1}{2\nu}\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \quad (38)$$

These representations are defined by arbitrary values of $\varkappa_0$, and $\lambda_0=0$, $B=-\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}$.

5. Generalized $(q;\alpha,\beta,\gamma;\nu)$-Deformed Oscillators and Nonlinear Quantum Optical Model

In this section, we study some aspects concerning the possible interpretation of $(q;\alpha,\beta,\gamma;\nu)$-deformed noninteracting systems describing non-deformed interacting systems. We consider an anharmonic oscillator in quantum optics defined by the Hamiltonian H to describe laser light in a nonlinear Kerr medium. In a lower order, it is of the form [18]

$$H_{\rm Kerr}=\frac{\hbar\omega_0}{2}(2N+1)+\frac{\kappa}{2}N(N-1), \quad (39)$$

where κ is the real constant related to the nonlinear susceptibility χ^3 of the Kerr medium. In the framework of the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra with the help of a corresponding choice of the deformation parameters, we shall construct operators approximating this Hamiltonian.

If $\alpha=\gamma$, we consider the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra (10) and define the corresponding Hamiltonian by

$$H=\frac{\hbar\omega_0}{2}(a^+a+aa^+), \quad (40)$$

or

$$H_N=\frac{\hbar\omega_0}{2}\Big[q^{\gamma(N-1)+\beta}\Big(N+2\nu(\frac{1-(-1)^N}{2}\Big)+$$

$$+q^{\gamma N+\beta}\Big(N+1+2\nu\frac{1-(-1)^(N+1)}{2}\Big)\Big]. \quad (41)$$

Assuming small values of γ and β in this operator, we obtain an approximation of this Hamiltonian:

$$H_N=\frac{\hbar\omega_0}{2}[2N+I+2\gamma(N-I)N+(2\gamma+2\beta+2\nu)N+$$

$$+O(\gamma^2, \beta^2, \beta\nu, \gamma\nu)]. \tag{42}$$

Comparing (39) and (42), we obtain their equivalence if

$$\gamma = \frac{\kappa}{2\hbar\omega_0}, \quad \beta + \nu = -\frac{\kappa}{2\hbar\omega_0}.$$

If $\gamma \neq \alpha \quad \nu = 0$, we consider the $(q; \alpha, \beta, \gamma; \nu)$-deformed oscillator algebra (10) and the Hamiltonian

$$H = \frac{\hbar\omega_0}{2} aa^+, \tag{43}$$

or

$$H_N = \frac{\hbar\omega_0}{2} \frac{q^{\alpha(N+1)} - q^{\gamma(N+1)}}{q^\alpha - q^\gamma}. \tag{44}$$

If we introduce the new deformation parameters $q = e, \alpha = \rho + \mu, \gamma = \rho - \mu, \beta = 0$, then Hamiltonian (44) takes the form

$$H_N = e^{\rho N} \frac{\hbar\omega_0}{2} \frac{e^{\mu(N+1)} - e^{-\mu(N+1)}}{e^\mu - e^{-\mu}} =$$

$$= e^{\rho N} \frac{\hbar\omega_0}{2} (e^{\mu N} + e^{\mu(N-1)} + \ldots + e^{-\mu(N-1)} + e^{-\mu N}). \tag{45}$$

Assuming small values of μ and ρ in this operator and using the expansion

$$e^{\mu N} = I + \mu N + \frac{\mu^2}{2} N^2 + \ldots,$$
$$\vdots$$
$$e^{-\mu N} = I - \mu N + \frac{\mu^2}{2} N^2 - \ldots,$$
$$e^{\rho N} \simeq I + \rho N + \ldots$$

we obtain

$$\frac{e^{\mu(N+I)} - e^{-\mu(N+I)}}{e^\mu - e^{-\mu}} \simeq (2N+I) + \frac{\mu^2}{2} \frac{N(N+1)(2N+1)}{6}$$

and

$$H_N = \frac{\hbar\omega_0}{2}[(2N+1) + (\frac{\mu^2}{2} + 2\rho)N(N-1) + (\frac{2}{3}\mu^2 + 3\rho)N +$$

$$+O(\rho^2, \rho\mu^2, \mu^4)]. \tag{46}$$

Comparing (39) and (46), we obtain their equivalence if

$$\mu^2 = -\frac{9}{2}\rho, \quad \rho = -\frac{2\kappa}{\hbar\omega_0}. \tag{47}$$

6. Generalized $(q, p; \alpha, \beta, l)$-Deformed Oscillator

We introduce a multiparameter generalization of the two-parameter deformed oscillator algebra [11] with $(p, q; \alpha, \beta, l)$-deformed canonical commutation relations

$$aa^+ - q^l a^+ a = p^{-\alpha N - \beta}, \quad aa^+ - p^{-l} a^+ a = q^{\alpha N + \beta},$$

$$[N, a] = -\frac{l}{\alpha} a, \quad [N, a^+] = \frac{l}{\alpha} a^+. \tag{48}$$

It is easy to see that the function $f(n)$ in this case has the form

$$f(n) = \frac{p^{-\alpha n - \beta} - q^{\alpha n + \beta}}{p^{-l} - q^l} \tag{49}$$

with α, $\beta, \in R, l \in Z$. The creation and annihilation operators a, a^+ and the operator N of relations (48) act on the Hilbert space $\mathcal{H}$ with the basis $\{|n\rangle\}, n = 0, 1, 2\ldots$ as follows:

$$a^+ |n\rangle = \left(\frac{p^{-\alpha-\beta-l} - q^{\alpha+\beta+l}}{p^{-l} - q^l}\right)^{1/2} |n + l/\alpha\rangle,$$

$$a|n\rangle = \left(\frac{p^{-\alpha-\beta} - q^{\alpha+\beta}}{p^{-l} - q^l}\right)^{1/2} |n - l/\alpha\rangle. \tag{50}$$

We define the difference operator (the Jackson derivative)

$$Df(z) = \frac{f(p^{-\alpha}z)p^{-\beta} - f(q^\alpha z)q^\beta}{(p^{-l} - q^l)z^{l/\alpha}}, \tag{51}$$

where $f(z)$ belongs to a space of functions (analytic ones if l/α is an integer). We have

$$Dz^n = \frac{z^n}{z^{l/\alpha}} \frac{p^{-\alpha n - \beta} - q^{\alpha n + \beta}}{p^{-l} - q^l} \frac{1}{(n)!} \frac{d^n z^n}{dz^n}.$$

If l/α is an integer, then, for an analytic function $f(z) = \sum_{n=0}^\infty a_n z^n$, we have

$$Df(z) = \sum_{n=1}^\infty \frac{z^n}{z^{l/\alpha}} \frac{p^{-\alpha n - \beta} - q^{\alpha n + \beta}}{p^{-l} - q^l} \frac{1}{n!} \frac{d^n}{dz^n} f(z). \tag{52}$$

In this space, we can give the "coordinate" realization of relations (48):

$$q^N : f \to q^{z\frac{d}{dz}} f = f(qz), \quad p^{-N} : f \to p^{-z\frac{d}{dz}} f = f(p^{-1}z),$$

$$a : f \to Df, \quad a^+ : f \to z^{l/\alpha} f, \quad N : f \to z\frac{d}{dz}. \tag{53}$$

Indeed, from (53), we obtain

$$Na^+f(z) = z\frac{d}{dz}(z^{l/\alpha}f(z)) = l/\alpha z^{l/\alpha}f + z^{1+l/\alpha}\frac{d}{dz}f(z),$$

$$a^+Nf(z) = l/\alpha z^{1+l/\alpha}\frac{d}{dz}f(z).$$

This yields

$$[N,a^+]f = l/\alpha\, a^+ f \quad (54)$$

and, analogously,

$$[N,a] = -l/\alpha\, a. \quad (55)$$

In a similar way, from (53), we obtain

$$a^+\, af(z) = \frac{f(p^{-\alpha}z)p^{-\beta} - f(q^{\alpha}z)q^{\beta}}{p^{-l} - q^{l}},$$

$$a\, a^+ f(z) = \frac{f(p^{-\alpha}z)p^{-l-\beta} - f(q^{\alpha}z)q^{l+\beta}}{p^{-l} - q^{l}} \quad (56)$$

the representation of relations (48).

7. Generalized $(q;a,b,c;0)$-Deformed Oscillator

The unified form of the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra is useful, because it gives a unified approach to the well-known deformed oscillator algebras 1–5 of Section 2 and new partial examples of the deformed oscillators with useful properties.

Let us consider the example of such oscillator algebra. It is convenient to introduce the new deformation parameters in (61):

$$\alpha = 2a + c - 1, \quad \beta = 2a + b, \quad \gamma = 2a. \quad (57)$$

We also assume $\nu = 0, \alpha \neq \gamma$ and obtain the commutation relations for a generalized deformed oscillator

$$[N,a] = -a, \quad [N,a^+] = a^+,$$

$$aa^+ - q^{2a}a^+a = q^{2a(N+1)+b}q'^N, \quad (58)$$

with the structure function of the deformation

$$f(n) = [n;q;a,b,c;0] = q^{2an+b}\Big(\frac{1-q^{(c-1)n}}{1-q^{(c-1)}}\Big)$$

$$= q^{2an+b}\Big(\frac{1-q'^n}{1-q'}\Big), \quad q' = q^{c-1}, \quad (59)$$

whose properties will be studied below.

8. Arik–Coon Oscillator with $q > 1$ and $(q;a,b,c;0)$-Deformation

Fixing the values of parameters in (58), we arrive at the oscillators well studied in the literature: $a = 1/2, b = -1, c = 0$ (the Arik–Coon oscillator with $q < 1$) connected with the Rogers $q-$ Hermite polynomials, $a = -1, b = 2, c = 2$ (the oscillator connected with the discrete q-Hermite II polynomials [33]). The replacement $q \to 1/q$ in (3) leads to the oscillator

$$[N,a] = -a, \quad [N,a^+] = a^+, aa^+ - q^{-1}a^+a = 1, \quad (60)$$

where $q < 1$, which is equivalent to oscillator (58), where $a = -1/2, b = 1, c = 2$, with the structure function of the deformation

$$f(n) = [n;q;-1/2,1,2;0] = q^{-n+1}\Big(\frac{1-q^n}{1-q}\Big), q < 1, \quad (61)$$

connected [34] with the Askey q^{-1}-Hermite polynomials [35].

As was shown in [34], the operator $Q = a^+ + a$ or

$$Q|n\rangle = r_n|n+1\rangle + r_{n-1}|n-1\rangle, \quad (62)$$

where

$$r_n = [n+1;q;a,b,c;0]^{1/2} = q^{-n/2}\Big(\frac{1-q^{n+1}}{1-q}\Big)^{1/2},$$

is an unbounded symmetric operator. Its closure $\bar{Q}$ is a nonself-adjoint operator and has the deficiency indices (1,1) [36]. Defining the generalized eigenfunction $Q|x\rangle = x|x\rangle$, where $|x\rangle = \sum_{n=0}^{\infty} P_n(x)|n\rangle$, we obtain the recurrence relation

$$r_{n-1}P_{n-1}(x) + r_nP_{n+1}(x) = xP_n(x). \quad (63)$$

The coefficients $P_n(x;q)$ of this equation satisfy the relation

$$xP_n(x;q) =$$

$$= q^{1/2}(1-q)^{-1/2}\Big(q^{-(n+1)}(1-q^{n+1})\Big)^{1/2}P_{n+1}(x;q) +$$

$$+ q^{1/2}(1-q)^{-1/2}\Big(q^{-n}(1-q^n)\Big)^{1/2}P_{n-1}(x;q). \quad (64)$$

Introducing the change of variables $2y = q^{-1/2}(1-$ $-q)^{1/2}x$, $\psi(x;a) = P(2q^{1/2}(1-q)^{-1/2}x)$ and

$$\psi_n(x;q) = \frac{h_n(x;q)}{q^{-n(n+1)/4}(q;q)_n^{1/2}}, \quad (65)$$

we obtain the recurrence relation

$$2xh_n(x;q) = h_{n+1}(x;q) + q^{-n}(1-q^n)h_{n-1}(x;q). \quad (66)$$

The solution of this equation with the initial conditions $h_0(x;q) = 1$, $h_1(x;q) = 2x$ is given by the $q^{-1}-$ Hermite polynomials [37]

$$h_n(x;q) = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \times$$

$$\times(-1)^k q^{k(k-n)}(x+\sqrt{x^2+1})^{n-k}. \quad (67)$$

The orthogonality relation for these polynomials is

$$\int_{-\infty}^{\infty} h_m(x;q)h_n(x;q)d\nu(x) = q^{-n(n+1)/2}(q;q)_n\delta_{m,n}. \quad (68)$$

(a) *Generalized Barut–Girardello coherent states.* By $\mathcal{H}_F$, we denote the Hilbert space spanned by the basis vectors $|n>= \psi_n(x;q), \quad n = 1,2,\ldots$ of the orthogonal polynomials (65). We consider $\mathcal{H}_F$ as the Fock space for the operators a^+, a. These operators (58) in the space $\mathcal{H}_F$ are represented as

$$a|n\rangle = r_{n-1}|n-1\rangle, a^+|n\rangle = r_n|n+1\rangle. \quad (69)$$

The coherent states of the Barut–Girardello type for this oscillator in the Fock space $\mathcal{H}_F$ are defined as eigenvectors of the annihilation operator a

$$a\,|z\rangle = z\,|z\rangle, \quad z \in \mathbb{C}. \quad (70)$$

They are given by the formula

$$|z\rangle = \mathcal{N}^{-1}(|z|^2)\sum_{n=1}^{\infty}\frac{z^n}{r_{n-1}!}|n\rangle, \quad (71)$$

where $\mathcal{N}$ is a normalization factor, and

$$r_n! = \begin{cases} 1, & \text{if} \quad n = 0; \\ r_n r_{n-1}\ldots r_1, & \text{if} \quad n = 1,2,\ldots. \end{cases}$$

We consider the coherent states of this oscillator, which are connected with q^{-1}-Hermite polynomials (67). They are given by expression (71), where

$$r_{n-1}! = \left(\frac{q}{1-q}\right)^{n/2} q^{-n(n+1)/4}(q;q)_n^{1/2}$$

and

$$|n\rangle = \psi_n(x;q) = \frac{h_n(x;q)}{q^{-n(n+1)/4}(q;q)_n^{1/2}}.$$

We obtain

$$|z\rangle = \mathcal{N}^{-1}(|z|^2)|z\rangle\sum_{n=0}^{\infty}\frac{z^n\left(\frac{1-q}{q}\right)^{n/2}}{q^{-n(n+1)/4}(q;q)_n^{1/2}}h_n(x;q)\times$$

$$\times\frac{1}{q^{-n(n+1)/4}(q;q)_n^{1/2}},$$

or

$$|z\rangle = \mathcal{N}^{-1}(|z|^2)\sum_{n=0}^{\infty}(\sqrt{1-q})^n q^{n^2/2}\frac{h_n(x;q)}{(q;q)_n}z^n. \quad (72)$$

With regard for relation (68), the normalization factor can be written as

$$\mathcal{N}^2(|z|^2) = \sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q;q)_n}(1-q)^n|z|^{2n} =$$

$$= (-(1-q)|z|^2;q)_\infty \quad (73)$$

or

$$\mathcal{N}^2(|z|^2) = {}_0\phi_0(-;-q;-(1-q)|z|^2). \quad (74)$$

Using the generating function [38]

$$\sum_{n=0}^{\infty}\frac{t^n q^{n(n-1)/2}}{(q;q)_n}h_n(x;q) =$$

$$= \left(-t(x+\sqrt{x^2+1}); t(\sqrt{x^2+1}-x)\right)_\infty \quad (75)$$

for the polynomials $h_n(x;q)$ and (72), we obtain

$$|z\rangle =$$

$$= \frac{\left(-z\sqrt{q(1-q)}(x+\sqrt{x^2+1}); z\sqrt{q(1-q)}(\sqrt{x^2+1}-x)\right)_\infty}{\left(-(1-q)|z|^2;q\right)_\infty^{1/2}}. \quad (76)$$

(b) *Completeness of generalized coherent states.* It is necessary to prove the formula presenting the resolution of identity,

$$\int\int_{\mathbb{C}}\hat{W}(|z|^2)|z\rangle\langle z|d^2z = \sum_{n=0}^{\infty}|n\rangle\langle n| = 1, \quad (77)$$

i.e., to construct a measure

$$d\mu(|z|^2) = \hat{W}(|z|^2)d^2z,\ d^2z = (\mathrm{Re}z)(\mathrm{Im}z). \quad (78)$$

In view of (71), relation (77) can be represented as

$$\sum_{n=0}^{\infty} \frac{\pi}{r_{n-1}^2} \int_0^{\infty} dx x^n \frac{\hat{W}(x)}{\mathcal{N}^2(x)} |n\rangle\langle n| = 1,\ (x = |z|^2). \quad (79)$$

Defining

$$\tilde{W}(x) = \pi \frac{\hat{W}(x)}{\mathcal{N}^2(x)}, \quad (80)$$

we arrive at the solution of the classical moment problem

$$\int_0^{\infty} dx x^n \tilde{W}(x) = r_{n-1}^2! = \left(\frac{1}{1-q}\right)^n q^{-n(n-1)/2}(q;q)_n. \quad (81)$$

The replacement of the variables $W(y) = \frac{1}{1-q}\tilde{W}(\frac{x}{1-q})$ leads (81) to the form

$$\int_0^{\infty} dy y^n W(y) = q^{-n(n-1)/2}(q;q)_n. \quad (82)$$

In order to solve the moment problem (82), we define a q-exponential function

$$e_q(x) = {}_0\phi_0(x;q) = \sum_{n=0}^{\infty} \frac{x^n}{c_{n-1}^2!} =$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{q^{-n+1}!(1-q^n)!} = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} x^n, \quad (83)$$

where [39]

$${}_0\phi_0(x;q) = \left(\begin{matrix} 0 & 0 \\ - & - \end{matrix} \middle| \ q; -x \right) = (-x;q)_\infty. \quad (84)$$

Defining the deformed derivative by

$$\left[\frac{d}{dx}\right]_q f(x) = \frac{f(q^{-1}x) - f(x)}{q^{-1}x}, \quad (85)$$

we obtain

$$\left[\frac{d}{dx}\right]_q x^n = \begin{cases} c_{n-1}^2 x^{n-1}, & \text{if} \quad n \geq 0; \\ 0, & \text{if} \quad n = 0. \end{cases} \quad (86)$$

Therefore,

$$\left[\frac{d}{dx}\right]_q e_q(x) = e_q(x). \quad (87)$$

The following Leibniz rule holds for this deformed derivative:

$$\left[\frac{d}{dx}\right]_q [u(x) \cdot v(x)] =$$

$$= \begin{cases} [\frac{d}{dx}]_q u(x) \cdot v(q^{-1}x) + u(x) \cdot [\frac{d}{dx}]_q v(x), \\ v(x) \cdot [\frac{d}{dx}]_q u(x) + u(q^{-1}x) \cdot [\frac{d}{dx}]_q v(x). \end{cases} \quad (88)$$

From this rule and the relation $e_q^{-1}(x)e_q(x) = 1$, we obtain

$$\left[\frac{d}{dx}\right]_q \left(e_q(x)e_q^{-1}(x)\right) =$$

$$= \left[\frac{d}{dx}\right]_q e_q(x) \cdot e_q^{-1}(q^{-1}x) + e_q(x) \cdot \left[\frac{d}{dx}\right]_q e_q^{-1}(x) = 0, \quad (89)$$

i.e.,

$$\left[\frac{d}{dx}\right]_q e_q^{-1}(x) = -e_q^{-1}(q^{-1}x). \quad (90)$$

We now introduce the Jackson integral corresponding to derivative (85):

$$\int_0^{\infty} f(t)dt_q = q^{-1} \sum_{l=0}^{\infty} q^{-l+1} f(q^{-l+1}) + q^{l+2} f(q^{l+2}). \quad (91)$$

The formula of integration by parts has the form

$$\int_0^{\infty} u(x) \cdot \left[\frac{d}{dx}\right]_q v(x) =$$

$$\int_0^{\infty} \left[\frac{d}{dx}\right]_q [u(x) \cdot v(x)] - \int_0^{\infty} \left[\frac{d}{dx}\right]_q u(x) \cdot v(q^{-1}x). \quad (92)$$

Let us consider the integral

$$I_n = \int_0^{\infty} x^n \left[\frac{d}{dx}\right]_q e_q^{-1}(x) = -\int_0^{\infty} x^n e_q^{-1}(q^{-1}x). \quad (93)$$

Using formula (92), we obtain

$$I_n = \int_0^{\infty} x^n e_q^{-1}(q^{-1}x) = \int_0^{\infty} x^{n-1} e_q^{-1}(q^{-1}x) c_{n-1}^2, \quad (94)$$

i.e.,

$$I_n = I_{n-1}c_{n-1}^2, \qquad n \geq 0, \tag{95}$$

or

$$I_n = c_{n-1}^2! = q^{-n(n-1)/2}(q,q)_n. \tag{96}$$

With regard for (91) and (95), we obtain the solution of the classical moment problem (82)

$$W(y) =$$

$$q^{-1}\sum_{l=0}^{\infty} y\Big[\delta(y-q^{-l+1})+\delta(y-q^{l+1})\Big]e_q(q^{-1}y), \tag{97}$$

so that

$$\bar{W}(x) = \frac{1-q}{q}e_q\Big(q^{-1}(1-q)x\Big)\times$$

$$\times\sum_{l=0}^{\infty} x\Big[\delta\Big((1-q)x-q^{-l+1}\Big)+\delta\Big((1-q)x-q^{l+1}\Big)\Big]. \tag{98}$$

The measure in (78) is given by

$$d\mu(|z|^2) = \frac{1-q}{\pi q}\Big(-(1-q)|z|^2;q)_\infty\Big)e_q\Big(q^{-1}(1-q)|z|^2\times$$

$$\times\sum_{l=0}^{\infty}|z|^2\Big[\delta\Big((1-q)|z|^2-q^{-l+1}\Big)+\delta\Big((1-q)|z|^2-q^{l+1}\Big)\Big]. \tag{99}$$

9. Conclusions

The aim of this article is the survey of the results obtained in [11, 13, 14, 28] on the generalized $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra. We study general properties of this algebra. The Arik–Coon oscillator with the main relation $aa^+ - qa^+a = 1$, where $q > 1$, is embedded in the framework of the unified $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra. In this last case, we also discuss the uniqueness of the solution of the Stieltjes moment problem. The following investigations of the properties of the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra and its applications can be found in works [8, 40–45].

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УЗАГАЛЬНЕНІ ДЕФОРМОВАНІ ОСЦИЛЯТОРИ В РАМКАХ ОБ'ЄДНАНОЇ $(q;\alpha,\beta,\gamma;\nu)$-ДЕФОРМАЦІЇ І ЇХ ОСЦИЛЯТОРНІ АЛГЕБРИ

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Р е з ю м е

Метою цієї статті є огляд наших результатів з побудови узагальнених $(q;\alpha,\beta,\gamma;\nu)$-деформованих осциляторів і їх осциляторних алгебр. Вивчено їхні незвідні представлення. Зокрема, осцилятор Аріка–Куна із головним співвідношенням $aa^+ - qa^+a = 1$, де $q > 1$, вкладається в ці рамки. Знайдено зв'язок цього осцилятора з ермітовими q^{-1}-деформованими поліномами Аскі. Побудовано сім'ю когерентних станів типу Барута–Джірарделло для цього осцилятора. За допомогою розв'язку відповідної класичної проблеми моментів Стільтєса ми доводимо властивість (переповненості) повноти цих станів.