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UDC 539.12: 537.8

QUANTUM-MECHANICAL DESCRIPTION OF THE FERMIONIC DOUBLET AND ITS LINK WITH THE DIRAC EQUATION[1]

A brief review of the different ways of the Dirac equation derivation is given. The foundations of the relativistic canonical quantum mechanics of a fermionic doublet on the basis of the Schrödinger–Foldy equation of motion are formulated. In our approach, the Dirac equation is derived from the Schrödinger–Foldy equation.

Keywords: fermionic doublet, Dirac equation, Schrödinger–Foldy equation.

1. Introduction

It may seem from the title of the article that the subject of our investigation is trivial. It may seem for everybody that he knows the quantum-mechanical description of spin $s = \frac{1}{2}$ doublet! Indeed, the old imagination that the Dirac equation gives the relativistic quantum-mechanical description of the fermionic doublet still is widespread.

Nevertheless, in some principal quantum-mechanical problems, such description is not satisfactory. L. Foldy and S. Wouthuysen [1] suggested the nonlocal canonical formulation of the equation for a spinor field. In this description [1–3], first of all for an electron having spin $\mathbf{s} = \frac{1}{2}\boldsymbol{\sigma}$, many quantum-mechanical details were clarified. The equation $i\partial_t f(x) = \sqrt{m^2 - \Delta} f(x)$, $f = \begin{vmatrix} f^1 \\ f^2 \end{vmatrix}$, was used. Such description [1–3] is adequate in the sense of its comparison with the nonrelativistic Schrödinger model of electron. The direct relativistic analogue of the Schrödinger equation is the spinless Salpeter equation [4–6] for the one-component wave function: $i\partial_t f(x) = \sqrt{m^2 - \Delta} f(x)$.

The relativistic equations mentioned above can be generalized in an obvious way for the particle multiplets with arbitrary spin. In the case of arbitrary spin and multicomponent wave functions, we suggested [7, 8] to call such type of equations as *the Schrödinger–Foldy equations*. The important contribution of L. Foldy [1–3] into the consideration and the analysis of a spinor field in the nonlocal canonical representation (and his analysis of the principles of heredity and correspondence with nonrelativistic quantum mechanics) was taken into account.

The analysis of a 4-component spinor field [1–3] enabled the authors of these papers to discern the quantum-mechanical interpretation of the Dirac equation. For these purposes, the Dirac equation was transformed into another representation $i\partial_t f(x) = \gamma^0 \sqrt{m^2 - \Delta} f(x)$, which is called the Foldy–Wouthuysen (FW) representation today.

In this article, the Schrödinger–Foldy equation for the 4-component wave function (in all essential details) is under consideration. The axiomatic formulation of the corresponding relativistic canonical quantum mechanics (RCQM) is briefly presented. The derivation of the Dirac equation directly from the Schrödinger–Foldy equation (without any additional assumptions) is given.

We started the consideration of this subject in paper [9]. Here, the basic principles of RCQM for a spin $s = \frac{1}{2}$ doublet and the derivation of the Dirac equation from this model are under the further consideration. The foundations of RCQM were given in [1–4, 7–9]. Here, the mathematically well-defined consideration on the level of modern axiomatic approaches to the field theory [10] is provided.

In Section 2, a brief review of various ways to derive the Dirac equation is given. Our motivation and goals are considered.

[1] This work is the contribution to Proceedings of the International Conference "Quantum Groups and Quantum Integrable Systems".

In Section 3, the main notations and definitions are fixed.

In Section 4, the difference between the Schrödinger–Foldy equation and the FW equation is demonstrated. The advantages of the Schrödinger–Foldy equation in the detailed quantum-mechanical description of the fermionic doublet are demonstrated. The reasons for our postulation of the Schrödinger–Foldy equation to be the RCQM equation of motion are shown.

In Section 5, the brief axiomatic foundations of RCQM are formulated.

In Section 6, we present our derivation of the Dirac equation. We start from the main principles of RCQM (directly from the Schrödinger–Foldy equation).

In Section 7, we formulate the main conclusions and discuss briefly the negative point of view of RCQM for the possibility of a negative mass of antiparticles.

2. Motivation and Goals

The significance of the Dirac equation and its wide-range application in different models of theoretical physics (QED, QHD, theoretical atomic and nuclear physics, solid systems) is well-known. Let us recall only that the first analysis of this equation enabled P. Dirac to give the theoretical prediction of a positron, which was discovered experimentally by C. Anderson in 1932. The recent well-known application of the massless Dirac equation to graphene ribbons is an example of possibilities of this equation. In our recent publications [11–15], we were able to extend the domain of applications of the Dirac equation. We proved [11–15] that this equation has not only fermionic but also the bosonic features and can describe not only the fermionic but also bosonic states.

Therefore, the new ways of derivation of the Dirac equation are the actual problems. They visualize automatically the ground principles that are in the foundation of the description of elementary particles on the basis of this equation. Hence, the active consideration of the various ways to derive the Dirac equation is the subject of many contemporary publications. The start from the different basic principles and assumptions is considered.

Below, a brief review of the different ways to derive the Dirac equation is given.

It is necessary to mark the elegant derivation given by P. Dirac in his book [16]. Till now, it is very interesting for readers to feel Dirac's thinking and to follow his logical steps. Nevertheless, Dirac's consideration of the Schrödinger–Foldy equation, which was essentially used in his derivation [16], was not correct. Especially, this concerns his assertion that the Schrödinger–Foldy equation is unsatisfactory from the point of view of the relativistic theory. Dirac's doubts were overcome in [1–3]. Today, this is confirmed by more than one hundred publications about the FW and spinless Salpeter equations, which have wide-range application in th contemporary theoretical physics.

In the well-known book [17], one can find the excellent review of the Dirac theory and two different ways to derive the Dirac equation. First, it is the presentation of the Klein–Gordon equation in the form of a first-order differential system of equations and the factorization of the Klein–Gordon operator. Second, the Lagrange approach is considered, and the Dirac equation is derived from the variational Euler–Lagrange least action principle.

In the van der Waerden–Sakurai derivation [18] of the Dirac equation, the electron spin is incorporated into the nonrelativistic theory. The representation of the nonrelativistic kinetic energy operator of a free spin-1/2 particle in the form $H^{\mathrm{KE}} = (\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{p})/2m$ and the relativistic expression $E^2 - \mathbf{p}^2 = m^2$ are used. After that, the procedure of transition from the 2-component to 4-component equation is fulfilled and explained.

In book [19], the Dirac equation is derived from the manifestly covariant transformational properties of a 4-component spinor.

The derivation of the Dirac equation from the initial geometric properties of the space-time and an electron together with the wide-range discussion of the geometric principles of the electron theory are the main content of book [20]. The ideas of V. Fock and D. Iwanenko [21, 22] of the geometric sense of the Dirac γ-matrices are in the basis of the approach.

The derivation of the Dirac equation based on the Bargmann–Wigner classification of the irreducible unitary representations of the Poincaré group should be mentioned as well (see, e.g., [23]). It is the illustrative demonstration of the possibilities of the group-theoretic approach to the elementary particle physics.

In Foldy's papers [1–3], one can easily find the inverse problem, in which the Dirac equation is obtained from the FW equation. Nevertheless, it is only the transition from one representation to another one.

H. Sallhofer [24, 25] derived the Dirac equation for the hydrogen spectrum, by starting from Maxwell's equations in a medium. Strictly speaking, only the stationary equations were considered.

In paper [26], a quaternion measurable process was introduced, and the Dirac equation was derived from the Langevin equation associated with a two-valued process.

L. Lerner [27] was able to derive the Dirac equation from the conservation law of a spin-1/2 current. The requirement that this current be conserved leads to a unique determination of the Lorentz invariant equation satisfied by a relativistic spin-1/2 field. Let us briefly comment that the complete list of conservation laws for the Dirac theory is the Noether consequence of the Dirac equation. Therefore, the validity of the inverse problem is really expected. Can it be considered as the independent derivation?

The Dirac equation has been derived [28] from the master equation of a Poisson process by analytic continuation. The extension to the case where a particle moves in an external field was given. It was shown that the generalized master equation is intimately connected with the three-dimensional Dirac equation in an external field.

In paper [29], a method of deriving the Dirac equation from the Newton's relativistic second law was suggested. Such derivation is possible in a new formalism, which connects the special form of relativistic mechanics with quantum mechanics. H. Cui suggested a concept of velocity field. At first, Newton's relativistic second law was rewritten as a field equation in terms of the velocity field, which directly reveals a new relationship connecting with quantum mechanics. After that, it was shown that the Dirac equation can be derived from the field equation in a rigorous and consistent manner.

In paper [30], a geometric derivation of the Dirac equation was given, by considering a spin-1/2 particle traveling with the speed of light in a cubic space-time lattice. The mass of the particle acts to flip the multicomponent wave function at the lattice sites. Starting with a difference equation for the case of one spatial and one time dimensions, the authors of paper [30] generalized the approach to higher dimensions. Interactions with external electromagnetic and gravitational fields were also considered. Nevertheless, the idea of such derivation is based on Dirac's observation that the instantaneous velocity operators of a spin-1/2 particle (hereafter called by the generic name "electron") have eigenvalues $\pm c$. This mistake of P. Dirac was demonstrated and overcome in [1].

Using the mathematical tool of Hamilton's biquaternions, the authors of [31] proposed a derivation of the Dirac equation from a geodesic equation. Such derivation is given in the program of application of the theory of scale relativity to microphysics aimed at recovering the quantum mechanics as a new nonclassical mechanics on a nonderivable space-time.

M. Evans was successful to express his equation of general relativity (generally covariant field equation for gravitation and electromagnetism [32]) in the spinor form, thus producing the Dirac equation in general relativity [33]. The Dirac equation in special relativity is recovered in the limit of the Euclidean or flat space-time.

Ten years ago, we already presented our own derivation of the Dirac equation [34–36]. The Dirac equation was derived from the slightly generalized Maxwell equations with gradient-like current and charge densities. This form of the Maxwell equations, which is directly linked with the Dirac equation, is the maximally symmetric variant of these equations. Such Maxwell equations are invariant with respect to a 256-dimensional algebra (the well-known algebra of the conformal group has only 15 generators). Of course, we derived only the massless Dirac equation.

Today, we present a new derivation of the Dirac equation. We derive the Dirac equation from the 4-component Schrödinger–Foldy equation of RCQM. We postulate the Schrödinger–Foldy equation and construct the corresponding formalism of RCQM as the most fundamental model of fermionic doublet. At first, the brief axiomatic formulation of the RCQM foundations is given. After that, the operator, which transforms the Schrödinger–Foldy equation into the Dirac equation, is given. Therefore, the new way to derive the Dirac equation is presented.

Our main goal is following.

To answer the question "Does there exist a more fundamental model of "particle doublet" (as an elementary fundamental object), from which the Dirac equation (and its content) would follow directly and unambiguously?", we are able to demonstrate that the axiomatically formulated RCQM of a particle-antiparticle doublet of spin $s = \frac{1}{2}$ should be chosen as such a model. Below, the specific detailed illustration of this assertion by the example of an electron-positron doublet, e^-e^+-doublet, is given.

3. Notations and Main Definitions

The model of RCQM for an elementary particle with $m > 0$ and spin $s = \frac{1}{2}$, which satisfies the Schrödinger–Foldy equation $i\partial_t f(x) = \sqrt{m^2 - \Delta} f(x)$; $x \in \in \mathrm{M}(1,3)$, $\int d^3x \, |\varphi(x)|^2 < \infty$, was suggested and approved in [1–3]. This model can be easily generalized to the case of an arbitrary $\mathbf{s}$-multiplet, i. e. the "elementary object" with mass m and spin $\mathbf{s} \equiv (s^j) = = (s_{23}, s_{31}, s_{12}) : [s^j, s^l] = i\varepsilon^{jln} s^n$, where ε^{jln} is the Levi-Civita tensor, and $s^j = \varepsilon^{j\ell n} s_{\ell n}$ are the Hermitian $\mathrm{M} \times \mathrm{M}$ matrices that are the generators of an M-dimensional representation of the spin group SU(2) (universal covering of the SO(3)⊂SO(1,3) group).

Here, we present the detalization of such generalization by the example of a fermionic doublet with the spin $s = \frac{1}{2}$. All mathematical and physical details related to the choice of a specific form of the spin-$\mathbf{s}$ doublet are illustrated by the example of an e^-e^+-doublet.

We choose the standard relativistic concepts, definitions, and notations in the form convenient for our consideration. For example, in the Minkowski space-time

$$\mathrm{M}(1,3) = \{x \equiv (x^\mu) = (x^0 = t, \mathbf{x} \equiv (x^j))\}; \qquad (1)$$

$$\mu = \overline{0,3},\ j = 1,2,3,$$

the x^μ are the Cartesian (covariant) coordinates of the points of the physical space-time in any fixed inertial frame of reference (IFR). We use the system of units $\hbar = c = 1$. The metric tensor is given by

$$g^{\mu\nu} = g_{\mu\nu} = g^\mu_\nu,\ (g^\mu_\nu) = \mathrm{diag}\,(1, -1, -1, -1); \qquad (2)$$

$$x_\mu = g_{\mu\nu} x^\mu,$$

where the summation over the twice repeated index is implied.

The analysis of the relativistic invariance of an arbitrary physical model involves, as the first step, the consideration of its invariance with respect to the proper orthochronous Lorentz $\mathrm{L}^\uparrow_+ = \mathrm{SO}(1,3) = = \{\Lambda = (\Lambda^\mu_\nu)\}$ and Poincaré $\mathrm{P}^\uparrow_+ = \mathrm{T}(4) \times) \mathrm{L}^\uparrow_+ \supset \mathrm{L}^\uparrow_+$ groups. This invariance in an arbitrary relativistic model is the realization of Einstein's relativity principle in the form of special relativity. Note that the mathematical correctness demands one to consider the invariance mentioned above as the invariance with respect to the universal coverings $\mathcal{L} = \mathrm{SL}(2,\mathrm{C})$ and $\mathcal{P} \supset \mathcal{L}$ of the groups $\mathrm{L}^\uparrow_+$ and $\mathrm{P}^\uparrow_+$, respectively.

For the group $\mathcal{P}$, we choose the real parameters $a = (a^\mu) \in$M(1,3) and $\varpi \equiv (\varpi^{\mu\nu} = -\varpi^{\nu\mu})$, whose physical meaning is well-known. For the standard $\mathcal{P}$ generators $(p_\mu, j_{\mu\nu})$, we use the commutation relations in the manifestly covariant form

$$\begin{gathered}[p_\mu, p_\nu] = 0,\ [p_\mu, j_{\rho\sigma}] = i g_{\mu\rho} p_\sigma - i g_{\mu\sigma} p_\rho, \\ [j_{\mu\nu}, j_{\rho\sigma}] = -i\,(g_{\mu\rho} j_{\nu\sigma} + g_{\rho\nu} j_{\sigma\mu} + g_{\nu\sigma} j_{\mu\rho} + g_{\sigma\mu} j_{\rho\nu}).\end{gathered} \qquad (3)$$

4. Canonical Equation of Motion of Relativistic Quantum Mechanics

In this section, we make comparison of the Schrödinger–Foldy and FW equations for a fermionic doublet. On this bases, we demonstrate why the Schrödinger–Foldy equation should be chosen as the main equation of motion in RCQM.

The Schrödinger–Foldy equation for a fermionic spin-1/2 doublet is given by

$$i\partial_t f(x) = \sqrt{m^2 - \Delta} f(x), \qquad (4)$$

where

$$f \equiv \mathrm{column}(f^1, f^2, f^3, f^4). \qquad (5)$$

This equation, similarly to the nonrelativistic 4-component Schrödinger equation

$$i\partial_t f(x) = \frac{\mathbf{p}^2}{2m} f(x) \qquad (6)$$

(involving also the internal degrees of freedom, spin, *etc.*) is considered in the quantum-mechanical Hilbert space

$$\mathrm{H}^{3,4} = \mathrm{L}_2(\mathrm{R}^3) \otimes \mathrm{C}^{\otimes 4} = \{f = (f^\alpha) : \mathrm{R}^3 \to \mathrm{C}^{\otimes 4}; \int d^3x |f(t, \mathbf{x})|^2 < \infty\}, \qquad (7)$$

where d^3x is the Lebesgue measure in the space $\mathrm{R}^3 \subset \mathrm{M}(1,3)$ of the eigenvalues of the position operator $\mathbf{x}$ of the Cartesian coordinate of the doublet

ISSN 0372-400X. Укр. фіз. журн. 2013. Т. 58, № 12

in an arbitrary fixed IFR. In (4)–(7) and below, the two upper components f^1, f^2 of the vector $f \in \mathrm{H}^{3,4}$ are the components of the electron wave function φ_-, and the two lower components f^3, f^4 are those of the positron wave function φ_+.

The general solution of the Schrödinger–Foldy equation (4), similarly to the general solution of the nonrelativistic 4-component Schrödinger equation (6), is given by

$$f(x) = \begin{vmatrix} f_{e^-} \\ f_{e^+} \end{vmatrix} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{-ikx} \times$$
$$\times \left[a_+^-(\mathbf{k})\mathrm{d}_1 + a_-^-(\mathbf{k})\mathrm{d}_2 + a_-^+(\mathbf{k})\mathrm{d}_3 + a_+^+(\mathbf{k})\mathrm{d}_4\right], \qquad (8)$$

where

$$kx \equiv \omega t - \mathbf{kx}, \quad \omega \equiv \sqrt{\mathbf{k}^2 + m^2}, \qquad (9)$$

the 4-columns d_α are the Cartesian orts in the space $\mathrm{C}^{\otimes 4} \subset \mathrm{H}^{3,4}$,

$$\mathrm{d}_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \mathrm{d}_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}, \mathrm{d}_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix}, \mathrm{d}_4 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}, \qquad (10)$$

the functions $a_+^-(\mathbf{k}), a_-^-(\mathbf{k})$ are the quantum-mechanical momentum-spin amplitudes of a particle with charge $-e$ and the spin projection eigenvalues $+1/2$ and -1/2; and $a_-^+(\mathbf{k})$, $a_+^+(\mathbf{k})$ are the quantum-mechanical momentum-spin amplitudes of an antiparticle with charge $+e$ and the spin projection eigenvalues -1/2 and +1/2, respectively.

In the general solution (8), we use the modern experimentally verified understanding of a positron as the "mirror mapping" of an electron. Such understanding leads to the specific postulation of the explicit forms of the charge sign and spin operators (see formula (24) in [7]), which determine the form of solution (8).

Contrary to the Schrödinger–Foldy equation (4), the FW equation has the form

$$i\partial_t f(x) = \gamma^0 \sqrt{m^2 - \Delta} f(x),$$
$$\gamma^0 = \begin{vmatrix} \mathrm{I}_2 & 0 \\ 0 & -\mathrm{I}_2 \end{vmatrix}, \quad \mathrm{I}_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \qquad (11)$$

and its general solution is given by

$$\phi(x) = \begin{vmatrix} \phi_{e^-} \\ \phi^*_{e^+} \end{vmatrix} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k\{e^{-ikx}[a_+^-(\mathbf{k})\mathrm{d}_1 +$$
$$+ a_-^-(\mathbf{k})\mathrm{d}_2] + e^{ikx}[a_-^{*+}(\mathbf{k})\mathrm{d}_3 + a_+^{*+}(\mathbf{k})\mathrm{d}_4]\}. \qquad (12)$$

Equations (4) and (11) are different due to the presence of γ^0 in (11). Owing to this fact, the general solutions (8) and (12) are also different. Solution (8) is the direct sum of the electron and positron quantum-mechanical wave functions. Solution (12) is the direct sum of the electron and complex conjugated positron wave functions. Furthermore, solution (8) contains only positive energetic states both for the electron and the positron, whereas solution (12) contains the positive energetic states of the electron and the negative energetic states of the positron.

Note that solution (8) for the wave function of a spin-1/2 doublet is expressed in terms of relativistic de Broglie waves for the electron and the positron,

$$\varphi_{\vec{k}\mathrm{A}}(t, \mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-i\omega t + i\vec{k}\vec{x}} \mathrm{d}_\mathrm{A}, \ \mathrm{A} = 1,2,3,4. \qquad (13)$$

Expressions (13) are the fundamental (basis) solutions of Eq. (4) and do not belong to the Hilbert space (7) (their $\mathrm{H}^{3,4}$-norms are equal to the infinity). The mathematical correctness of the consideration is ensured by applying the rigged Hilbert space

$$\mathrm{S}^{3,4} \equiv \mathrm{S}(\mathrm{R}^3) \times \mathrm{C}^4 \subset \mathrm{H}^{3,4} \subset \mathrm{S}^{3,4*}. \qquad (14)$$

Here, $\mathrm{S}^{3,4}$ is the 4-component Schwartz test function space over the space $\mathrm{R}^3 \subset \mathrm{M}(1,3)$, and $\mathrm{S}^{3,4*}$ is the space of 4-component Schwartz generalized functions, which is conjugated to the Schwartz test function space $\mathrm{S}^{3,4}$ by the corresponding topology (see, e.g., [37]). Strictly speaking, the mathematical correctness of the consideration demands us to make calculations in the space $\mathrm{S}^{3,4*}$ of generalized functions, i.e. with the application of cumbersome functional analysis.

Nevertheless, let us take into account that the Schwartz test function space $\mathrm{S}^{3,4}$ in triple (14) is *kernel*. This means that $\mathrm{S}^{3,4}$ is dense both in the quantum-mechanical space $\mathrm{H}^{3,4}$ and in the space of generalized functions $\mathrm{S}^{3,4*}$. Therefore, any physical state $f \in \mathrm{H}^{3,4}$ can be approximated with arbitrary precision by the corresponding elements of the Cauchy sequence in $\mathrm{S}^{3,4}$, which converges to the given $f \in \mathrm{H}^{3,4}$. Further, with regard for the requirement to measure the arbitrary value of the model with nonabsolute precision, this means that all specific calculations can be fulfilled within the Schwartz test function space $\mathrm{S}^{3,4}$.

Note that if the general solution $f(x)$ (8) belongs to $\mathrm{S}^{3,4}$, then the amplitudes $a^-_+(\mathbf{k})$, $a^-_-(\mathbf{k})$, $a^+_-(\mathbf{k})$, and $a^+_+(\mathbf{k})$ also belong to $\mathrm{S}^{3,4}$. It is enough for the approximation of any experimental situation.

Contrary to this situation, even if the amplitudes a from (12) belong to the $\mathrm{S}^{3,4}$, the general solution (12) of the FW equation (11) does not belong to the quantum-mechanical Hilbert space (7) (due to the indefinite metric in the space of solutions). Nevertheless, the mathematical correctness of the consideration is achieved (ensured) in the space $\mathrm{S}^{3,4} \subset \mathrm{S}^{3,4*}$ due to the fact that $\mathrm{S}^{3,4}$ is dense in $\mathrm{S}^{3,4*}$.

Therefore, we are able to demonstrate the difference between the Schrödinger–Foldy equation (4) and the FW equation (11) in the quantum-mechanical description of the fermionic doublet. Despite the fact that one can discern some quantum-mechanical aspects in the FW representation, however, the FW equation does not guarantee generally the detailed quantum-mechanical description of the fermionic doublet (as well as the Dirac equation).

Hence, we postulate the Schrödinger–Foldy equation (4) to be the RCQM equation of motion for a fermionic spin-1/2 doublet and construct the corresponding canonical formalism.

5. Relativistic Canonical Quantum Mechanics of a Fermi Doublet

The *axioms of the model* are formulated on the level of correctness of von Neumann's monograph [38]. The requirements of such physically verified principles as *the principle of relativity with respect to the tools of cognition (PRTC)*, *principle of heredity (PH)* with both classical mechanics of single mass point and nonrelativistic quantum mechanics (and *the principle of correspondence (PC)* with these theories), and *the Einstein principle of relativity (EPR)* are taken into consideration. The last principle requires, first of all, *the special relativity (SR)* to be taken into account.

The *basic axioms* of the model (we present here the brief consideration) as mathematical assertions have the form.

On the space of states. The space of states of an isolated e^-e^+-doublet in the arbitrarily fixed IFR in its $\mathbf{x}$-realization is the Hilbert space $\mathrm{H}^{3,4}$ (7) of complex-valued 4-component square-integrable functions of $x \in \mathrm{R}^3 \subset \mathrm{M}(1,3)$ (similarly, in momentum, $\mathbf{p}$-realization). Here, $\mathbf{x}$ and $\mathbf{p}$ are the operators of canonically conjugated dynamical variables of the e^-e^+-doublet, and the vectors f, $\tilde{f}$ in $\mathbf{x}$ and $\mathbf{p}$ realizations are linked by the 3-dimensional Fourier transformation (the variable t is the parameter of time-evolution).

The mathematical correctness of the consideration demands the application of the rigged Hilbert space (14), where the Schwartz test function space $\mathrm{S}^{3,4}$, which is the verified tool of the PRTC realization, is kernel (i.e., it is dense both in $\mathrm{H}^{3,4}$ and in the space $\mathrm{S}^{3,4*}$ of the generalized Schwartz functions). Such application allows us to fulfill, without any loss of generality, all necessary calculations in the space $\mathrm{S}^{3,4}$ on the level of correct differential and integral calculus. The more detailed consideration is given in paragraphs after formula (14).

On the time evolution of the state vectors. The time dependence of the state vectors $f \in \mathrm{H}^{3,4}$ (time t is the parameter of evolution) is given either in the integral form by the unitary operator

$$u\left(t_0, t\right) = \exp\left[-i\widehat{\omega}(t - t_0)\right]; \quad \widehat{\omega} \equiv \sqrt{-\Delta + m^2} \quad (15)$$

(below we put $t_0 = t$) or in the differential form by the Schrödinger–Foldy equation of motion (4). Here, the operator $\widehat{\omega} \equiv \sqrt{-\Delta + m^2}$ is the relativistic analog of the energy operator (Hamiltonian) of nonrelativistic quantum mechanics. The Minkowski space-time M(1,3) is pseudo-Euclidean with the metric $g = \mathrm{diag}(+1, -1, -1, -1)$.

On the fundamental dynamical variables. The dynamical variable $\mathbf{x} \in \mathrm{R}^3 \subset$M(1,3)(as well as the variable $\mathbf{k} \in \mathrm{R}^3_{\vec{k}}$) represents the external degrees of freedom of the e^-e^+-doublet. The spin $\mathbf{s}$ of the e^-e^+-doublet is the first in the list of carriers of the internal degrees of freedom. In view of the Pauli principle and the fact that the positron is experimentally observed as the mirror reflection of the electron, the operators of charge sign and spin of the e^-e^+-doublet are taken in the form

$$g \equiv -\gamma^0 = \begin{vmatrix} -\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_2 \end{vmatrix}, \ \mathbf{s} = \frac{1}{2}\begin{vmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -C\boldsymbol{\sigma}C \end{vmatrix}, \ \mathrm{I}_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \quad (16)$$

where $\boldsymbol{\sigma}$ are the standard Pauli matrices, and C is the operator of complex conjugation. The spin matrices (16) satisfy the commutation relations of the algebra of SU(2) group

$$\mathbf{s} \equiv \left(s^j\right) = \left(s_{23}, s_{31}, s_{12}\right) : \ \left[s^j, s^l\right] = i\varepsilon^{jln} s^n;$$

$$\varepsilon^{123} = +1. \tag{17}$$

Here, ε^{jln} is the Levi-Civita tensor, and $s^j = \varepsilon^{j\ell n} s_{\ell n}$ are the Hermitian 4×4 matrices that are the generators of a 4-dimensional reducible representation of the spin group SU(2) (universal covering of the SO(3)⊂SO(1,3) group).

On the algebra of observables. Using the operators of canonically conjugated coordinate $\mathbf{x}$ and momentum $\mathbf{p}$ (where $[x^j, p^\ell] = i\delta_{j\ell}$ in $\mathrm{H}^{3,4}$), being completed by the operators $\mathbf{s}$ and g, we construct the algebra of observables (according to the PH) as the Hermitian functions of 10 ($\mathbf{x}$, $\mathbf{p}$, $\mathbf{s}$, $-\gamma^0$) generating elements of the algebra.

On the relativistic invariance of the theory. This invariance (realization of the SR) is ensured by the proof of the invariance of the Schrödinger–Foldy equation (4) with respect to the unitary representation of the universal covering $\mathcal{P} \supset \mathcal{L}$=SL(2,C) of the proper orthochronous Poincaré group $\mathrm{P}_+^\uparrow = \mathrm{T}(4) \times \mathrm{L}_+^\uparrow \supset \mathrm{L}_+^\uparrow$. Here, $\mathcal{L}$ = SL(2,C) is the universal covering of the proper orthochronous Lorentz group $\mathrm{L}_+^\uparrow$.

The generators of the fermionic $\mathcal{P}^\mathrm{f}$ representation of the group $\mathcal{P}$, with respect to which the Schrödinger–Foldy equation (4) is invariant, are given by

$$\widehat{p}_0 = \widehat{\omega} \equiv \sqrt{-\Delta + m^2},\ \widehat{p}_l = i\partial_l,$$

$$\widehat{j}_{ln} = x_l \widehat{p}_n - x_n \widehat{p}_l + s_{ln} \equiv \widehat{m}_{ln} + s_{ln}, \tag{18}$$

$$\widehat{j}_{0l} = -\widehat{j}_{l0} = t\widehat{p}_l - \frac{1}{2}\{x_l, \widehat{\omega}\} - \frac{s_{ln}\widehat{p}_n}{\widehat{\omega} + m}, \tag{19}$$

in the $\mathbf{x}$-realization of the space $\mathrm{H}^{3,4}$ (7) and

$$p_0 = \omega,\ p_l = k_l,\ \widetilde{j}_{ln} = \widetilde{x}_l k_n - \widetilde{x}_n k_l + s_{ln}; \tag{20}$$

$$\left(\widetilde{x}_l = -i\widetilde{\partial}_l,\ \widetilde{\partial}_l \equiv \frac{\partial}{\partial k^l}\right),$$

$$\widetilde{j}_{0l} = -\widetilde{j}_{l0} = tk_l - \frac{1}{2}\{\widetilde{x}_l, \omega\} - \frac{s_{ln}k_n}{\omega + m}, \tag{21}$$

in the momentum $\mathbf{k}$-realization $\widetilde{\mathrm{H}}^{3,4}$ of the space of doublet states, respectively. The explicit form of the spin operators s_{ln} in formulae (18)–(21), which is used for the e^-e^+-doublet, is given in formula (16).

In spite of manifestly noncovariant forms (18)–(21) of the $\mathcal{P}^\mathrm{f}$-generators, they satisfy the commutation relations of the $\mathcal{P}$ algebra in the manifestly covariant form (3).

The $\mathcal{P}^\mathrm{f}$-representation of the group $\mathcal{P}$ in the space $\mathrm{H}^{3,4}$ (7) is given by the exponential series convergent in this space,

$$\mathcal{P}^\mathrm{f}: (a, \varpi) \to U(a, \varpi) = \exp(-ia^0\widehat{\omega} - i\mathbf{a}\widehat{\mathbf{p}} - \frac{i}{2}\varpi^{\mu\nu}\widehat{j}_{\mu\nu}), \tag{22}$$

or, in the momentum space $\widetilde{\mathrm{H}}^{3,4}$, by the corresponding exponential series given in terms of generators (20) and (21).

We emphasize that the modern definition of $\mathcal{P}$ invariance (or $\mathcal{P}$ symmetry) of the equation of motion (4) in $\mathrm{H}^{3,4}$ is given by the following assertion (see, e.g., [39]). *The set* $\mathrm{F} \equiv \{f\}$ *of all possible solutions of Eq. (4) is invariant with respect to the* $\mathcal{P}^\mathrm{f}$*-representation of the group* $\mathcal{P}$*, if, for an arbitrary solution* f *and the arbitrarily fixed parameters* (a, ϖ)*, the assertion*

$$(a, \varpi) \to U(a, \varpi)\{f\} = \{f\} \equiv \mathrm{F} \tag{23}$$

is valid. Furthermore, assertion (23) is ensured by the fact that, (as is easy to verify), all the $\mathcal{P}$-generators (18) and (19) commute with the operator $i\partial_t - \sqrt{-\Delta + m^2}$ of Eq. (4).

Not a matter of fact that many manifestly noncovariant objects are used in RCQM, *the model under consideration is relativistic invariant in the sense of the definition given above.*

On the main and additional conservation laws. Similarly to the nonrelativistic quantum mechanics, *the conservation laws are found in the form of quantum-mechanical mean values of the operators, which commute with the operator of the equation of motion.*

The important physical consequence of the assertion about the relativistic invariance is the fact that 10 integral dynamical variables of the doublet

$$(P_\mu,\ J_{\mu\nu}) \equiv \int d^3x f^\dagger(t, \mathbf{x})(\widehat{p}_\mu,\ \widehat{j}_{\mu\nu}) f(t, \mathbf{x}) = \mathrm{Const} \tag{24}$$

do not depend on the time, i.e., they are the constants of motion for this doublet.

Note that the external and internal degrees of freedom for the free e^-e^+-doublet are independent. Therefore, the operator $\mathbf{s}$ (16) commutes not only with the operators $\widehat{\mathbf{p}}$ and $\mathbf{x}$, but also with the orbital part $\widehat{m}_{\mu\nu}$ of the total angular momentum operator. Moreover, both operators $\mathbf{s}$ and $\widehat{m}_{\mu\nu}$ commute with the operator $i\partial_t - \sqrt{-\Delta + m^2}$ of Eq. (4). Therefore, besides 10 main (consequences of 10 Poincaré generators) conservation laws (24), 12 additional constants of motion exist for the free e^-e^+-doublet. These additional conservation laws are the consequences of the operators of the following observables:

$$s_j,\, \breve{s}_l = \frac{s_{ln}p_n}{\widehat{\omega}+m},\, \widehat{m}_{ln} = x_l\widehat{p}_n - x_n\widehat{p}_l,$$
$$\widehat{m}_{0l} = -\widehat{m}_{l0} = t\widehat{p}_l - \frac{1}{2}\{x_l, \widehat{\omega}\}. \qquad (25)$$

Thus, the following assertions can be proved. In the space $\mathrm{H}^{\mathrm{A}} = \{A\}$ of the quantum-mechanical amplitudes, the 10 main conservation laws (24) have the form

$$(P_\mu, J_{\mu\nu}) = \int d^3k A^\dagger(\mathbf{k})(\widetilde{p}_\mu, \widetilde{j}_{\mu\nu})A(\mathbf{k}), \quad A(\mathbf{k}) \equiv \begin{vmatrix} a_{\mathrm{r}}^- \\ a_{\mathrm{f}}^+ \end{vmatrix}, \qquad (26)$$

where the density generators of $\mathcal{P}^{\mathrm{A}}$, $(\widetilde{p}_\mu, \widetilde{j}_{\mu\nu})$ of (26) are given by

$$\widetilde{p}_0 = \omega,\, \widetilde{p}_l = k_l,\, \widetilde{j}_{ln} = \widetilde{x}_l k_n - \widetilde{x}_n k_l + s_{ln};\ \left(\widetilde{x}_l = -i\frac{\partial}{\partial k^l}\right), \qquad (27)$$

$$\widetilde{j}_{0l} = -\widetilde{j}_{l0} = -\frac{1}{2}\{\widetilde{x}_l, \omega\} - \left(\breve{\widetilde{s}}_l \equiv \frac{s_{ln}k_n}{\omega+m}\right). \qquad (28)$$

Note that operators (26)–(28) satisfy the Poincaré commutation relations in the manifestly covariant form (3).

It is evident that 12 additional conservation laws

$$(M_{\mu\nu}, S_{\mu\nu}) \equiv \int d^3x f^\dagger(t,\mathbf{x})(\widehat{m}_{\mu\nu}, s_{\mu\nu})f(t,\mathbf{x}) = \mathrm{Const} \qquad (29)$$

generated by operators (25) are the separate terms in expressions (26)–(28) of the principal (main) conservation laws.

On the Clifford–Dirac algebra. The Clifford–Dirac algebra of the γ-matrices must be introduced into the FW representations. The reasons are as follows.

The part of the Clifford–Dirac algebra operators is directly related to the spin-1/2 doublet operators $(\frac{1}{2}\gamma^2\gamma^3,\, \frac{1}{2}\gamma^3\gamma^1,\, \frac{1}{2}\gamma^1\gamma^2)$. Only in the FW representation, these spin operators commute with the Hamiltonian and with the operator of the equation of motion. In the Pauli–Dirac representation, these operators do not commute with the Dirac equation operator. Only the sums of the orbital and such spin operators commute with the Diracian. So. *if we want to relate the orts of the Clifford–Dirac algebra with the real spin, we must introduce this algebra into the FW representation.*

In the quantum-mechanical representation (i.e., in the space of solutions of the Schrödinger–Foldy equation), the γ-matrices are obtained by the transformation v given in formulae (33) and (34) of Section 6.

Moreover, we use the generalized Clifford–Dirac algebra over the field of real numbers. This algebra was introduced in papers [11–15]. The use of 29 orts of this *proper extended real Clifford–Dirac algebra* gives the additional possibilities in comparison with only 16 elements of the standard Clifford–Dirac algebra (see, e.g., [11–15]).

The definition of spin matrices (16) determines *de facto* the so-called "quantum-mechanical" representation of the Dirac matrices

$$\bar{\gamma}^\mu : \bar{\gamma}^\mu\bar{\gamma}^\nu + \bar{\gamma}^\nu\bar{\gamma}^\mu = 2g^{\mu\nu};\ \bar{\gamma}_0^{-1} = \bar{\gamma}_0,\, \bar{\gamma}_l^{-1} = -\bar{\gamma}_l. \qquad (30)$$

The matrices $\bar{\gamma}^\mu$ (30) of this representation are linked to the Dirac matrices γ^μ in the standard Pauli–Dirac (PD) representation:

$$\bar{\gamma}^0 = \gamma^0,\, \bar{\gamma}^1 = \gamma^1 C,\, \bar{\gamma}^2 = \gamma^0\gamma^2 C,$$
$$\bar{\gamma}^3 = \gamma^3 C,\, \bar{\gamma}^4 = \gamma^0\gamma^4 C; \qquad (31)$$
$$\bar{\gamma}^\mu = v\gamma^\mu v, \quad v \equiv \begin{vmatrix} \mathrm{I}_2 & 0 \\ 0 & C\mathrm{I}_2 \end{vmatrix} = v^{-1}, \quad C\phi = \phi^*,$$

where the standard Dirac matrices γ^μ are given by

$$\gamma^0 = \begin{vmatrix} \mathrm{I}_2 & 0 \\ 0 & -\mathrm{I}_2 \end{vmatrix},\ \gamma^k = \begin{vmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{vmatrix},\ \mu = 0,1,2,3. \qquad (32)$$

Note that, in terms of $\bar{\gamma}^\mu$ matrices (31), the spin operator (16) have the form $\mathbf{s} = \frac{i}{4}(\bar{\gamma}^2\bar{\gamma}^3,\, \bar{\gamma}^3\bar{\gamma}^1,\, \bar{\gamma}^1\bar{\gamma}^2)$.

The $\bar{\gamma}^\mu$ matrices (31) together with the matrix $\bar{\gamma}^4 \equiv \bar{\gamma}^0\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3$, imaginary unit $i \equiv \sqrt{-1}$, and the operator C of complex conjugation in $\mathrm{H}^{3,4}$ generate the quantum-mechanical representations of the extended real Clifford–Dirac algebra and the proper extended real Clifford-Dirac algebra, which were put into consideration in [11] (see also [12–15]).

On the principles of heredity and correspondence. The explicit forms (24)–(29) of the main and additional conservation laws demonstrate evidently that the model of RCQM satisfies the principles of heredity and correspondence with the nonrelativistic classical and quantum theories. The deep analogy between RCQM and these theories for the physical system with a finite number degrees of freedom (where the values of the free dynamical conserved quantities are additive) is also evident.

6. Derivation of the Foldy–Wouthuysen and the Standard Dirac Equations

We consider briefly the derivation of the FW and Dirac equations on the basis of the start from the the Schrödinger–Foldy equation (4). This means that the Dirac equation is a consequence of the quantum-mechanical spin-1/2 doublet model.

The link between the the Schrödinger–Foldy equation (4) and the FW equation (11) is given by the operator v,

$$v = \begin{vmatrix} \mathrm{I}_2 & 0 \\ 0 & C\mathrm{I}_2 \end{vmatrix}; \quad v^2 = \mathrm{I}_4, \quad \mathrm{I}_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \tag{33}$$

C is the operator of complex conjugation, the operator of involution in the space $\mathrm{H}^{3,4}$. The operator v (33) transforms an arbitrary operator q of RCQM into the operator Q in the FW representation for the spinor field and *vice versa*:

$$Q = vqv \leftrightarrow q = vQv. \tag{34}$$

The only warning is that formula (34) is valid only for the anti-Hermitian operators! This means that, in order to avoid the mistakes, one must apply this formula only for the prime (anti-Hermitian) energy-momentum, angular momentum, and spin quantities. The justification and the use of the conception of the prime generators of the Lie groups are given in [11–15].

The role of anti-Hermitian operators in physics is well-known. As well as the physical parameters of groups and algebras are real, it is convenient to associate just the anti-Hermitian generators with them. For example, the real parameters a^μ, $\varpi^{\mu\nu}$ of translations and rotations of the Poincaré group are associated with the anti-Hermitian generators $\widehat{p}_\mu$, $\widehat{j}_{\mu\nu}$, where $\widehat{p}_\mu = \partial_\mu$, *etc.* The mathematical correctness of appealing to the anti-Hermitian generators is considered in [40, 41] in detail. In our papers, just the use of the anti-Hermitian generators allowed us [11–15] to find the additional bosonic properties of the FW and Dirac equations. The details are not the subject of this consideration.

Here, in order to work with the mathematically well-defined relationship between the Schrödinger–Foldy and FW equations, we slightly rewrite these equations and present them in completely equivalent forms in terms of the anti-Hermitian operators. Thus, we consider the Schrödinger–Foldy equation (4) in the form

$$(\partial_0 + i\widehat{\omega})\, f(t, \mathbf{x}) = 0; \tag{35}$$

$$\widehat{\omega} \equiv \sqrt{\mathbf{p}^2 + m^2} = \sqrt{-\Delta + m^2} \geq m > 0,$$

and the FW equation in the form

$$\left(\partial_0 + i\gamma^0\widehat{\omega}\right) \phi(t, \mathbf{x}) = 0. \tag{36}$$

We also rewrite the Dirac equation similarly in the form

$$(\partial_0 + i(\alpha \cdot \mathbf{p} + \beta m))\, \psi(t, \mathbf{x}) = 0 \tag{37}$$

only for the reasons of analogy and orderliness. Note that the FW transformation between the FW and Dirac models

$$V^{\pm} \equiv \frac{\pm i\gamma^l \partial_l + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega} + m)}} \tag{38}$$

is well-defined both for the Hermitian and anti-Hermitian operators.

It is easy to verify that the FW equation (36) follows from the Schrödinger–Foldy equation (35)

$$v\,(\partial_0 + i\widehat{\omega})\, v = \left(\partial_0 + i\gamma^0\widehat{\omega}\right) \leftrightarrow v\,\left(\partial_0 + i\gamma^0\widehat{\omega}\right) v =$$

$$= (\partial_0 + i\widehat{\omega})\,, \tag{39}$$

and the general solution of the FW equation (36) follows from the general solution (8) of the Schrödinger–Foldy equation (35)

$$\phi(t, \mathbf{x}) = vf(t, \mathbf{x}) \leftrightarrow f(t, \mathbf{x}) = v\phi(t, \mathbf{x}). \tag{40}$$

Corresponding links between the FW and Dirac equations are well-known from [1].

Thus, we are able to find the general transformation, which gives relationship directly between RCQM and the Dirac model

$$W = V^+v,\ W^{-1} = vV^-;\ WW^{-1} = W^{-1}W = 1, \quad (41)$$

and to derive the Dirac equation from RCQM (from the Schrödinger–Foldy equation)

$$W\left(\partial_0 + i\widehat{\omega}\right)W^{-1} = \partial_0 + i(\alpha \cdot \mathbf{p} + \beta m), \quad (42)$$

$$\psi(t, \mathbf{x}) = Wf(t, \mathbf{x}). \quad (43)$$

The inverse links also exist as well-defined mathematical transformations

$$W^{-1}(\partial_0 + i(\alpha \cdot \mathbf{p} + \beta m))W = \partial_0 + i\widehat{\omega}, \quad (44)$$

$$f(t, \mathbf{x}) = W^{-1}\psi(t, \mathbf{x}), \quad (45)$$

but are not so interesting for our purposes as the direct transformations (42) and (43). The direct transformations derive the Dirac equation from a more elementary model of the same physical reality.

7. Conclusions

The model of relativistic canonical quantum mechanics on the level of axiomatic approaches to the quantum field theory is considered. The main intuitive physical principles, reinterpreted on the level of modern physical methodology, are mapped mathematically correctly into the basic assertions (axioms) of the model. The Einstein principle of relativity is mapped as a requirement of special relativity. The principles of heredity and correspondence of the model with respect to the nonrelativistic classical and quantum mechanics are supplemented by the clarifications of the carriers of external and internal degrees of freedom. The principle of relativity of the model with respect to the means of cognition is realized by the applications of the rigged Hilbert space. The Schwartz test function space $\mathrm{S}^{3,4}$ is shown to be sufficient to satisfy the requirements of the principle of relativity of the model with respect to the means of cognition. Moreover, the fulfilment of calculations in $\mathrm{S}^{3,4}$ does not lead to the loss of generality of the consideration.

It is shown that the algebra of experimentally observed quantities, which is associated with the Poincaré-invariance of the model, is determined by the nine functionally independent operators $\mathbf{x}, \mathbf{p}, \mathbf{s}$, which have the unambiguous physical sense in the relativistic canonical quantum mechanics model of a doublet. It is demonstrated that the application of the stationary complete sets of operators of the experimentally measured physical quantities guarantees the visualization and the completeness of the consideration.

The derivation of the Foldy–Wouthuysen and Dirac equations from the Schrödinger–Foldy equation of relativistic canonical quantum mechanics is presented and briefly discussed. We prove that the Dirac equation is the consequence of a more elementary model of the same physical reality. The relativistic canonical quantum mechanics is suggested to be such fundamental model of the physical reality. Moreover, it is suggested to be the most fundamental model of the Fermi spin-1/2 doublet.

An important assertion is that *an arbitrary physical and mathematical information, which is contained in the model of relativistic canonical quantum mechanics, is translated directly and unambiguously into the information of the same physical content in the field model of the Dirac equation.*

Hence, the Dirac equation is the unambiguous consequence of the relativistic canonical quantum mechanics of the Fermi spin-1/2 doublet (e.g., e^-e^+-doublet). Nevertheless, the model of relativistic canonical quantum mechanics of the Fermi doublet has the evident independent application.

We pay attention to the fact that the model of relativistic canonical quantum mechanics of a Fermi doublet does not need the application of the positron negative mass concept [42–45]. It is natural due to the following reasons. It is only the energy that depends on the mass. The total energy together with the momentum are associated with the external degrees of freedom, which are common and the same for the particle and the antiparticle (for the electron and the positron). The difference between e_- and e_+ is contained only in the internal degrees of freedom such as the spin $\mathbf{s}$ and the charge sign $g = -\gamma^0$. Thus, if the mass of the particle is taken positive in the relativistic canonical quantum mechanics, then the mass of the antiparticle must be taken positive too.

On the other hand, the comprehensive analysis [43] of the Dirac equation for a doublet had led the authors of paper [43] to the concept of the negative mass of the antiparticle. Therefore, our consideration in the last paragraph gives the additional arguments that the Dirac model (or the Foldy–Wouthuysen model associated with it) is not the quantum-mechanical one. Furthermore, in the problem of the relativistic hydrogen atom, the use of negative-frequency part $\psi^{-}(x) = e^{-i\omega t}\psi(\mathbf{x})$ of the spinor $\psi(x)$ in the "role of the quantum-mechanical object" is not a valid. In this case, neither $|\psi(\mathbf{x})|^2$, nor $\overline{\psi}(\mathbf{x})\psi(\mathbf{x})$ is the probability distribution density with respect to the eigenvalues of the Fermi doublet coordinate operator. It is due to the fact [1] that, in the Dirac model, the $\mathbf{x}$ is not the experimentally observable Fermi doublet coordinate operator.

The application of the relativistic canonical quantum mechanics can be useful for the analysis of the experimental situation found in [46]. Such analysis is interesting due to the fact that (as it is demonstrated here in Sections 4 and 5) the relativistic canonical quantum mechanics is the most fundamental model of a Fermi-doublet.

Another interesting application of the relativistic canonical quantum mechanics is inspired by paper [47], where the quantum electrodynamics is reformulated in the Foldy–Wouthuysen representation. The author of paper [47] used essentially the result of paper [43] about the negative mass of antiparticles. Starting from the relativistic canonical quantum mechanics, we are able not to appeal to the conception of the negative mass of antiparticles.

1. L. Foldy and S. Wouthuysen, Phys. Rev. **78**, 29 (1950).
2. L. Foldy, Phys. Rev. **102**, 568 (1956).
3. L. Foldy, Phys. Rev. **122**, 275 (1961).
4. E. Salpeter, Phys. Rev. **87**, 328 (1952).
5. W. Lucha and F. Schobert, Phys. Rev. A. **54**, 3790 (1996).
6. Y. Chargui and A. Trabelsi, Phys. Lett. A **377**, 158 (2013).
7. I. Krivsky, V. Simulik, I. Lamer, and T. Zajac, arXiv: 1301.6343 [math-ph] 27 Jan 2013.
8. I. Krivsky, V. Simulik, I. Lamer, and T. Zajac, TWMS J. App. Eng. Math. **3**, 62 (2013).
9. I. Krivsky, V. Simulik, T. Zajac, and I. Lamer, in: *Proceedings of the 14-th Int. Conference "Mathematical Methods in Electromagnetic Theory", August 28–30, 2012* (Institute of Radiophysics and Electronics, Kharkiv, 2012), pp. 201–204.
10. N. Bogoliubov, A. Logunov, and I. Todorov, *Foundations of the Axiomatic Approach in Quantum Field Theory* (Nauka, Moscow, 1969) (in Russian).
11. V. Simulik and I. Krivsky, Dopov. NAN Ukr., No. 5, 82 (2010).
12. I. Krivsky and V. Simulik, Cond. Matt. Phys. **13**, 43101 (2010).
13. V. Simulik and I. Krivsky, Phys. Lett. A **375**, 2479 (2011).
14. V. Simulik, I. Krivsky, and I. Lamer, TWMS J. App. Eng. Math. **3**, 46 (2013).
15. V. Simulik and I. Krivsky, Ukr. J. Phys. **58**, 523 (2013).
16. P. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1958).
17. N. Bogoliubov and D. Shirkov, *Introduction to the Theory of Quantized Fields* (Wiley, New York, 1980).
18. J. Sakurai, *Advanced Quantum Mechanics* (Addison–Wesley, New York, 1967).
19. L. Ryder, *Quantum Field Theory* (Cambridge Univ. Press, Cambridge, 1996).
20. J. Keller, *Theory of the Electron. A Theory of Matter from START* (Kluwer, Dordrecht, 2001).
21. V. Fock and D. Iwanenko, Z. Phys. **54**, 798 (1929).
22. V. Fock, Z. Phys. **57**, 261 (1929).
23. A. Wightman, in *Dispersion Relations and Elementary Particles*, edited by C. De Witt and R. Omnes (Wiley, New York, 1960), p. 291.
24. H. Sallhofer, Z. Naturforsch. A **33**, 1378 (1978).
25. H. Sallhofer, Z. Naturforsch. A **41**, 468 (1986).
26. S. Strinivasan and E. Sudarshan, J. Phys. A **29**, 5181 (1996).
27. L. Lerner, Eur. J. Phys. **17**, 172 (1996).
28. T. Kubo, I. Ohba, and H. Nitta, Phys. Lett. A **286**, 227 (2001).
29. H. Cui, arXiv: quant-ph/0102114, Aug. 15, 2001.
30. Y.J. Ng and H. van Dam, arXiv: hep-th/0211002, Feb. 4, 2003.
31. M. Calerier and L. Nottale, Electromagn. Phen. **3(9)**, 70 (2003).
32. M. Evans, Found. Phys. Lett. **16**, 369 (2003).
33. M. Evans, Found. Phys. Lett. **17**, 149 (2004).
34. V.M. Simulik and I.Yu. Krivsky, Rep. Math. Phys. **50**, 315 (2002).
35. V.M. Simulik and I.Yu. Krivsky, Electromagn. Phen. **3(9)**, 103 (2003).
36. V.M. Simulik, in *What is the Electron?*, edited by V.M. Simulik (Apeiron, Montreal, 2005), p. 109.
37. V. Vladimirov, *Methods of the Theory of Generalized Functions* (Taylor and Francis, London, 2002).
38. von J. Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932).
39. G. Bluman and S. Anco, *Symmetry and Integration Methods for Differential Equations* (Springer, New York, 2002).
40. J.P. Elliott and P.J. Dawber, *Symmetry in Physics* (Macmillan Press, London, 1979), Vol. 1.
41. B.G. Wybourne, *Classical Groups for Physicists* (Wiley, New York, 1974).
42. H. Bondi, Rev. Mod. Phys. **29**, 423 (1957).
43. E. Recami, and G. Zino, Nuovo Cim. A **33**, 205 (1976).
44. G. Landis, Journ. Propulsion and Power **7**, 304 (1991).

45. R. Wayne, Turk. J. Phys. **36**, 165 (2012).
46. W. Kuellin, P. Gaal, K. Reimann, T. Worner, T. Elsaesser, and R. Hey, Phys. Rev. Lett. **104**, 146602(1-4) (2010).
47. V.P. Neznamov, Phys. Part. Nucl. **37**, 86 (2006).

Received 24.09.2013

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КВАНТОВОМЕХАНІЧНИЙ ОПИС ФЕРМІОННОГО ДУБЛЕТУ ТА ЙОГО ЗВ'ЯЗОК З РІВНЯННЯМ ДІРАКА

Р е з ю м е

Представлено короткий огляд різних способів виводу рівняння Дірака. На базі рівняння руху Шредінгера–Фолді сформульовано основи релятивістської канонічної квантової механіки ферміонного дублету. Рівняння Дірака у нашому підході виведено з рівняння Шредінгера–Фолді.

В.М. Симулик, И.Ю. Кривский

КВАНТОВОМЕХАНИЧЕСКОЕ ОПИСАНИЕ ФЕРМИОННОГО ДУБЛЕТА И ЕГО СВЯЗЬ С УРАВНЕНИЕМ ДИРАКА

Р е з ю м е

Представлен краткий обзор разных способов вывода уравнения Дирака. На базе уравнения движения Шредингера–Фолди сформулированы основы релятивистской канонической квантовой механики фермионного дублета. Уравнение Дирака в нашем подходе выведено из уравнения Шредингера–Фолди.