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UNUSUAL MANIFESTATIONS OF THE PAULI PRINCIPLE IN SCATTERING OF ATOMIC NUCLEI

Within the microscopic model based on the algebraic version of the resonating group method, the role of the Pauli principle in the formation of the continuum wave function of the $\alpha+$ $+t+n$ nuclear system has been investigated. The norm kernel for the $\alpha+t+n$ three-cluster system has been constructed in the Fock–Bargmann space. The complete classification of the eigenfunctions and the eigenvalues of the ^8Li norm kernel by the eigenvalues of the $^7Li=\alpha+t$, $^5He=\alpha+n$, and $^4H=t+n$ binary subsystems has been given.

K e y w o r d s: Pauli principle, resonating group method, Fock–Bargmann space, wave function.

1. Introduction

In the present paper, we will analyze the role of the Pauli principle in the formation of the continuum wave function of the $\alpha+t+n$ nuclear system within the microscopic model based on the algebraic version of the resonating group method. Our principal concern has been with the study of exchange effects contained in the genuine three-cluster norm kernel, i.e., taking the eigenvalues of Pauli-allowed states into account.

Near the neutron and proton drip-lines, novel behavior modes of nuclei, such as the cluster decay, two-proton radioactivity, neutron halo, disappearance of magic numbers, were observed. Hence, the investigation of the structure and the decay of neutron-rich and proton-rich nuclei is very important for understanding the properties of nuclear matter under extreme conditions.

The majority of states in neutron-rich nuclei are unbound. For example, ^{8}Li nucleus has only two states below the $^8\mathrm{Li}\to{}^7\mathrm{Li}+n$ decay threshold (see Figure). In view of the proximity of the latter threshold to the ground state of ^{8}Li nucleus, ^{8}Li can be considered as a $^7\mathrm{Li}+n$ two-cluster system in the states with the energy falling in the range $E_{\mathrm{g.s.}} < E < 4.5$ MeV. At the energy $E = 4.5$ MeV above the ground state of ^{8}Li, the $^8\mathrm{Li}\to\alpha+t+n$ decay becomes possible, and, hence, all states of ^{8}Li nucleus located above this threshold have three-cluster structure. The main difficulty in studies of resonances in a three-cluster system consists in the formulation of correct asymptotic boundary conditions for a wave function in the continuum. Such boundary conditions should ensure a continuous transition from the region of small distances between the clusters forming the system in question, where exchange effects are operative, to the asymptotic region, where the formation of the scattering matrix occurs.

In Ref. [2], we have shown that, in the case of three-cluster systems composed of an s-cluster and two neutrons, the correct asymptotic boundary conditions can be found, by employing a complete basis of Pauli-allowed harmonic-oscillator states (classified with the use of the SU(3) symmetry indices and defined in the Fock–Bargmann space) along with their eigenvalues. Analyzing the structure of the eigenfunctions and the behavior of the eigenvalues of the antisymmetrization operator of $^3\mathrm{H}+n+n$ system, we observed that the asymptotic behavior of basis functions consistent with the requirements of the Pauli principle gives an indication of possible decay channels of ^{5}H nucleus and allows us to specify the most

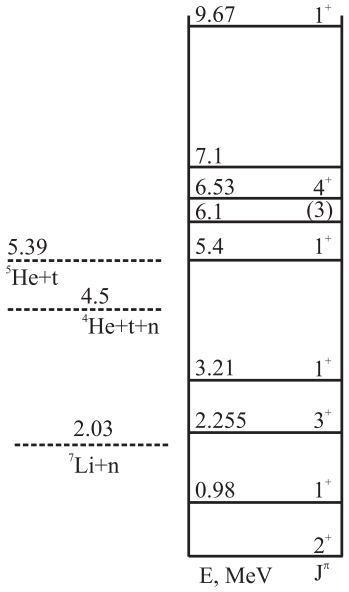
important decay channels. We have shown that such asymptotic behavior corresponds to the subsequent decay $^5\mathrm{H} \to n{+}^4\mathrm{H} \to n + n{+}^3\mathrm{H}$ rather than to the direct decay of ^{5}H nucleus.

In Ref. [3], the complete classification of the eigenfunctions and the eigenvalues of the ^{12}C norm kernel by the ^{8}Be$= \alpha + \alpha$ eigenvalues has been given for the first time. We have demonstrated that the Pauli-allowed states of the 3α system can be arranged into branches and families. The eigenvalues belonging to a given branch tend to the same limit value $\lambda_{2k}^{^8\mathrm{Be}}$ of the 2α subsystem with the number of quanta increasing. The branches, which share limit eigenvalues, are combined in a family of eigenstates, which asymptotically corresponds to a certain binary decay channel of the 3α system into a 2α subsystem occurring in a particular harmonic-oscillator state and a remaining α-particle. Hence, the excited states of ^{12}C nucleus should decay via the subsequent stage $^{12}\mathrm{C} \to {}^8\mathrm{Be} + \alpha \to \alpha + \alpha + \alpha$ rather than in a "democratic" way $^{12}\mathrm{C} \to \alpha + \alpha + \alpha$ usually associated with hyperspherical harmonics.

^{8}Li nucleus is an example of a three-cluster system composed of three different clusters. We have shown that the eigenvalues of binary subsystems of a three-cluster system play a crucial role in the classification of the eigenvalues of the three-cluster norm kernel and rather determination of dominant decay channels of a three-cluster system. Based on the results of Refs. [2, 3], we can conclude that, in the case where a binary subsystem of the three-cluster system is characterized by unit eigenvalues of the antisymmetrization operator, the corresponding binary decay channel of a three-cluster system into the above-mentioned subsystem and a remaining cluster is not realized. In ^{8}Li nucleus, all possible binary decay channels can be realized, because the eigenvalues of the antisymmetrization operator in all two-cluster subsystems differ from unity.

2. Theoretical Approach

Following the resonating group method (RGM) [4], it will be supposed hereafter that the considered nuclear system consists of three clusters. An RGM wave function is built in the form of an antisymmetrized product of cluster internal wave functions and a wave function of their relative motion. The internal wave functions of the clusters are fixed[1], and the wave function of relative motion of the clusters, which depends only on two Jacobi vectors of the considered three-cluster system, is found by solving an integro-differential equation. The latter is obtained by substitution of the RGM wave function into the Schrödinger equation followed by integration with respect to single-particle coordinates. The integro-differential equation can be transformed into a system of linear equations by expanding the wave function of the cluster relative motion into the complete basis of the Pauli-allowed harmonic oscillator states, as the RGM suggests. Another important simplification can be achieved by transformation from the coordinate space to the space of complex generator parameters (the Fock–Bargmann space [5]), in which the basis functions are of an especially simple form and are expressed via powers of complex vectors. Thus, the wave functions of the considered discrete representation are reduced to power series with an infinitely large convergence radius. The validity of this statement is indicated by the fact that all the wave functions in the Fock–Bargmann space are entire and analytic. Therefore, the series of these functions in pow-



Energy levels of ^{8}Li nucleus taken from [1]

[1] Here, we shall assume the intrinsic cluster wave functions to be the simplest functions of a translation-invariant shell model.

ers of complex vectors converge in any finite region of the complex space.

First and foremost, the RGM calls for the construction of a complete basis of Pauli-allowed harmonic oscillator states and their classification. This is accomplished by solving the problem of eigenvalues and eigenfunctions for the norm kernel, i.e., the overlap integral of the two Slater determinants composed of the single-particle orbitals:

$$I(\{\mathbf{S}_j\},\{\mathbf{R}_j\}) = \int \Phi(\{\mathbf{S}_j\},\mathbf{r})\Phi(\{\mathbf{R}_j\},\mathbf{r})d\tau. \qquad (1)$$

Here, the integration is performed over all single-particles vectors, $\{\mathbf{R}_j\}$ (or $\{\mathbf{S}_j\}$) identifies the collection of three complex vectors determining the position of the center-of-mass of clusters in the Fock–Bargmann space. For the spatial part of the single-particle wave functions, we used the modified Bloch–Brink orbitals:

$$\phi(\mathbf{r}_i) = \frac{1}{\pi^{3/4}} \exp\left(-\frac{1}{2}\mathbf{r}_i^2 + \sqrt{2}(\mathbf{R}_j\mathbf{r}_i) - \frac{1}{2}\mathbf{R}_j^2\right),\ i \in A_j,$$

where A_j is the number of nucleons in the j-th cluster. Each of these orbitals is an eigenfunction of the coordinate operator $\hat{\mathbf{r}}_i$:

$$\hat{\mathbf{r}}_i = \frac{1}{\sqrt{2}}(\mathbf{R}_j + \boldsymbol{\nabla}_{\mathbf{R}_j});\quad \mathbf{R}_j = \frac{\boldsymbol{\xi}_j + i\boldsymbol{\eta}_j}{\sqrt{2}},$$

which is defined in the Fock–Bargmann space and corresponds to the eigenvalue $\mathbf{r}_i$, and $\boldsymbol{\xi}_j$ and $\boldsymbol{\eta}_j$ are the vectors of coordinate and momentum, respectively. At the same time, the orbital $\phi(\mathbf{r}_i)$ is the kernel of the integral transformation from the coordinate representation to the Fock–Bargmann representation [5] and the generating function for the harmonic-oscillator basis [6]. The center-of-mass motion are factored out (and dropped out from now on) by transition from the generator parameters $\{\mathbf{R}_j\}$ to the Jacobi vectors:

$$\mathbf{R}_{\rm cm} = \frac{1}{\sqrt{A}}(A_1\mathbf{R}_1 + A_2\mathbf{R}_2 + A_3\mathbf{R}_3),$$

$$\mathbf{a} = \sqrt{\frac{A_1(A_2+A_3)}{A}}\left(\mathbf{R}_1 - \frac{A_2\mathbf{R}_2 + A_3\mathbf{R}_3}{A_2+A_3}\right),$$

$$\mathbf{b} = \sqrt{\frac{A_2A_3}{A_2+A_3}}(\mathbf{R}_2 - \mathbf{R}_3).$$

As a result, the overlap of two Slater determinants composed of the modified Bloch–Brink orbitals generates a complete basis of Pauli-allowed harmonic-oscillator functions along with their eigenvalues:

$$I(\{\mathbf{S}_j\},\{\mathbf{R}_j\}) \Rightarrow I(\mathbf{a},\mathbf{b};\tilde{\mathbf{a}},\tilde{\mathbf{b}}) =$$

$$= \sum_n \Lambda_n \Psi_n(\mathbf{a},\mathbf{b})\Psi_n(\tilde{\mathbf{a}},\tilde{\mathbf{b}}).$$

The functions Ψ_n are defined in the Fock–Bargmann space and orthonormalized with the Bargmann measure $d\mu_{\rm B}$:

$$d\mu_{\rm B} = \exp\{-(\mathbf{a}\tilde{\mathbf{a}})\}\,\frac{d\boldsymbol{\xi}_a d\boldsymbol{\eta}_a}{(2\pi)^3}\,\exp\{-(\mathbf{b}\tilde{\mathbf{b}})\}\,\frac{d\boldsymbol{\xi}_b\, d\boldsymbol{\eta}_b}{(2\pi)^3},$$

n stands for the set of the quantum numbers of basis functions. The quantum numbers and the structure of functions Ψ_n will be discussed in the next section. Let us discuss now the set of quantum numbers of the Pauli-allowed states $\Psi_n(\mathbf{a},\mathbf{b})$. For the three-cluster system ^{8}Li considered here, n includes the number of oscillator quanta ν, the indices (λ,μ) of their SU(3) symmetry, the additional quantum number k if there are several different (λ,μ) multiplets, the orbital momentum L, its projection M, and, if necessary, one more additional quantum number α_L. The latter is needed to label the states with the same L in a given (λ,μ) multiplet. It is well known that the diagonalization of the norm kernel requires the basis to be labeled with the quantum indices (λ,μ) of irreducible representations of the SU(3) group [7]. The eigenvalues of the norm kernel depend on the total number of the oscillator quanta and (λ,μ) and do not depend on the angular momenta of the basis states. We restrict our consideration to the states with unit orbital momentum $L = 1$ and positive parity. Hence, the number of oscillator quanta should be even and equal 2ν, and the second index SU(3)-symmetry should be odd and equal $2\mu+1$. The quantum numbers L and M will be dropped from now on.

3. Eigenvalues and Eigenfunctions of the Norm Kernel of ^{8}Li $= \alpha + t + n$ System

By definition, the states $\Psi_n(\mathbf{a},\mathbf{b})$ are the eigenfunctions of the antisymmetrization operator $\hat{A}$:

$$\hat{A}\Psi_{(2\nu-4\mu-2,2\mu+1)_k} =$$

$$= \Lambda_{(2\nu-4\mu-2,2\mu+1)_k}\Psi_{(2\nu-4\mu-2,2\mu+1)_k}.$$

In the Fock–Bargmann space, the latter functions are superpositions of the eigenfunctions of the second-order Casimir operator:

$$\Psi_{(2\nu-4\mu-2,2\mu+1)_k}(\mathbf{a}_i,\mathbf{b}_i) = \\ = \sum_{m=2\mu+1}^{2\nu-2\mu-1} D^{m-2\mu-1}_{(2\nu-4\mu-2,2\mu+1)_k}\psi^{m-2\mu-1}_{(2\nu-4\mu-2,2\mu+1)}(\mathbf{a}_i,\mathbf{b}_i);$$

where the index $1 \le i \le 3$ enumerates different Jacobi trees. The eigenvalues of the norm kernel do not depend on the choice of a Jacobi tree, whereas the structure of the eigenfunctions does. The Pauli-allowed basis states of ^{8}Li$=\alpha+t+n$ system take the simplest form in the $(\mathbf{a}_3,\mathbf{b}_3)$-tree, where the vector $\mathbf{b}_3$ describes the relative distance between the α-cluster and tritium, while the vector $\mathbf{a}_3$ indicates the distance from the valence neutron to the center-of-mass of the $\alpha+t$ subsystem:

$$\Psi_{(2\nu-2,1)_k}(\mathbf{a}_3,\mathbf{b}_3) = \sum_{m=3}^{2\nu-1} D^{m-1}_{(2\nu-2,1)_k}\psi^{m-1}_{(2\nu-2,1)}(\mathbf{a}_3,\mathbf{b}_3).$$

The Pauli-allowed basis function belonging to the leading SU(3) representation $(2\nu-2,1)$ contains two terms less in the $(\mathbf{a}_3,\mathbf{b}_3)$-tree than in other Jacobi trees for the reason that the minimally possible number of quanta along the vector $\mathbf{b}_3$ equals three, while the minimum Pauli-allowed number of quanta along the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ equals one. Hence, namely the $(\mathbf{a}_3,\mathbf{b}_3)$-tree is best suited to the construction and the analysis of these states.

The quantities $\mathbf{a}_1,\mathbf{b}_1$ and $\mathbf{a}_2,\mathbf{b}_2$ are the Jacobi vectors of two alternative Jacobi trees, with the vector $\mathbf{b}_1$ ($\mathbf{b}_2$) describing the relative distance between the valence neutron and tritium (α-cluster). Respectively, the vector $\mathbf{a}_1$ ($\mathbf{a}_2$) determines the position of the remaining cluster relative to the center-of-mass of ^{4}H (^{5}He) subsystem. The Jacobi vectors of each Jacobi tree are related to others via a unitary transformation:

$$\mathbf{a}_3 = \cos\alpha_{1,2}\,\mathbf{a}_{1,2} + \sin\alpha_{1,2}\,\mathbf{b}_{1,2};$$

$$\mathbf{b}_3 = \mp\sin\alpha_{1,2}\,\mathbf{a}_{1,2} \pm \cos\alpha_{1,2}\,\mathbf{b}_{1,2}.$$

It was found in Ref. [8] that the eigenfunctions of the Casimir operator, with powers of the vectors $\mathbf{a}$ and $\mathbf{b}$ being fixed, are expressible in terms of the hypergeometric functions ${}_2F_1(\alpha,\beta;\gamma;Z)$ of the variable

$$Z = \frac{[\mathbf{ab}]^2}{\mathbf{a}^2\mathbf{b}^2}.$$

Namely,

$$\psi^{2m-2\mu}_{(2\nu-4\mu-2,2\mu+1)}(\mathbf{a},\mathbf{b}) = N^{2m-2\mu}_{(2\nu-4\mu-2,2\mu+1)}\times$$

$$\times\,\mathbf{a}^{2\nu-2m-2\mu-2}\mathbf{b}^{2m-2\mu}[\mathbf{ab}]^{2\mu+1}\times$$

$$\times\,{}_2F_1\left(-\nu+m+\mu+1,-m+\mu;-\nu+2\mu+\frac{3}{2};Z\right);$$

$$\psi^{2m-2\mu-1}_{(2\nu-4\mu-2,2\mu+1)}(\mathbf{a},\mathbf{b}) = N^{2m-2\mu-1}_{(2\nu-4\mu-2,2\mu+1)}\times$$

$$\times\,\mathbf{a}^{2\nu-2m-2\mu-2}\mathbf{b}^{2m-2\mu-2}(\mathbf{ab})[\mathbf{ab}]^{2\mu+1}\times$$

$$\times\,{}_2F_1\left(-\nu+m+\mu+1,-m+\mu+1;-\nu+2\mu+\frac{3}{2};Z\right).$$

In two-cluster systems, the eigenvalues of the antisymmetrizer tend to 1 as $\nu\to\infty$, and the deviations from 1 are due to the Pauli exclusion principle. Contrastingly, the eigenvalues of three-cluster systems tend to eigenvalues of a two-cluster subsystem with increasing the number of oscillator quanta ν:

$$\lim_{\nu-2\mu\to\infty}\Lambda^{^8\mathrm{Li}=\alpha+t+n}_{(2\nu-4\mu-2,2\mu+1)_{k_1}} \to \lambda^{^4\mathrm{H}=t+n}_{k_1} = 1-\left(-\frac{1}{3}\right)^{k_1},$$

$$\lim_{\nu-2\mu\to\infty}\Lambda^{^8\mathrm{Li}=\alpha+t+n}_{(2\nu-4\mu-2,2\mu+1)_{k_2}} \to \lambda^{^5\mathrm{He}=\alpha+n}_{k_2} = 1-\left(-\frac{1}{4}\right)^{k_2},$$

$$\lim_{\nu-2\mu\to\infty}\Lambda^{^8\mathrm{Li}=\alpha+t+n}_{(2\nu-4\mu-2,2\mu+1)_{k_3}} \to \lambda^{^7\mathrm{Li}=\alpha+t}_{k_3} =$$

$$= 1-\left(-\frac{3}{4}\right)^{k_3} - 3\left(\frac{5}{12}\right)^{k_3} + 3\left(-\frac{1}{6}\right)^{k_3}, \quad k_3 \ge 3.$$

Here, as in ^{5}H and the ^{12}C three-cluster systems, the number of quanta k_i in binary subsystems serves as an additional quantum number of the SU(3) degenerate three-cluster states of ^{8}Li, with the only difference that, instead of one number k, we have three different k_i by the number of possible binary subsystems.

As the eigenvalues $\Lambda^{^8\mathrm{Li}=\alpha+t+n}_{(2\nu-4\mu-2,2\mu+1)_{k_i}}$ of the norm kernel approach the limit values of binary subsystems, the corresponding eigenvectors $\Psi_{(2\nu-4\mu-2,2\mu+1)_{k_i}}$ take a simple analytic form:

$$\Psi_{(2\nu-4\mu-2,2\mu+1)_{k_i}}(\mathbf{a}_3,\mathbf{b}_3) \to \psi^{k_i-2\mu-1}_{(2\nu-4\mu-2,2\mu+1)}(\mathbf{a}_i,\mathbf{b}_i), \tag{2}$$

Such asymptotic behavior takes place with proviso that $\nu \gg k_i$.

The remarkable feature of the asymptotic relations (2) lies in the fact that, in the limit $\nu \gg k_{1,2}$, the expansion coefficients $D^{m-2\mu-1}_{(2\nu-4\mu-2,2\mu+1)_{k_{1,2}}}$ can be identified with the Krawtchuk polynomials of a discrete variable [9]:

$$D^{m-2\mu-1}_{(2\nu-4\mu-2,2\mu+1)_{k_{1,2}}} \to \mathcal{K}^{(p_{1,2})}_{k_{1,2}-2\mu-1}(m-2\mu-1)\times$$
$$\times \frac{\sqrt{\rho_{m-2\mu-1}}}{d_{k_{1,2}-2\mu-1}},$$
$$p_{1,2} = \sin^2\alpha_{1,2}, \quad q = 1-p.$$

The Krawtchuk polynomials $\mathcal{K}^{(p)}_k(m)$ of a discrete variable m are specified on the interval $0 \le m \le 2\nu$ and orthogonal with the weighting function ρ_m and the norm d_k:

$$\mathcal{K}^{(p)}_k(m)\frac{\sqrt{\rho_m}}{d_k} = (-1)^k \frac{(2\nu)!}{\sqrt{(2\nu-k)!(2\nu-m)!m!k!}}\times$$
$$\times \left(\frac{p}{q}\right)^{\frac{m+k}{2}} q^{\nu}\,{}_2F_1\left(-k,-m;-2\nu;\frac{1}{p}\right).$$

Note that ν, m, and k are the natural integers, $0 \le \le k \le 2\nu$. The Krawtchuk polynomials are a discrete analog of the Hermitian polynomials.

As for the asymptotic behavior of the expansion coefficients $D^{m-2\mu-1}_{(2\nu-4\mu-2,2\mu+1)_{k_3}}$, it has exceptionally simple form:

$$D^{m-2\mu-1}_{(2\nu-4\mu-2,2\mu+1)_{k_3}} \to \delta_{k_3,m-2\mu-1}.$$

4. Asymptotic Equations for Expansion Coefficients

We seek the wave function of the considered three-cluster system in the form of an expansion over the SU(3) basis of Pauli-allowed states

$$\Upsilon_{\kappa\,(E)}(\mathbf{a},\mathbf{b}) = \sum_n \sqrt{\Lambda_n} C^{\kappa\,(E)}_n \Psi_n(\mathbf{a},\mathbf{b}).$$

The coefficients in the expansions of both discrete states having energies in the region $E_\kappa = -\kappa^2/2 < 0$ and continuum states having energies in the region $E > 0$ are a solution of the system of linear equations

$$\sum_{n'} \langle n|\hat{H}|n'\rangle C_{n'} - E\Lambda_n C_n = 0. \qquad (3)$$

In the limit $\nu \gg k$, the variables in Eqs. (3) are separated with a result that this system of equations can be represented in the form of two systems of equations. One of the two systems of equations describes the relative motion of ^{4}H, ^{5}He, or ^{7}Li subsystem and a remaining cluster:

$$-\frac{1}{4}\sqrt{(2\nu-k_i-l_i)(2\nu-k_i+l_i+1)}C^{l_i}_{\nu-1,k_i}+$$
$$+\frac{1}{2}\left(2\nu-k_i+\frac{3}{2}-2(E-\varepsilon)\right)C^{l_i}_{\nu,k_i}-$$
$$-\frac{1}{4}\sqrt{(2\nu-k_i-l_i+2)(2\nu-k_i+l_i+3)}C^{l_i}_{\nu+1,k_i} = 0.$$

The second system of equations characterizes $^4\mathrm{H} = {}^3\mathrm{H}+n$, $^5\mathrm{He} = {}^4\mathrm{He}+n$, or the $^7\mathrm{Li} = {}^4\mathrm{He}+t$ binary subsystem:

$$-\sqrt{\frac{\lambda_{k_i-2}}{\lambda_{k_i}}}\frac{1}{4}\sqrt{(k_i-l_i)(k_i+l_i+1)}C^{l_i}_{\nu-1,k_i-2}+$$
$$+\frac{1}{2}\left(k_i+\frac{3}{2}-2\varepsilon\right)C^{l_i}_{\nu,k_i}-$$
$$-\sqrt{\frac{\lambda_{k_i}}{\lambda_{k_i+2}}}\frac{1}{4}\sqrt{(k_i-l_i+2)(k_i+l_i+3)}C^{l_i}_{\nu+1,k_i+2} = 0.$$

Here, we set the nucleon mass, Planck constant, and oscillator length to be equal to 1. The quantum number l is the orbital angular momentum of ^{4}H, ^{5}He, or the ^{7}Li binary subsystem; it is equal to the orbital angular momentum of the relative motion of the remaining cluster and the above-mentioned subsystem. The total energy E is equal to the sum of the internal energy ε of the binary subsystem and the energy of the relative motion of the remaining cluster and the binary subsystem.

The form of these equations indicates that the problem of the decay of the ^{8}Li three-cluster system to an α-particle, tritium, and a neutron reduces to the multichannel problem of the decay of ^{8}Li by three different channels: $^8\mathrm{Li}\to{}^7\mathrm{Li}+n$, $^8\mathrm{Li}\to{}^4\mathrm{H}+\alpha$, and $^8\mathrm{Li}\to{}^5\mathrm{He}+t$. Therewith, each of three binary subsystems is in a localized state determined by the maximum number of quanta corresponding to the relative motion of the two clusters forming a binary subsystem. This number is equal, in turn, to the maximum number $k_i^{\max}$ of families that were taken into account.

When we consider a limited number of families, we thereby prevent one of the clusters from going away from another cluster beyond some fixed distance. As a result, the spectrum of any binary subsystem becomes quasidiscrete; that is, the binary subsystem

from the decay of ^{8}Li may prove to be in one of the states of the continuous spectrum with energy ε_i and orbital angular momentum l_i. The greater the extent to which the subsystem is localized, the larger the number of levels and the smaller the spacing between them. If we take a fixed number of $k_i^{\max}$ families into account, the asymptotic behavior of the expansion coefficients takes the form:

$$C^{l_i}_{k_i,\,\nu}(E-\varepsilon_n) = c^{l_i}_{k_i}(\varepsilon_n)\times$$

$$\times\left\{H^-_{l_i+1/2}\left(\sqrt{2\,(E-\varepsilon_n)}\sqrt{4\nu-2k_i+3}\right)+\right.$$

$$\left.+\,S_{l_i,\,\varepsilon_n\to l_i,\,\varepsilon_n}H^+_{l_i+1/2}\left(\sqrt{2\,(E-\varepsilon_n)}\sqrt{4\nu-2k_i+3}\right)\right\}+$$

$$+\sum_{i=1}^{3}\sum_{m=1}^{k_i^{\max}} c^{l'_i}_{\varepsilon'_m}(k_i)\times$$

$$\times\, S_{l_i,\,\varepsilon_n\to l'_i,\,\varepsilon'_m}H^+_{l'_i+1/2}\left(\sqrt{2\,(E-\varepsilon'_m)}\sqrt{4\nu-2k_i+3}\right).$$

Here, $c^{l_i}_{k_i}(\varepsilon_n)$ are the coefficients in the expansion of the wave function for the binary subsystem, which were obtained by diagonalizing the Hamiltonian for the binary subsystem, and l_i, ε_n is the entrance channel here.

5. Binary Cluster Configurations of ^{8}Li Nucleus

The state that corresponds to the translation-invariant shell-model function projected onto the completely antisymmetrized wave function for a cluster system is the simplest among the allowed basis states ensuring the description of the relative motion of the clusters, whose intrinsic wave functions are fixed. In the case where the number of oscillator quanta is minimal, the same allowed eigenfunction in the coordinate representation (i.e., the shell-model wave function for the compound nucleus in the ground state) corresponds to each of the cluster configurations. In the representation of the discrete basis of the RGM, however, the eigenfunctions of different configurations are expressed in terms of Jacobi vectors belonging to different trees.

In this section, three main binary cluster configurations of ^{8}Li nucleus are considered: ^{8}Li $=^7$Li$+n$, ^{8}Li $=^5$He$+t$, and ^{8}Li $=^4$H$+\alpha$. All these configurations correspond to the decay of ^{8}Li into a binary subsystem, being in the lowest oscillator shell model state, and a remaining cluster. These families appears first and allows one to understand an asymptotic behavior of the simplest families of Pauli-allowed states of the three-cluster system $\alpha+t+n$.

5.1. ^{8}Li $=^7$Li$+n$

The translation-invariant norm kernel of the two-cluster system ^{7}Li$+n$ takes the form

$$I_{^7\mathrm{Li}+n} = \frac{(\mathbf{b}_3\tilde{\mathbf{b}}_3)^3}{3!}\left\{\exp(\mathbf{a}_3\tilde{\mathbf{a}}_3)-\right.$$

$$\left.-\left(1+\frac{8}{7}(\mathbf{a}_3\tilde{\mathbf{a}}_3)\right)\exp\left(-\frac{1}{7}(\mathbf{a}_3\tilde{\mathbf{a}}_3)\right)\right\}+$$

$$+\frac{8}{7}\frac{(\mathbf{b}_3\tilde{\mathbf{b}}_3)^2([\mathbf{a}_3\mathbf{b}_3][\tilde{\mathbf{a}}_3\tilde{\mathbf{b}}_3])}{3!}\exp\left(-\frac{1}{7}(\mathbf{a}_3\tilde{\mathbf{a}}_3)\right).$$

The third power of the vector $\mathbf{b}_3$ points to the fact that ^{7}Li cluster has one proton and three neutrons in the p-shell.

Restricting ourselves to the states of positive parity and $L=1$, after the projection onto states with a definite SU(3)-symmetry, we obtain

$$I_{^7\mathrm{Li}+n} = \sum_{\nu=2}\lambda_{(2\nu-2,1)}\langle(2\nu-2,1)|(2\nu-2,1)\rangle+$$

$$+\sum_{\nu=3}\lambda_{(2\nu-6,3)}\langle(2\nu-6,3)|(2\nu-6,3)\rangle.$$

At a fixed number of oscillator quanta, there are two Pauli-allowed states belonging to the $(2\nu-2,1)$ and $(2\nu-6,3)$ SU(3)-representations:

$$\lambda_{(2\nu-2,1)} = 1+\frac{75-32\nu}{3}\left(\frac{1}{7}\right)^{2\nu-3},$$

$$\lambda_{(2\nu-6,3)} = 1+9\left(\frac{1}{7}\right)^{2\nu-3}.$$

The eigenvalues $\lambda_{(2\nu-2,1)}$ tend to 1 from above, which indicates the attraction between a neutron and ^{7}Li cluster. The eigenvalues $\lambda_{(2\nu-6,3)}$ approach 1 from below, which corresponds to the repulsion between a neutron and ^{7}Li cluster.

For the minimum number of quanta, the norm kernel of ^{7}Li$+n$ reduces to the leading SU(3) irreducible representation $(2,1)$:

$$I_{^7\mathrm{Li}+n} = \frac{32}{21}\frac{1}{8}(\mathbf{b}_3\tilde{\mathbf{b}}_3)^2([\mathbf{a}_3\mathbf{b}_3][\tilde{\mathbf{a}}_3\tilde{\mathbf{b}}_3])+\ldots.$$

This norm kernel contains states with the total orbital momentum $L = 1, 2, 3$ of ^{8}Li. The eigenvalue corresponding to the lowest Pauli-allowed basis function for ^{7}Li+n configuration equals 32/21.

5.2. ^{8}Li = ^{5}He+t

The norm kernel of ^{5}He+t configuration looks like

$$I_{^5\mathrm{He}+t} = \exp\left(-\frac{3}{5}(\mathbf{a}_2\tilde{\mathbf{a}}_2)\right)\left(\exp\left(\frac{8}{15}(\mathbf{a}_2\tilde{\mathbf{a}}_2)\right)-1\right)^2\times$$
$$\times\left\{(\mathbf{b}_2\tilde{\mathbf{b}}_2)\left(\exp\left(\frac{8}{15}(\mathbf{a}_2\tilde{\mathbf{a}}_2)\right)-1-\frac{8}{15}(\mathbf{a}_2\tilde{\mathbf{a}}_2)\right)+\right.$$
$$\left.+\frac{8}{15}(\mathbf{a}_2\tilde{\mathbf{a}}_2)([\mathbf{a}_2\mathbf{b}_2][\tilde{\mathbf{a}}_2\tilde{\mathbf{b}}_2])\right\}.$$

This norm kernel contains the first power of the vector $\mathbf{b}_2$, because ^{5}He cluster has one neutron in the p-shell.

Since we restrict our consideration to the states of positive parity and unit orbital momentum, we arrive at the following expression for the SU(3)-projected norm kernel of the ^{5}He+t cluster system:

$$I_{^5\mathrm{He}+t} = \sum_{\nu=2}\lambda_{(2\nu-2,1)}\langle(2\nu-2,1)|(2\nu-2,1)\rangle.$$

Here, the eigenvalues

$$\lambda_{(2\nu-2,1)} = 1-\frac{13}{7}\left(\frac{7}{15}\right)^{2\nu-3}-$$
$$-19\left(\frac{1}{15}\right)^{2\nu-3}+\frac{17}{9}\left(\frac{3}{5}\right)^{2\nu-3}$$

also approach 1 from above. Such behavior of the eigenvalues indicates that ^{5}He and tritium experience the effective attraction due to the exchange effects.

At a minimum number of quanta, we again arrive at the same leading $(2,1)$ SU(3)-representation

$$I_{^5\mathrm{He}+^3\mathrm{H}} = \left(\frac{16}{15}\right)^3\frac{1}{8}(\mathbf{a}_2\tilde{\mathbf{a}}_2)^2([\mathbf{a}_2\mathbf{b}_2][\tilde{\mathbf{a}}_2\tilde{\mathbf{b}}_2])+\ldots,$$

but a different lowest eigenvalue, which equals 16/15 in this case.

5.3. ^{8}Li = ^{4}H+α

The vector $\mathbf{b}_1$ enters the norm kernel of ^{4}H+α configuration of ^{8}Li nucleus only in the first power, because ^{4}H cluster also has one neutron in the p-shell:

$$I_{^4\mathrm{H}+\alpha} = \exp\left(-\frac{(\mathbf{a}_1\tilde{\mathbf{a}}_1)}{2}\right)\left(\exp\left(\frac{(\mathbf{a}_1\tilde{\mathbf{a}}_1)}{2}\right)-1\right)^2\times$$
$$\times\left\{(\mathbf{b}_1\tilde{\mathbf{b}}_1)\left(\exp\left(\frac{(\mathbf{a}_1\tilde{\mathbf{a}}_1)}{2}\right)-1-\frac{(\mathbf{a}_1\tilde{\mathbf{a}}_1)}{2}\right)+\right.$$
$$\left.+\frac{1}{2}([\mathbf{a}_1\mathbf{b}_1][\tilde{\mathbf{a}}_1\tilde{\mathbf{b}}_1])\right\}.$$

The Pauli-allowed states of positive parity and unit momentum are generated by the norm kernel possessing the $(2\nu-2,1)$ SU(3)-symmetry:

$$I_{^4\mathrm{H}+\alpha} = \sum_{\nu=2}\lambda_{(2\nu-2,1)}\langle(2\nu-2,1)|(2\nu-2,1)\rangle$$

with eigenvalues

$$\lambda_{(2\nu-1,1)} = 1+\frac{1-2\nu}{4^{\nu-1}}$$

approaching 1 from below. This is an indication of the effective repulsion between α-cluster and ^{4}H-cluster due to the Pauli principle.

Finally, the norm kernel generating the lowest shell-model function for ^{4}H+α configuration of ^{8}Li nucleus appears as

$$I_{^4\mathrm{H}+\alpha} = \frac{1}{4}\frac{1}{8}(\mathbf{a}_1\tilde{\mathbf{a}}_1)^2([\mathbf{a}_1\mathbf{b}_1][\tilde{\mathbf{a}}_1\tilde{\mathbf{b}}_1])+\ldots.$$

The eigenvalue of the normalization kernel of ^{4}H+α featuring the minimum number of quanta takes the lowest value 1/4.

6. Pauli-Allowed States of $\alpha + t + n$ System

The norm kernel of $\alpha+t+n$ system at a minimum number of quanta allowed by the Pauli principle is of the following form:

$$I_{\alpha+t+n} = \frac{49}{27}\frac{1}{8}(\mathbf{b}_3\tilde{\mathbf{b}}_3)^2([\mathbf{a}_3\mathbf{b}_3][\tilde{\mathbf{a}}_3\tilde{\mathbf{b}}_3]).$$

The appearance of this norm kernel is quite similar to that of the ^{7}Li+n two-cluster configuration. The difference is only in the eigenvalue, which is equal to 49/27 for the three-cluster model.

Therefore, assuming any two-cluster or three-cluster configuration of ^{8}Li, we arrive at a conceptually identical expression for the norm kernel at the minimally possible number of quanta. Only eigenvalues are different. In [10], we showed that the probability of the presence of one or another cluster configuration in the wave function for a binary cluster

Table 1. **Eigenvalues $\Lambda_{(2\nu-4\mu-2,2\mu+1)_{k_3}}$ of the norm kernel for $\alpha+t+n$ system**

ν	$k_3=3$		$k_3=4$		$k_3=5$			$k_3=6$			$k_3=7$			
	$\mu=0$	$\mu=1$	$\mu=0$	$\mu=1$	$\mu=0$	$\mu=1$	$\mu=2$	$\mu=0$	$\mu=1$	$\mu=2$	$\mu=0$	$\mu=1$	$\mu=2$	$\mu=3$
2	1.8148													
3	1.0388	1.2223	0.6473											
4	1.1986	1.1911	0.5968	0.5942	1.0484	1.2466		0.9406						
5	1.1927	1.1910	0.5950	0.5955	1.2091	1.2052	1.2037	0.7849	0.8165			1.1482		
6	1.1919	1.1910	0.5954	0.5955	1.2038	1.1996	1.1993	0.8024	0.8062	0.8064	1.1324	1.1396	1.1310	
7	1.1909	1.1910	0.5955	0.5955	1.1982	1.1993	1.1992	0.8057	0.8063	0.8064	1.1322	1.1283	1.1267	1.1274
8	1.1910	1.1910	0.5955	0.5955	1.1992	1.1992	1.1992	0.8063	0.8064	0.8064	1.1280	1.1270	1.1269	1.1269
9	1.1910	1.1910	0.5955	0.5955	1.1992	1.1992	1.1992	0.8064	0.8064	0.8064	1.1271	1.1269	1.1269	1.1269
10	1.1910	1.1910	0.5955	0.5955	1.1992	1.1992	1.1992	0.8064	0.8064	0.8064	1.1269	1.1269	1.1269	1.1269

system is proportional to the eigenvalue of an individual configuration.

So, we could compare our three-cluster model of ^{8}Li with the approximation of three coupled cluster configurations of this nucleus: ^{7}Li+n, ^{4}H + α, and ^{5}He+t. By virtue of the fact that ^{7}Li+n configuration is characterized by the largest eigenvalue, the contribution of precisely this cluster configuration to the wave function of ^{8}Li system would be dominant.

Of course, the three-cluster model cannot be reduced to a two-cluster model by truncation of the three-cluster model space. But a parallel can be drawn between the three-cluster model and the approximation of coupled binary configurations.

The Pauli-allowed states of the $\alpha+t+n$ system can be arranged into branches and families, with all the states of a particular branch having the common SU(3)-symmetry index μ, but differing in the first SU(3) index λ. The eigenvalues belonging to a given branch tend to the same eigenvalue λ_{k_i} of the $\alpha+t$, $\alpha+n$, or $t+n$ binary subsystems with increasing the number of quanta. The branches which share limit eigenvalues are combined in a family of eigenstates, which is completely determined by the number of oscillator quanta k_i in the two-cluster subsystem. Each family of Pauli-allowed states asymptotically corresponds to a certain binary decay channel of ^{8}Li into a two-cluster subsystem occurring in the ground or excited harmonic-oscillator state and a remaining cluster. Such asymptotic behavior gives an indication of possible decay channels of a three-cluster ^{8}Li nucleus and allows us to specify the most important decay channels of the nucleus under consideration.

Table 2. **Eigenvalues $\Lambda_{(2\nu-2,1)_{k_i}}$ of the norm kernel for $\alpha+t+n$ system, $i=1,2$**

ν	$k_1=1$	$k_1=2$	$k_2=1$	$l_2=2$
3	1.7131			
4	1.5826			
5	1.4797		1.1144	0.9726
6	1.4116		1.1764	0.9547
7	1.3714	0.8680	1.2185	0.9461
8	1.3501	0.8784	1.2360	0.9388
9	1.3402	0.8850	1.2445	0.9365
10	1.3360	0.8876	1.2479	0.9364

Eigenvalues belonging to the first five families corresponding to the ^{8}Li$\to^7$Li$+n$ decay channel are given in Table 1. Eigenvalues belonging to the first four families corresponding to the ^{8}Li$\to^4$H$+\alpha$ and ^{8}Li$\to^5$He$+t$ decay channels are given in Table 2.

One can readily see from Tables 1 and 2 that the ^{8}Li norm kernel eigenvalues belonging to the k_3 family tend to the eigenvalues of the two-cluster subsystem ^{7}Li with increasing the number of oscillator quanta ν, while the ^{8}Li norm kernel eigenvalues belonging to the k_1 and k_2 families tend to the eigenvalues of ^{4}H subsystem and ^{5}He subsystem, respectively. Obviously, the index k_i makes sense of the number of oscillator quanta accounted for by one of three binary subsystems. It is worth noting the fact that $\lambda_{k_i=2\tilde{k_i}+1}>1$, while $\lambda_{k_i=2\tilde{k_i}}<1$. This testifies to the attraction between clusters forming a binary subsystem in the states with odd number of quanta $k_i=2\tilde{k_i}+1$ and the repulsion in the states with

even number of quanta $k_i = 2\tilde{k}_i$. Moreover, the eigenvalues $\Lambda_{(2\nu-4\mu-2,2\mu+1)_{2\tilde{k}_i+1}}$ approach the limit eigenvalues $\lambda_{2\tilde{k}_i+1}$ mainly from above. As was concluded in [2], there are strong grounds to believe that the families of states characterized by the odd values of quantum number k_i dominate in the wave function of ^{8}Li system.

It is evident from Table 1 that the majority of Pauli-allowed states of the $\alpha + t + n$ system are related to the ^{8}Li$\to^7$Li$+n$ decay. This conclusion is consistent with the fact that the largest eigenvalue belongs to the ^{7}Li$+n$ binary configuration of ^{8}Li. The approximation of three coupled binary configurations of ^{8}Li would roughly correspond to such truncation of the $\alpha + t + n$ model space, when only the first three families of Pauli-allowed states are retained, namely, $k_3 = 3$, and $k_1 = k_2 = 1$. These three families should dominate in the wave function of ^{8}Li system.

7. Conclusions

Within a microscopic model based on the algebraic version of the resonating group method, the role of the Pauli principle in the formation of the continuum wave function of the $\alpha+t+n$ nuclear system has been investigated. Our principal concern has been with the study of the exchange effects contained in a genuine three-cluster norm kernel, by taking the eigenvalues of Pauli-allowed states into account.

The norm kernel for the $\alpha + t + n$ three-cluster system has been constructed in the Fock–Bargmann space. The careful analysis of the structure of the eigenfunctions and the behavior of the eigenvalues of the $\alpha + t + n$ norm kernel has been performed for the first time. The eigenvalues of the $\alpha + t + n$ three-cluster system are shown to tend to those of two-cluster subsystems, as the number of oscillator quanta ν increases. At the same time, the corresponding eigenvectors take a simple analytic form, as the number of oscillator quanta increases. We suggest a way of resolving the problem of SU(3) degeneracy of the Pauli-allowed states.

The complete classification of the eigenfunctions and the eigenvalues of the ^{8}Li norm kernel by the eigenvalues of the ^{7}Li $= \alpha + t$, ^{5}He $= \alpha + n$ and ^{4}H $= t + n$ binary subsystems has been given. We have demonstrated that, for the $\alpha + t + n$ system, such classification is unique in that it is consistent with the requirements of the Pauli exclusion principle both in the region of small intercluster distances and in the asymptotic region, where the scattering matrix elements are produced. Due to the difference of eigenvalues of the antisymmetrization operator of the $\alpha + t + n$ system, the corresponding eigenfunctions are uniquely determined. Any unitary transformation applied to the latter SU(3)-basis functions would disrupt the diagonal form of the norm kernel of ^{8}Li nucleus and, hence, is inappropriate in this case.

The Pauli-allowed states of $\alpha + t + n$ system can be arranged into branches and families, with all the states of a particular branch having the common SU(3)-symmetry index μ, but differing in the first SU(3) index λ. The eigenvalues belonging to a given branch tend to the same eigenvalue λ_{k_i} of $\alpha + t$, $\alpha + n$, or $t + n$ binary subsystems, as the number of quanta increases. The branches with limit eigenvalues are combined in a family of eigenstates, which is completely determined by the number of oscillator quanta k in a two-cluster subsystem. Each family of Pauli-allowed states corresponds asymptotically to a certain binary decay channel of ^{8}Li into a two-cluster subsystem occurring in the ground or excited harmonic-oscillator state and a remaining cluster. Such asymptotic behavior gives an indication of possible decay channels of a three-cluster ^{8}Li nucleus and allows us to specify the most important decay channels of the nuclei under consideration.

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НЕЗВИЧНІ ПРОЯВИ ПРИНЦИПУ ПАУЛІ
В РЕАКЦІЯХ РОЗСІЯННЯ АТОМНИХ ЯДЕР

Р е з ю м е

У рамках мікроскопічної моделі, що ґрунтується на алгебраїчній версії методу резонуючих груп досліджено роль принципу Паулі у формуванні хвильової функції неперервного спектра ядерної системи $\alpha + t + n$. В просторі Фока–Баргманна побудовано ядро нормування трикластерної системи $\alpha + t + n$. Запропоновано повну класифікацію власних функцій і власних значень ядра нормування ^{8}Li за допомогою власних значень бінарних підсистем $^7\mathrm{Li} = \alpha + t$, $^5\mathrm{He} = \alpha + n$ і $^4\mathrm{H} = t + n$.

Ю.А. Лашко, Г.Ф. Филиппов

НЕОБЫЧНЫЕ ПРОЯВЛЕНИЯ ПРИНЦИПА ПАУЛИ
В РЕАКЦИЯХ РАССЕЯНИЯ АТОМНЫХ ЯДЕР

Р е з ю м е

В рамках микроскопической модели, которая базируется на алгебраической версии метода резонирующих групп исследована роль принципа Паули в формировании волновой функции непрерывного спектра ядерной системы $\alpha + t + n$. В пространстве Фока–Баргманна построено ядро нормирования трикластерной системы $\alpha + t + n$. Предложена полная классификация собственных функций и собственных значений ядра нормирования ^{8}Li с помощью собственных значений бинарных подсистем $^7\mathrm{Li} = \alpha + t$, $^5\mathrm{He} = \alpha + n$ и $^4\mathrm{H} = t + n$.