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NON-PERTURBATIVE ANHARMONIC CORRECTION TO MEHLER'S PRESENTATION OF THE HARMONIC OSCILLATOR PROPAGATOR

We find the possibility of a non-perturbative anharmonic correction to Mehler's formula for the propagator of a harmonic oscillator. The conditional Wiener measure functional integral with a fourth-order term in the exponent is evaluated using a method alternative to the conventional perturbative approach. In contrast to the conventional perturbation theory, we expand the term linear in the integration variable in the exponent into a power series. The case where the starting point of the propagator is zero is discussed. The results are presented in analytical form for positive and negative frequencies.

K e y w o r d s: Non-perturbative anharmonic correction, Mehler's formula, harmonic oscillator.

1. Introduction

As to the definition of a Wiener path integral, there are essentially two approaches:

– to define the path integral via a finite-dimensional approximation. Then the path integral is an appropriate continuum limit, when the number of time slices goes to infinity;

– to define the Wiener measure in the frame of the axiomatic probabilistic measure theory as a Gauss-type measure on the set of trajectories.

Quantum theory is rather pointed toward the integration method, when it comes to deal with the conditional Wiener measure. We would like to use the path integral formalism to obtain the non-perturbative analytical description of an anharmonic oscillator in quantum mechanics or possibly to describe quantum field theory systems in this way. In the quantum theory with imaginary time, we see the formal connection with the path integral formalism for the Brownian motion. The main difference between the classical description of the Brownian motion as a random process and the quantum description of the particle motion via a path integral inheres in the interpretation of results. In classical physics, we interpret the results of the path integral as a probability of the displacement of a particle from position i to position f. In quantum theory, we evaluate the amplitude of a propagation of the particle by a path integral, and this should not be confused with the statistical probability of the underlying Brownian motion.

The transition probability for a Brownian particle under an external harmonic oscillator force is given by Mehler's formula

$$W(x_i, t_i; x_f, t_f) = \left(\frac{k}{2\pi \sinh(\nu)}\right)^{1/2} \times$$
$$\times \exp\left\{-\frac{k(x_i^2 + x_f^2)}{2\tanh(\nu)} + \frac{k x_i x_f}{\sinh(\nu)}\right\}. \qquad (1)$$

This formula was derived by F. Mehler (1866) who investigated the diffusion equation in the presence of a harmonic oscillator force, i.e. with the unit mass harmonic oscillator Hamiltonian (see [1])

$$H = -\frac{1}{2}\triangle + \frac{1}{2}k^2x^2, \ \nu = k(\tau_f - \tau_i) \qquad (2)$$

on the right-hand side of the diffusion equation. The same result was derived for the probability of a stochastic movement of a Brownian particle in an external harmonic potential using the conditional measure Wiener path integral methods (see, e.g., Hille [2], Doob [3]). Alternatively, Eq. (1) can be obtained also as the propagator of a harmonic oscillator in quantum mechanics (i.e., with τ replaced by it). R. Feynman obtained it in 1948 within his path integral approach to quantum mechanics (see [4] and [5]).

R. Feynman expressed the quantum mechanical transition amplitude for a general potential $V(x)$ as a path integral of the following form:

$$W_{QM}(q_i,t_i;q_f,t_f) = \int\limits_{\text{all paths}} \prod_{\tau=t_i}^{t_f} \frac{dx(t)}{\sqrt{\frac{2i\pi\hbar}{m}dt}} \times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_{t_i}^{t_f} dt\left(\frac{1}{2}m\dot{x}^2 - V(x)\right)\right\}. \tag{3}$$

Later, M. Kac rigorously justified the imaginary time analog of a Feynman path integral (see [6, 7]) for a broad class of potentials $V(x)$. The imaginary time propagator can be represented by a conditional measure Wiener path integral defined by the continuum limit of time-sliced finite-dimensional integrals.

The path integral approach is frequently used in quantum mechanics and quantum field theory, since it provides a way to efficiently derive/incorporate standard perturbative expansions and even indicates steps beyond perturbative methods (see, e.g., [8], [9], [10]). However, there are few path integrals that allow the explicit evaluation. Such are, for example, the systems of harmonic oscillators and the free (relativistic or Euclidean) fields. The corresponding transition probabilities/amplitudes represent multidimensional generalizations of Mehler's formula.

Our aim is to evaluate the transition probability for the motion of a Brownian particle in a quartic anharmonic external potential given by a conditional measure Wiener integral. In the quantum mechanical formalism with imaginary time, such a system corresponds to a symmetric anharmonic oscillator. There are various approximative or numerical estimates of various quantities, e.g., the eigenenergies of systems, that go beyond standard perturbative methods. However, to our best knowledge, there is little known directly about the anharmonic oscillator transition probability (propagator).

Our ambition in this article is pointed toward a non-perturbative correction to the Mehler's formula for a harmonic oscillator. To simplify the evaluations, we fixed the start point of the propagator to zero. The evaluation of the N dimensional integral is given in Section 2, where we present the precise result expressed in the form of the parabolic cylinder functions, but we have still $N-1$ fold summations in Eq. (12) as a consequence of Taylor's expansions during the evaluation. These summations are the only point in our calculations, where some approximation appears. The procedure used to deal with such summations is described in Section 2; the result is Eq. (16). In Section 3, we evaluate the continuum limit of the N-dimensional integral. The final formula for the conditional Wiener measure path integral with a term of the fourth order in the exponent (see Eq. (16)) is a product of the Mehler's formula for a harmonic oscillator (17) with fixed start point to be zero and the anharmonic correction to this formula (21). Our result, in contrast to the conventional perturbative approach, describes the propagator for an anharmonic oscillator for the positive or negative frequency (in our model, the parameter b). In Section 4, we show the evaluation of a non-perturbative correction to the exponential factor of Mehler's formula for a harmonic oscillator.

2. Evaluation of the Path Integral

Below, we shall present the non-perturbative evaluation of a conditional measure Wiener path integral with quartic addition to the harmonic oscillator potential. There is no reason to assume that this anharmonicity is small. Let us briefly describe the idea of the evaluation of a finite-dimensional integral, which was explained in our previous article [11] in detail.

Let us first consider the one-dimensional integral with the fourth order term in the exponent, which is going to appear frequently:

$$J(a,b,c) = \int\limits_{-\infty}^{+\infty} dx \exp\left\{-(ax^4 + bx^2r + cx)\right\}, \tag{4}$$

where $\operatorname{Re} a > 0$. This integral is not given by a (simple) formula. The standard perturbation approach corresponds to the expansion in powers of a:

$$J(a,b,c) \doteq \sum_{n=0}^{\infty} Q_n(-a)^n, \quad Q_n = \frac{(4n)!}{n!}\left(\frac{c}{4b}\right)^{4n} \times$$
$$\times \exp\left(\frac{c^2}{b}\right)\sqrt{\frac{\pi}{b}}\sum_{k=0}^{2n}\frac{1}{(4n-2k)!k!}\left(\frac{b}{c^2}\right)^k. \tag{5}$$

The behavior of Q_n for large n indicates that (5) represents just a singular power expansion with zero radius of convergence (which is, in fact, Borel-summable). However, $J(a,b,c)$ is an entire function for any complex values of b and c, since there exist all integrals

$$\partial_c^n \partial_b^m J(a,b,c) = = (-1)^{n+m} \int\limits_{-\infty}^{+\infty} dx\, x^{2m+n} \exp\{-(ax^4 + bx^2 + cx)\}.$$

As a result, the power expansions of $J(a,b,c)$ in c and/or b have an infinite radius of convergence. Consequently, they are uniformly convergent on any compact set of values of c and/or b. The power expansion in c is

$$J(a,b,c) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!} \int\limits_{-\infty}^{+\infty} dx\, x^n \exp\{-(ax^4 + bx^2)\}. \tag{6}$$

For n odd, integrals (6) are zero. For n even ($n = 2m$), the integrals can be expressed in terms of the parabolic cylinder function $D_\nu(z)$, $\nu = -m - 1/2$, (see, e.g., [12, 13]):

$$J(a,b,c) = \frac{\Gamma(1/2)}{(2a)^{1/4}} \sum_{m=0}^{\infty} \frac{(\xi)^m}{m!} e^{z^2/4} D_{-m-1/2}(z),$$

$$\xi = \frac{c^2}{4\sqrt{2a}}, \quad z = \frac{b}{\sqrt{2a}}. \tag{7}$$

This sum is convergent for any values of c, b, and $a > 0$.

Using the expansions based on (7), we are going to evaluate the *conditional Wiener measure* path integral defined as (see [9, 10]):

$$\mathcal{W} = \int [D\varphi(\tau)]\ \exp\left(-E[\varphi]\right), \tag{8}$$

where

$$E[\varphi] = \int\limits_0^\beta d\tau \left[c/2 \left(\frac{\partial \varphi(\tau)}{\partial \tau}\right)^2 + b\varphi(\tau)^2 + a\varphi(\tau)^4\right]. \tag{9}$$

In the *conditional Wiener measure* path integral, the values $\varphi(0) = x_i$ and $\varphi(\beta) = x_f$ are fixed by definition. The path integral (8) can be defined as the limit of time-sliced finite dimensional integrals [9]:

$$\mathcal{W}_N = \left(\frac{1}{\sqrt{\frac{2\pi\triangle}{c}}}\right)^N \int\limits_{-\infty}^{+\infty} \prod_{i=1}^{N-1} d\varphi_i \exp(-E_N), \tag{10}$$

with

$$E_N = \sum_{i=1}^{N} \triangle \left[c/2 \left(\frac{\varphi_i - \varphi_{i-1}}{\triangle}\right)^2 + b\varphi_i^2 + a\varphi_i^4\right] \tag{11}$$

representing the standard time-slice discretization of $E[\varphi]$. The *conditional Wiener measure* path integral is defined by the limit:

$$\mathcal{W} = \lim_{N \to \infty} \mathcal{W}_N.$$

To simplify the evaluation, we fix the initial point $\varphi(0) = x_i = 0$. Performing successively all one-dimensional integrals in (10), we are dealing with integrals of the form (6) all the time. The evaluation of the $N-1$-dimensional integral (10) is described in details in the long version of this article [14]. The result reads

$$\mathcal{W}_N = \left(\frac{2\pi\triangle}{c}\right)^{-1/2} \left[\sqrt{2\pi\left(1 + \frac{b\triangle^2}{c}\right)}\right]^{-N+1} \times$$

$$\times \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_{N-1}=0}^{\infty} \prod_{i=1}^{N-2} \left\{ \frac{\left(1 + \frac{b\triangle^2}{c}\right)^{-2k_i}}{(2k_i)!} \times \right.$$

$$\left. \times\, \Gamma(k_{i-1} + k_i + 1/2) \mathcal{D}_{-k_{i-1}-k_i-1/2}(z) \right\} \times$$

$$\times \frac{\left(1 + \frac{b\triangle^2}{c}\right)^{-k_{N-1}}}{(2k_{N-1})!} \left(\frac{c}{\triangle}\varphi_N^2\right)^{k_{N-1}} \times$$

$$\times\, \Gamma(k_{N-2} + k_{N-1} + 1/2) \mathcal{D}_{-k_{N-2}-k_{N-1}-1/2}(z) \times$$

$$\times \exp\left\{-a\triangle\varphi_N^4 - \left(\frac{c}{2\triangle} + b\triangle\right)\varphi_N^2\right\}. \tag{12}$$

Here, $a, b, c, \varphi_N = x_f, \triangle = \beta/N$ are constants of the model, the variable z is defined as

$$z = \frac{c\left(1 + \frac{b\triangle^2}{c}\right)}{\sqrt{2a\triangle^3}},$$

and the functions $\mathcal{D}_{-\nu-1/2}(z)$ are related to the parabolic cylinder functions $D_{-\nu-1/2}(z)$ by the relation

$$\mathcal{D}_{-\nu-1/2}(z) = z^{\nu+1/2} \ \exp\left\{\frac{z^2}{4}\right\} D_{-\nu-1/2}(z).$$

The result in Eq. (12) is an exact expression; we have not used any approximation in the evaluation. As we have shown in [11], the multiple sums are uniformly convergent.

Our aim is to separate the multiple summations over k_i into the leading term and the remainder disappearing in the continuum limit $\triangle \to 0$, when $z \approx \triangle^{-3/2}$. The individual summation over given k_i in product (12) is

$$\sum_{k_i=0}^{\infty}\left\{\frac{\left(1+\frac{b\triangle^2}{c}\right)^{-2k_i}}{(2k_i)!}\Gamma(k_{i-1}+k_i+1/2)\times\right.$$
$$\times\,\mathcal{D}_{-k_{i-1}-k_i-1/2}(z)\Gamma(k_i+k_{i+1}+1/2)\times$$
$$\left.\times\,\mathcal{D}_{-k_i-k_{i+1}-1/2}(z)\right\}. \tag{13}$$

We shall divide this sum into the leading part and the remainder:

$$\sum_{j=0}^{\mathcal{J}} \frac{(-1)^j}{j!\ (2z^2)^j} \sum_{k_i=0}^{\infty} \Gamma(k_{i-1}+k_i+1/2)\times$$
$$\times\,\Gamma(k_i+k_{i+1}+2j+1/2)\times$$
$$\times\,\mathcal{D}_{-k_{i-1}-k_i-1/2}(z)+R(\mathcal{J},K_0).$$

To manage this task, we introduce the first and the only one approximation in our calculation, when one of the parabolic cylinder functions is replaced by the asymptotic Poincaré-type expansion (see [15], [16]) of the parabolic cylinder functions valid for a finite index and a large argument z:

$$\mathcal{D}_{-\nu-1/2}(z) \equiv z^{\nu+1/2} e^{z^2/4} D_{-\nu-1/2}(z) =$$
$$= \sum_{j=0}^{\mathcal{J}} (-1)^j \frac{(\nu+1/2)_{2j}}{j!(2z^2)^j} + \varepsilon_{\mathcal{J}}(\nu, z), \tag{14}$$

where $\varepsilon_{\mathcal{J}}(\nu, z)$ is the remainder of the Poincaré-type expansion. Here, $(\nu)_k = \nu(\nu+1)...(\nu+k-1)$ is the Pochhammer symbol. This asymptotic expansion is particularly useful in the continuum limit $\triangle \to 0$, when $z \approx \triangle^{-3/2}$, and the functions $\mathcal{D}_{-\nu-1/2}(z) \to 1$. The first term in (14) contributes to the leading part of (13), whereas the second part generates the remainder. The k_i summations of the leading part can be performed using the Taylor expansion formula for parabolic cylinder functions [13], which takes the form

$$e^{x^2/4} \sum_{k=0}^{\infty} \frac{(\nu)_k}{k!} t^k D_{-\nu-k}(x) = e^{(x-t)^2/4} D_{-\nu}(x-t). \tag{15}$$

The estimate of the remainder part of (12) can be found in [11]. The detailed evaluation and discussion of the summation over indices k_i for the *conditional Wiener measure* path integral can be found in [14], where the following expression has been given for the leading term of the $N-1$-dimensional integral (12):

$$\mathcal{W}_N^{\text{leading}} = \frac{1}{\sqrt{\left(\frac{2\pi\triangle}{c}\right)\prod_{i=0}^{N-2} 2\omega_i(1+b\triangle^2/c)}}\times$$
$$\times\,\exp\left\{-a\triangle\varphi_N^4 - \left(\frac{c}{2\triangle}+b\triangle\right)\varphi_N^2 \ + \ \xi\right\}\times$$
$$\times\sum_{\nu=0}^{\mathcal{J}} (-1)^\nu \ \frac{1}{\nu!\ (2z^2)^\nu} \ \sum_{p=0}^{2\nu} (\xi)^p \ \ (N-1)_p^{2\nu}\,. \tag{16}$$

The new symbols ξ and ω_i are defined as

$$\xi = \frac{1}{\omega_{N-2}} \ \frac{c}{4\triangle\left(1+\frac{b\triangle^2}{c}\right)} \ \varphi_N^2,$$

$$\omega_{i+1} = 1 - \frac{\sigma^2}{\omega_i}, \quad \omega_0 = 1,$$

and the expression $(N-1)_p^{2\nu}$ is defined by the recurrence relation in Appendix A.

3. Path Integral as the Continuum Limit of the $N-1$-Dimensional Integral.

The evaluation of $\mathcal{W}_N$ was the target of the preceding sections, where we have found relation (16) for the leading term of an $N-1$-dimensional integral. In this section, we will discuss its continuum limit. The continuum limit of the first line in (16) is evaluated in Appendices B and C, giving the result

$$\lim_{N\to\infty} \frac{1}{\sqrt{\left(\frac{2\pi\triangle}{c}\right)\prod_{i=0}^{N-2} 2\omega_i(1+b\triangle^2/c)}}\times$$

$$\times \exp\left\{-a\triangle\varphi_N^4 - \left(\frac{c}{2\triangle} + b\triangle\right)\varphi_N^2 \ + \ \xi\right\} =$$

$$= \frac{1}{\sqrt{\frac{2\pi}{c}\ \frac{\sinh(\gamma\beta)}{\gamma}}}\ \exp\left\{-\frac{c\gamma}{2}\coth(\gamma\beta)\varphi_N^2\right\},$$

$$\gamma = \sqrt{2b/c}. \tag{17}$$

Formula (17) represents Mehler's formula for the imaginary time [9], [10] for the propagator of a harmonic oscillator with starting point zero and end point φ_N. The anharmonic content of the oscillator is stored in the continuum limit of the second line in (16):

$$\sum_{\nu=0}^{\mathcal{J}} (-1)^\nu \frac{1}{\nu!\,(2z^2)^\nu} \sum_{p=0}^{2\nu} (\xi)^p\ (N-1)_p^{2\nu}, \tag{18}$$

where the variable z was defined in the previous section as

$$z = \frac{c\left(1 + \frac{b\triangle^2}{c}\right)}{\sqrt{2a\triangle^3}}.$$

We see that z diverges as $\triangle^{-3/2}$ in the continuum limit $\triangle \to 0$. We rewrite Eq. (18) into the form

$$\sum_{\nu=0}^{\mathcal{J}} (-1)^\nu \frac{1}{\nu!\,(2z^2\triangle^3)^\nu} \left(\triangle^{3\nu}\sum_{p=0}^{2\nu} (\xi)^p\ (N-1)_p^{2\nu}\right). \tag{19}$$

The term $(2z^2\triangle^3)^\nu$ is finite in the continuum limit; and we are interested in the continuum limit of

$$\triangle^{3\nu}\sum_{p=0}^{2\nu} (\xi)^{2\nu-p}\ (N-1)_{2\nu-p}^{2\nu}. \tag{20}$$

The detailed evaluation of the above expression is done in [14]; here, we summarize the final result only.

The anharmonic correction to Mehler's formula defined in Eq. (19) in the continuum limit reads

$$\sum_{\nu=0}^{\mathcal{J}} (-a)^\nu \sum_{p=0}^{2\nu} \frac{c^{-p}}{1/2(2\nu-p)} \left(\frac{1}{Q^2(\beta)}\varphi_N^2\right)^{2\nu-p} \times$$

$$\times \sum_{\substack{\{m_1,\dots,m_\nu\} \\ m_1+\dots+m_\nu=p}} \prod_{j=1}^{\nu} \Sigma(m_j,j,p_j) I_{m_1,\dots,m_\nu}(0), \tag{21}$$

where

$$I_{m_1,\dots,m_\nu}(\tau) = \int_\tau^\beta d\tau_1 \int_{\tau_1}^\beta d\tau_2 \dots \int_{\tau_{\nu-1}}^\beta d\tau_\nu \times$$

$$\times\, d^{m_1}(\tau_1) d^{m_2}(\tau_2) \dots d^{m_\nu}(\tau_\nu)\ Q^4(\tau_\nu)\dots Q^4(\tau_2)\ Q^4(\tau_1). \tag{22}$$

The multiple summations in (21) are meant to be over all sets of indices $0 \le m_j \le 4$ satisfying the condition $m_1 + \dots + m_\nu = p$. The dependence of $\Sigma(m_j, j, p_j)$ (where $p_j = p - m_1 - \dots - m_j$) on the values m_j is given in the Table 1.

Equation (21) is the key formula. For any given $\mathcal{J}$, it gives the anharmonic correction as a finite sum. The integrals $I_{m_1,\dots,m_\nu}(\tau)$ are analyzed in the next section, where we derive various recurrence relations that allow us to analyze anharmonic corrections successively in the parameter p. The symbols $d(\tau)$ and $Q(\tau)$, following the definitions in Appendices, read:

$$d(\tau) = \frac{1}{2\gamma}\left(\coth(\gamma\tau) - \coth(\gamma\beta)\right),$$

$$Q(\tau) = 2\sinh(\gamma\tau), \quad \gamma = \sqrt{\frac{2b}{c}}.$$

4. Analysis of the Anharmonic Correction

In this section, we will show the evidence that the anharmonic corrections in (21) give a non-perturbative contribution to Mehler's formula for the propagator of a harmonic oscillator. In order to extract as much information as possible, we interchange the order of finite summations in Eq. (21):

$$\sum_{\nu=0}^{\mathcal{J}} \sum_{p=0}^{2\nu} \to \sum_{p=0}^{2\mathcal{J}} \sum_{\nu=\lfloor\frac{p+1}{2}\rfloor}^{\mathcal{J}}.$$

Table 1. **Values of $\Sigma(m_j, j, p_j)$ for m_j**

m_j	$\Sigma(m_j,j,p_j)$
0	$(2(\nu-j)-p_j+1/2)(2(\nu-j)-p_j+3/2)$
1	$4(2(\nu-j)-p_j+1/2)(2(\nu-j)-p_j+3/4)$
2	$6(2(\nu-j)-p_j)(2(\nu-j)-p_j-1)+9(2(\nu-j)-p_j)+3/4$
3	$4(2(\nu-j)-p_j-1/4)(2(\nu-j)-p_j)$
4	$(2(\nu-j)-p_j-1)(2(\nu-j)-p_j)$

For finite p and ν high enough, the product

$$\prod_{i=1}^{\nu} \Sigma(m_i, i, p_i)$$

in Eq.(21) contains many terms with $m_i = 0$.

Let $m_j \neq 0$, $m_k \neq 0$, and $m_i = 0$ for $j \leq i \leq k$. As a result, $p_i = p_k$. Then the product in question is

$$\Pi(k, j, p_k) = \prod_{i=k+1}^{j-1} (2(\nu - i) - p_k + 1/2) \times$$
$$\times (2(\nu - i) - p_k + 3/2) = \frac{\Gamma(2(\nu - k) - p_k + 1/2)}{\Gamma(2(\nu - j) - p_k + 5/2)}.$$

Here, we have used the identity

$$\Gamma(x)\Gamma(x + 1/2) = \frac{\sqrt{\pi}}{2^{2x-1}}\Gamma(2x).$$

Let $m_{j_i} \neq 0$ for $j_i = j_1, ..., j_\mu$. Then

$$\prod_{i=1}^{\nu} \Sigma(m_i, i, p_i) = \Pi(0, j_1, p)\Sigma(m_{j_1}, j_1, p_{j_1}) \times$$
$$\times \Pi(j_1, j_2, p_{j_1})...\Sigma(m_{j_\mu}, j_\mu, 0)\Pi(j_\mu, \nu + 1, 0).$$

This expression can be rewritten in the form

$$\prod_{i=1}^{\nu} \Sigma(m_i, i, p_i) = \frac{\Gamma(2\nu - p + 1/2)}{\Gamma(1/2)} \times$$
$$\times \frac{\Gamma(2(\nu - j_1) - p_{j_1} + 1/2)}{\Gamma(2(\nu - j_1) - p + 5/2)}\Sigma(m_{j_1}, j_1, p_{j_1}) ...$$
$$... \frac{\Gamma(2(\nu - j_i) - p_{j_i} + 1/2)}{\Gamma(2(\nu - j_i) - p_{j_{i-1}} + 5/2)}\Sigma(m_{j_i}, j_i, p_{j_i}) ...$$
$$... \frac{\Gamma(2(\nu - j_\mu) + 1/2)}{\Gamma(2(\nu - j_\mu) - p_{j_{\mu-1}} + 5/2)}\Sigma(m_{j_\mu}, j_\mu, 0). \quad (23)$$

Table 2. **Dependence of the algebraic factor on m_{j_k}**

m_{j_k}	$F(j_k, m_{j_k}, p_{j_k})$
1	$4(2(\nu - j_k) - p_{j_k} + 3/4)$
2	$6(2(\nu - j_k) - p_{j_k} + 1/4 + i/4)(2(\nu - j_k) - p_{j_k} + 1/4 - i/4)$
3	$4(2(\nu - j_k) - p_{j_k})(2(\nu - j_k) - p_{j_k} - 1/2)(2(\nu - j_k) - p_{j_k} - 1/4)$
4	$(2(\nu - j_k) - p_{j_k})(2(\nu - j_k) - p_{j_k} - 1)(2(\nu - j_k) - p_{j_k} - 1/2)(2(\nu - j_k) - p_{j_k} - 3/2)$

We can rewrite expression (23) as a product of algebraic factors, which depend on all $m_i \neq 0$:

$$F(j_i, m_{j_i}, p_{j_i}) =$$
$$= \frac{\Gamma(2(\nu - j_i) - p_{j_i} + 1/2)}{\Gamma(2(\nu - j_i) - p_{j_i} - m_{j_i} + 5/2)}\Sigma(m_{j_i}, j_i, p_{j_i}). \quad (24)$$

In the above definition, the identity $p_{j_i} = p_{j_{i-1}} - m_{j_i}$ has been used.

With this definition, we can rewrite Eq. (23) in terms of nonzero m_i:

$$\prod_{i=1}^{\nu} \Sigma(m_i, i, p_i) = (1/2)_{2\nu-p} \prod_{i=1}^{\mu} F(j_i, m_{j_i}, p_{j_i}). \quad (25)$$

The dependence of the values of algebraic factor $F(j_i, m_{j_i}, p_{j_i})$ on the values of $m_{j_i} \neq 0$ is summarized in Table 2:

We stress the important interesting characteristics of integrals in the form (21):

$$I_{m_1,...,m_n}(\tau) = \int_\tau^\beta d\tau_1 \int_{\tau_1}^\beta d\tau_2 ...$$
$$... \int_{\tau_{n-1}}^\beta d\tau_n J_{m_1}(\tau_1) ... J_{m_n}(\tau_n). \quad (26)$$

Putting $\tau = 0$ and $J_a(\tau) = d^a(\tau)Q^4(\tau)$, the connection to integrals in Eq. (21) is evident. The crucial feature is the identity

$$I_{a,b}(\tau) + I_{b,a}(\tau) = I_a(\tau)I_b(\tau). \quad (27)$$

In the second term, we change the order of integrations and then rename the integration variables $x \leftrightarrow y$:

$$I_{a,b}(\tau) + I_{b,a}(\tau) = \int_\tau^\beta dx \int_x^\beta dy J_a(x)J_b(y) +$$
$$+ \int_\tau^\beta dx \int_x^\beta dy J_b(x)J_a(y) = \int_\tau^\beta dx \int_x^\beta dy J_a(x)J_b(y) +$$
$$+ \int_\tau^\beta dy \int_y^\beta dx J_b(y)J_a(x) = \int_\tau^\beta \int_\tau^\beta dxdy J_a(x)J_b(y) =$$
$$= I_a(\tau)I_b(\tau).$$

Various identities related to the products of such integrals can be proven using (27) such as, for instance,

$$I_{m_1,\ldots,m_{n-1}} I_m = I_{m,m_1,\ldots,m_{n-1}} + I_{m_1,m,m_2,\ldots,m_{n-1}} + \\ + \ldots + I_{m_1,\ldots,m_{j-1},m,m_j,\ldots,m_{n-1}} + \ldots + I_{m_1,\ldots,m_{n-1},m}.$$

We obtain n terms with n indices each as a result of the product of an integral with one index m and another integral with $n-1$ indices $m_1, m_2, \ldots, m_{n-1}$. The index m takes successively all the positions in the string of n indices, the indices m_i don't permute among themselves.

In order to evaluate the anharmonic correction in (21), we consider such a product of integrals, where one of the integrals has n indices of the same value. A well-known identity is obtained by induction:

$$I_{\underbrace{a,\ldots,a}_{n}}(\tau) = \frac{I_a^n(\tau)}{(n)!}. \tag{28}$$

The application of (27) gives

$$I_\alpha(\tau) I_{\underbrace{a,\ldots,a}_{n-1}}(\tau) = \sum_{j=1}^{n} I_{\underbrace{a,\ldots,a,\alpha_j,a,\ldots,a}_{n}}(\tau). \tag{29}$$

The subscript j in α indicates the position of the index α among the indices of the integrals $I_{a,\ldots,a,\alpha_j,a,\ldots,a}$. In order to evaluate (21), we need

$$I_{\alpha,\beta}(\tau) I_{\underbrace{a,\ldots,a}_{n-2}}(\tau) = \\ = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} I_{\underbrace{a,\ldots,a,\alpha_j,a,\ldots,a,\beta_k,a,\ldots,a}_{n}}(\tau). \tag{30}$$

We will have to deal with the case where $\beta = a$, then the integrals $I_{\underbrace{a,\ldots,a,\alpha_j,a,\ldots,a}_{n}}(\tau)$ in the above expression are independent of the summation index k, and we obtain

$$I_{\alpha,a}(\tau) I_{\underbrace{a,\ldots,a}_{n-2}}(\tau) = \\ = \sum_{j=1}^{n} (n-j) I_{\underbrace{a,\ldots,a,\alpha_j,a,\ldots,a}_{n}}(\tau). \tag{31}$$

We have expanded the summation over the index j up to n by adding the zero term for $j = n$. The evaluations of some other useful relations are given in Appendix D.

Applying this new notation to the anharmonic correction (21), we have

$$\sum_{p=0}^{2\mathcal{J}} c^{-p} \sum_{\nu=[\frac{p+1}{2}]}^{\mathcal{J}} (-a)^\nu \ (X_N)^{2\nu-p} \times \\ \times \sum_{\substack{\{m_1,\ldots,m_\nu\} \\ m_{j_1}+\ldots+m_{j_\mu}=p}} \prod_{i=1}^{\mu} F(j_i, m_{j_i}, p_{j_i}) \times \\ \times I_{\underbrace{0,\ldots,0,m_{j_1},0,\ldots,0,m_{j_2},0,\ldots,0,m_{j_\mu},0,\ldots,0}_{\nu}}(0). \tag{32}$$

Here,

$$X_N = \frac{\varphi_N^2}{Q^2(\beta)}$$

has been introduced to simplify the formulas.

Let us analyze this result. We can see that, for $p = 0$, the contribution to (21) can be written as

$$\sum_{\nu=0}^{\mathcal{J}} (-a)^\nu \, (X_N)^{2\nu} \frac{I_0^\nu(0)}{\nu!} \approx \exp\left\{-\frac{aI_0(0)\varphi_N^4}{Q^4(\beta)}\right\}. \tag{33}$$

For sufficiently large $\mathcal{J}$, the contribution of the terms with $\nu > \mathcal{J}$,

$$\sum_{\nu=\mathcal{J}+1}^{\infty} (-a)^\nu \, (X_N)^{2\nu} \frac{I_0^\nu(0)}{\nu!} < \frac{\theta \frac{a\varphi_N^4 I_0(0)}{Q^4(\beta)}}{\mathcal{J}\mathcal{J}!}, \ 0 < \theta < 1,$$

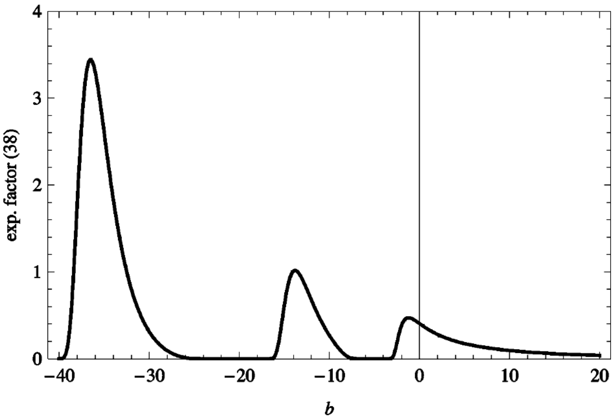
can be neglected with a sufficient precision preserved.

The contribution for $p = 1$ can be expressed as

$$\sum_{\nu=1}^{\mathcal{J}} (-a)^\nu \ (X_N)^{2\nu-1} \times \\ \times \sum_{j_1=1}^{\nu} F(j_1, 1, 0) I_{\underbrace{0,\ldots,0,1_{j_1},0,\ldots,0}_{\nu}}(0). \tag{34}$$

Inserting $F(j_1, 1, 0)$ from Table 2, we get

$$(-aX_N) \sum_{\nu=1}^{\mathcal{J}} (-a)^{\nu-1} \ (X_N)^{2\nu-2} \times \\ \times \sum_{j_1=1}^{\nu} (8(\nu - j_1) + 3) I_{\underbrace{0,\ldots,0,1_{j_1},0,\ldots,0}_{\nu}}(0). \tag{35}$$



The b dependence of the exponential part of Mehler's formula for the anharmonic oscillator (40), when the model parameters are fixed as $\beta = c = 1,\ a = 2$

Following Eqs. (29),(31), we have

$$(-aX_N)\sum_{\nu=1}^{\mathcal{J}}(-a)^{\nu-1}(X_N)^{2\nu-2}\times$$
$$\times\left(8I_{1,0}(0)\frac{I_0^{\nu-2}(0)}{(\nu-2)!}+3I_1(0)\frac{I_0^{\nu-1}(0)}{(\nu-1)!}\right)\approx$$
$$\approx\exp\left\{-\frac{aI_0(0)\varphi_N^4}{Q^4(\beta)}\right\}\times$$
$$\times\left\{-3a(X_N)I_1(0)+8a^2(X_N)^3I_{1,0}(0)\right\}. \quad (36)$$

As to the contribution for $p = 2$, we must take into account that it is divided into two parts, one for the case where one $m_j = 2$ and another where two $m_j = 1$, and $m_k = 1$ are nonzero. We have

$$\sum_{\nu=1}^{\mathcal{J}}(-a)^{\nu}\ (X_N)^{2\nu-2}\times$$
$$\times\left\{\sum_{j_1=1}^{\nu}F(j_1,2,0)I_{\underbrace{0,\ldots,0,2_{j_1},0,\ldots,0}_{\nu}}(0)+\right.$$
$$+\sum_{j_1=1}^{\nu-1}\sum_{j_2=j_1+11}^{\nu}F(j_1,1,1)F(j_2,1,0)\times$$
$$\left.\times I_{\underbrace{0,\ldots,0,1_{j_1},0,\ldots,0,1_{j_2},0,\ldots,0}_{\nu}}(0)\right\}. \quad (37)$$

In the spirit of the previous calculations, the contribution to the anharmonicity correction for $p = 2$ is obtained as

$$\exp\left\{-\frac{aI_0(0)\varphi_N^4}{Q^4(\beta)}\right\}\{3/4(-a)I_2(0)+(-a)^2\times$$
$$\left.\times\left(30I_{2,0}(0)+21I_{1,1}(0)\right)(X_N)^2\right)+(-a)^3\times$$
$$\times\left(48I_{2,0,0}(0)+144I_{1,1,0}(0)+24I_{1,0,1}(0)\right)(X_N)^4+$$
$$\left.+64(-a)^4(I_{1,0,1,0}(0)+2I_{1,1,0,0}(0))(X_N)^6\right\}. \quad (38)$$

Calculations can be extended to any value of p. The common characteristic of all calculations is the universal non-perturbative exponential correction to Mehler's formula given by the exponential factor

$$\exp\left\{-\frac{aI_0(0)\varphi_N^4}{Q^4(\beta)}\right\},$$

where

$$\frac{I_0(0)}{Q^4(\beta)}=\{3\gamma\beta-4\cosh(\gamma\beta)\sinh(\gamma\beta)+$$
$$+\cosh^3(\gamma\beta)\sinh(\gamma\beta)+\cosh(\gamma\beta)\sinh^3(\gamma\beta)\}/$$
$$/\{8\gamma\sinh^4(\gamma\beta)\}. \quad (39)$$

The second factor, given in braces in Eq. (38), is p-dependent and is represented for any p as a polynomial of degree $2p$ in the variable $-a$.

5. Conclusions

We present an analytical method of evaluation of the conditional Wiener measure path integral with the fourth-order term in the action. In contrast to the methods used in the conventional perturbative approach, the linear part of the kinetic term of the action is expanded. We obtain the analytical results representing the anharmonic correction to Mehler's formula for the propagator of a harmonic oscillator. The most vital parts of this article are related to the recurrence relations in Section IV leading to the exponential correction in (40). The universal non-perturbative exponential correction to the exponent of Mehler's formula is the most important result. For the anharmonic oscillator, the exponential factor can be written as

$$\exp\left\{-\frac{c\gamma}{2}\coth(\gamma\beta)\varphi_N^2 - a\,\frac{I_0(0)}{Q^4(\beta)}\varphi_N^4\right\}. \qquad (40)$$

In Figure, the dependence of the exponential term (40) on the parameter b is shown (for positive and negative values). The remaining anharmonic corrections come from polynomials of order $2p$ in the variable a that multiply the exponential factor in (36) and (38). The corrections for $p = 0, 1, 2$ are presented in detail. Of course, the same can be done systematically for any p. An interesting feature is the information given by Eq. (40), when the frequency is negative (this can occur for $b < 0$.). The exponential factor in Eq. (40) approaches $-\infty$, when $\gamma\beta = \beta\sqrt{2b/c} \to ik\pi$. This means that the propagator for this particular b, β vanishes, and the particle is frozen at the origin $x_i = 0$, because it cannot propagate to any other point of the space.

APPENDIX A
Evaluation of the $\mathcal{W}_N$ by the recurrent summation over k_i indices

This Appendix is a short version of a more detailed evaluation given in [14]. Let us begin with Eq. (16), which can be rewritten as

$$\mathcal{W}_N = \left(\frac{2\pi\triangle}{c}\right)^{-1/2}\left[\sqrt{2\left(1+\frac{b\triangle^2}{c}\right)}\right]^{-N+1}\times$$
$$\times\prod_{i=1}^{N-2}\left\{\sum_{k_i=0}^{\infty}\frac{\left[2\left(1+\frac{b\triangle^2}{c}\right)\right]^{-2k_i}}{k_i!}(k_{i-1}+1/2)_{k_i}\times\right.$$
$$\left.\times\,\mathcal{D}_{-k_{i-1}-k_i-1/2}(z)\right\}\sum_{k_{N-1}=0}^{\infty}\frac{\left[4\left(1+\frac{b\triangle^2}{c}\right)\right]^{-k_{N-1}}}{(k_{N-1})!\;1/2k_{N-1}}\times$$
$$\times\left(\frac{c}{\triangle}\varphi_N^2\right)^{k_{N-1}}(k_{N-2}+1/2)_{k_{N-1}}\mathcal{D}_{-k_{N-2}-k_{N-1}-1/2}(z)\times$$
$$\times\exp\left\{-a\triangle\varphi_N^4-\left(\frac{c}{2\triangle}+b\triangle\right)\varphi_N^2\right\}. \qquad (A1)$$

The summation relation for the parabolic cylinder function is needed now, see [13]:

$$e^{x^2/4}\sum_{k=0}^{\infty}\frac{(\nu)_k}{k!}t^k D_{-\nu-k}(x) = e^{(x-t)^2/4}D_{-\nu}(x-t). \qquad (A2)$$

As to the functions $\mathcal{D}_{-\nu-k}(x)$, the above formula leads to

$$\sum_{k=0}^{\infty}\frac{(\nu)_k}{k!}t^k\mathcal{D}_{-\nu-k}(x) = \left(\frac{x}{x-xt}\right)^{\nu}\mathcal{D}_{-\nu}(x-xt). \qquad (A3)$$

The direct use of this identity in Eq. (A1) is not possible, because each function $\mathcal{D}_{-\nu-k}(x)$ has, in fact, two summation indices. So, we have to evaluate the sum

$$\frac{1}{\sqrt{2\left(1+\frac{b\triangle^2}{c}\right)}}\sum_{k_1=0}^{\infty}\frac{\left[2\left(1+\frac{b\triangle^2}{c}\right)\right]^{-2k_1}}{k_1!}(1/2)_{(k)_1}\times$$
$$\times\mathcal{D}_{-k_1-1/2}(z)(k_1+1/2)_{(k_2)}\mathcal{D}_{-k_1-k_2-1/2}(z). \qquad (A4)$$

In our previous article [11], it has been shown that the sum in Eq. (A4) is uniformly convergent. Therefore, we can approximate the infinite sum up to the desired precision by replacing it by a finite one. It is of importance that, in a finite sum of the type Eq.(A4), we can benefit from the Poincaré-type expansion of the parabolic cylinder function, which means

$$\mathcal{D}_{-k_1-1/2}(z) \equiv z^{k_1+1/2}e^{z^2/4}D_{-k_1-1/2}(z) =$$
$$=\sum_{j=0}^{\mathcal{J}}(-1)^j\frac{(k_1+1/2)_{2j}}{j!(2z^2)^j}+\varepsilon_{\mathcal{J}}(k_1,z). \qquad (A5)$$

In the last relation, $\mathcal{J}$ denotes the number of terms of the asymptotic expansion taken into account, while $\varepsilon_{\mathcal{J}}(k_1,z)$ is the remainder. We have discussed the problem of this remainder in our previous paper [11], where we have shown that it converges to zero more rapidly than $1/N$. This means that all the contributions to the summations over indices k_i containing such remainder, or products of such remainders, disappear in the continuum limit. Our evaluations and estimates concerning the upper limit of this remainder follow from works [17] and [16] dealing with estimates of the upper bounds of remainders of the Poincaré-type expansions of the parabolic cylinder functions.

When the Poincaré-type asymptotic expansion is applied to the function $\mathcal{D}_{-k_1-1/2}(z)$, then the following holds for the leading term (i.e. without the remainder) of the finite sum, which approximates the infinite one in Eq. (A4):

$$\frac{1}{\sqrt{2\left(1+\frac{b\triangle^2}{c}\right)}}\sum_{k_1=0}^{M}\frac{\left[2\left(1+\frac{b\triangle^2}{c}\right)\right]^{-2k_1}}{k_1!}\times$$
$$\times\left\{\sum_{j=0}^{\mathcal{J}}(-1)^j\frac{(k_1+1/2)_{2j}}{j!(2z^2)^j}\right\}(1/2)_{k_2+k_1}\mathcal{D}_{-k_1-k_2-1/2}(z). \qquad (A6)$$

Swapping the order of the summations leads to

$$\frac{1}{\sqrt{2\left(1+\frac{b\triangle^2}{c}\right)}}\sum_{j=0}^{\mathcal{J}}(-1)^j\frac{1}{j!(2z^2)^j}\times$$
$$\times\sum_{k_1=0}^{M}\frac{\left[2\left(1+\frac{b\triangle^2}{c}\right)\right]^{-2k_1}}{k_1!}(k_1+1/2)_{2j}\times$$
$$\times(1/2)_{k_2+k_1}\mathcal{D}_{-k_1-k_2-1/2}(z). \qquad (A7)$$

In the previous paper [11], the following relation has been proven:

$$(k_1+1/2)_{2j} = \sum_{i=0}^{\min(2j,k_1)}a_i^{2j}\frac{(k_1)!}{(k_1-i)!}. \qquad (A8)$$

The coefficients a_i^{2j} are given by

$$a_i^j = \binom{j}{i}\frac{(1/2)_j}{(1/2)_i}. \qquad (A9)$$

Inserting these relations into Eq. (A7), we get

$$\frac{1}{\sqrt{2\left(1+\frac{b\triangle^2}{c}\right)}}\sum_{j=0}^{\mathcal{J}}(-1)^j\frac{1}{j!(2z^2)^j}\sum_{i=0}^{2j}a_i^{2j}\times$$

$$\times\sum_{k_1=i}^{M}\frac{\left[2\left(1+\frac{b\triangle^2}{c}\right)\right]^{-2k_1}}{(k_1-i)!}(1/2)_{k_2+k_1}\mathcal{D}_{-k_1-k_2-1/2}(z). \qquad (A10)$$

The sum over k_i is uniformly convergent, and we can extend the summation to be as precise as we want; even to infinity, if required. Let us define

$$\sigma = \left[2\left(1+\frac{b\triangle^2}{c}\right)\right]^{-1}.$$

We have

$$\frac{1}{\sqrt{2\left(1+\frac{b\triangle^2}{c}\right)}}\sum_{j=0}^{\mathcal{J}}(-1)^j\frac{1}{j!(2z^2)^j}\times$$

$$\times\sum_{i=0}^{2j}a_i^{2j}(1/2)_{k_2+i}\,\sigma^{2i}\sum_{k_1=i}^{\infty}\frac{\sigma^{2k_1-2i}}{(k_1-i)!}\times$$

$$\times(k_2+i+1/2)_{k_1-i}\mathcal{D}_{-(k_1-i)-(k_2+i)-1/2}(z). \qquad (A11)$$

Recalling the identity from Eq.(A3), the leading term of Eq. (A4) is

$$\frac{1}{\sqrt{2\left(1+\frac{b\triangle^2}{c}\right)}}\sum_{j=0}^{\mathcal{J}}(-1)^j\frac{1}{j!(2z^2)^j}\sum_{i=0}^{2j}a_i^{2j}\,\sigma^{2i}\times$$

$$\times\left(\frac{z}{z-z\sigma^2}\right)^{k_2+i+1/2}(1/2)_{k_2+i}\mathcal{D}_{-k_2-i-1/2}(z-z\sigma^2). \qquad (A12)$$

We have neglected the remainders of two uniformly convergent series in this calculation. Such an approximation can be done, still maintaining the desired precision. In the continuum limit, the truncated series approaches the original one. In order to simplify dealing with the recurrence procedure, we define some new variables:

$\sigma_1 = \sigma^2,$

$z_1 = z(1-\sigma^2),$

$\omega_1 = \frac{z_1}{z} = 1-\sigma_1,$

$(1)_i^{2j} = a_i^{2j}.$

The summation over all indices k_i is done in [14], with the recurrence evaluation described step by step.

The following lemma can be proven by induction:

Lemma. The leading term of the partial sum in Eq. (A1) *over the indices* $k_1, k_2, \ldots, k_\Lambda$, $\Lambda \le N-2$ *is given by*

$$Z_\Lambda = \frac{1}{\sqrt{\frac{2\pi\triangle}{c}\prod_{i=1}^{\Lambda}\left(2\omega_i(1+b\triangle^2/c)\right)}}\times$$

$$\times\sum_{\mu=0}^{\mathcal{J}}(-1)^\mu\frac{1}{\mu!(2z^2)^\mu}\sum_{i_\Lambda=0}^{2\mu}(\Lambda)_{i_\Lambda}^{2\mu}\left(\frac{\sigma_\Lambda}{1-\sigma_\Lambda}\right)^{i_\Lambda}\times$$

$$\times\left(\frac{z}{z_\Lambda}\right)^{k_{\Lambda+1}}(1/2)_{k_{\Lambda+1}+i_\Lambda}\,\mathcal{D}_{-k_{\Lambda+1}-i_\Lambda-1/2}(z_\Lambda). \qquad (A13)$$

The symbol $(\Lambda)_\mu^\nu$ is defined via the recurrence relation

$$\Lambda)_{i_\Lambda}^{2\mu} = \sum_{j_\Lambda=0}^{\mu}\binom{\mu}{j_\Lambda}\left(\frac{1}{\omega_{\Lambda-1}}\right)^{2j_\Lambda}\sum_{i=\max(0,\ i_\Lambda-2j_\Lambda)}^{2\mu-2j_\Lambda}a_{i_\Lambda}^{2j_\Lambda+i}\times$$

$$\times(\Lambda-1)_i^{2\mu-2j_\Lambda}\left(\frac{\sigma_{\Lambda-1}}{1-\sigma_{\Lambda-1}}\right)^i. \qquad (A14)$$

The first term is

$(1)_i^{2\mu} = a_i^{2\mu}.$

There are also some more recurrence definitions to note:

$$\sigma_{i+1} = \frac{\sigma^2}{1-\sigma_i}\ ,\ \sigma_1 = \sigma^2,$$

$$z_i = z(1-\sigma_i), \quad \omega_i = 1-\sigma_i$$

or

$$\omega_{i+1} = 1-\frac{\sigma^2}{\omega_i}, \quad \omega_1 = 1-\sigma^2, \quad \omega_0 = 1.$$

After performing the summation over all indices k_i, we finally find the leading term of Eq (A1):

$$\mathcal{W}_N = \frac{1}{\sqrt{\left(\frac{2\pi\triangle}{c}\right)\prod_{i=0}^{N-2}2\omega_i(1+b\triangle^2/c)}}\times$$

$$\times\exp\left\{-a\triangle\varphi_N^4-\left(\frac{c}{2\triangle}+b\triangle\right)\varphi_N^2\ +\ \xi\right\}\times$$

$$\times\sum_{\nu=0}^{\mathcal{J}}(-1)^\nu\frac{1}{\nu!\,(2z^2)^\nu}\sum_{p=0}^{2\nu}(\xi)^p\ (N-1)_p^{2\nu}\,. \qquad (A15)$$

All non-leading terms disappear due to the remainders of the Poincaré expansion of the parabolic cylinder function in the continuum limit [11]. The continuum limit of Eq. (A15) is to be discussed in the next appendices.

APPENDIX B
Continuum Limit of the Square-Root Factor in Eq. (A15)

In our paper [11], we have evaluated the continuum limit of the leading part of the N-dimensional integral $\mathcal{W}_N$, using the generalized Gelfand–Yaglom equation. We defined the function F_N connected with the N-dimensional integral by the relation

$$\mathcal{W}_N = \frac{1}{\sqrt{F_N}}.$$

Due to the recurrence relations for the quantities in $\mathcal{W}_N$, we can evaluate the difference equation for the values F_k, where $k = 1,2,\ldots,N$. The aim of the Gelfand–Yaglom construction is to find the continuum limit of the difference equation for the function F_k. The solution of this differential equation is connected to the continuum path integral by

$$\mathcal{W}(\beta)=\frac{1}{\sqrt{F(\beta)}},$$

where β is the upper bound of the time interval in the action.

We could use the same method to evaluate the continuum limit of the N-dimensional integral, when dealing with the conditional measure Wiener integral, but we would like to present a slightly different method here. In the Gelfand–Yaglom method, we have to evaluate the difference equation. Then, after imposing the continuum limit, we obtain the differential equation and can find its solution. In the new approach, we evaluate directly this function. We will present the evaluation of the continuum limit:

$$\left(\frac{2\pi\triangle}{c}\right)\prod_{i=0}^{N-2}2\omega_i(1+b\triangle^2/c), \tag{B1}$$

where ω_i obeys the recurrence relation

$$\omega_{i+1}=1-\frac{\sigma^2}{\omega_i},$$

where

$$\omega_0=1,\quad \omega_1=1-\sigma^2,\quad \sigma=\left[2\left(1+\frac{b\triangle^2}{c}\right)\right]^{-1}.$$

Let us define

$$\Omega_n=\prod_{i=0}^{n}\omega_i.$$

Using the recurrence relation for ω_n, we obtain the recurrence relation for Ω_n:

$$\Omega_n=\omega_n\omega_{n-1}\Omega_{n-2}=$$
$$=(\omega_{n-1}-\sigma^2)\Omega_{n-2}=\Omega_{n-1}-\sigma^2\Omega_{n-2}, \tag{B2}$$

with the first two values

$$\Omega_0=1,\quad \Omega_1=1-\sigma^2.$$

The methods of difference calculus [18] propose to search for a solution of the recurrence equation (B2) in the form

$$\Omega_n=w_1\varrho_1^n+w_2\varrho_2^n. \tag{B3}$$

The characteristic equation for ϱ is

$$\varrho^2-\varrho+\sigma^2=0,$$

with the solution

$$\varrho_{1,2}=\frac{1\pm\sqrt{1-4\sigma^2}}{2}.$$

The coefficients w_1 and w_2 will be obtained from the values Ω_0 , Ω_1, and we can write

$$w_{1,2}=1/2\left(1\pm\frac{1-2\sigma^2}{\sqrt{1-4\sigma^2}}\right).$$

For expression (B1), the following holds:

$$\left(\frac{2\pi\triangle}{c}\right)\left[2(1+b\triangle^2/c)\right]^{N-1}\Omega_{N-2}=\left(\frac{2\pi\triangle}{c}\right)\times$$
$$\times\left[2(1+b\triangle^2/c)\right]^{N-1}(w_1\rho_1^{N-2}+w_2\rho_2^{N-2}). \tag{B4}$$

Inserting $w_{1,2}$, $\rho_{1,2}$ and performing some calculations, we get

$$\frac{4\pi\triangle}{c}\left(1+\frac{b\triangle^2}{c}\right)\left\{\frac{1}{2}\left(1+\frac{1-2\sigma^2}{\sqrt{1-4\sigma^2}}\right)\times\right.$$
$$\times\left[\left(1+\frac{b\triangle^2}{c}\right)(1+\sqrt{1-4\sigma^2})\right]^{N-2}+$$
$$+\frac{1}{2}\left(1-\frac{1-2\sigma^2}{\sqrt{1-4\sigma^2}}\right)\times$$
$$\left.\times\left[\left(1+\frac{b\triangle^2}{c}\right)\left(1-\sqrt{1-4\sigma^2}\right)\right]^{N-2}\right\}. \tag{B5}$$

Let us define

$$\gamma=\sqrt{2b/c},$$

and

$$\triangle=\frac{\beta}{N}.$$

Now we are left with the following result for (B5) in the continuum limit $\lim N\to\infty$:

$$\frac{2\pi}{c}\,\frac{\sinh(\gamma\beta)}{\gamma}.$$

The same outcome is acquired by the Gelfand–Yaglom method.

APPENDIX C
The Continuum Limit of the Exponential Factor in Eq. (A15)

We are going to evaluate the exponent in Eq. (A15):

$$\exp\left\{-a\triangle\varphi_N^4-\left(\frac{c}{2\triangle}+b\triangle\right)\varphi_N^2\ +\ \xi\right\}, \tag{C1}$$

where ξ is defined as

$$\xi=\frac{1}{\omega_{N-2}}\frac{c}{4\triangle\left(1+\frac{b\triangle^2}{c}\right)}\varphi_N^2,$$

and ω_{N-2} obeys the recurrence relation

$$\omega_{i+1}=1-\frac{\sigma^2}{\omega_i},\quad \omega_1=1-\sigma^2,\quad \omega_2=\frac{1-2\sigma^2}{1-\sigma^2}.$$

We are going to evaluate ω_i. Following the method of the n-th convergent [18], we define

$$\omega_n=\frac{p_n}{q_n}=a_n+\frac{b_n}{\omega_{n-1}},$$

p_n and q_n are bound to satisfy

$$p_n=a_np_{n-1}+b_np_{n-2},$$

$$q_n=a_nq_{n-1}+b_nq_{n-2}.$$

Solutions of the above recurrence relations can be written in the form

$$p_n=u_1\rho_1^n+u_2\rho_2^n,$$

$$q_n=\tilde{u}_1\rho_1^n+\tilde{u}_2\rho_2^n.$$

Here, ρ has to satisfy the characteristic equation

$$\rho^2 - a_n\rho - b_n = 0, \quad a_n = 1, \quad b_n = -\sigma.$$

The solution reads

$$\rho_{1,2} = \frac{1 \pm \sqrt{1-4\sigma^2}}{2}. \tag{C2}$$

The coefficients $u_{1,2}$ and $\tilde{u}_{1,2}$ are obtained from the conditions

$$\omega_1 = \frac{p_1}{q_1} = 1 - \sigma^2, \quad \omega_2 = \frac{p_2}{q_2} = \frac{1-2\sigma^2}{1-\sigma^2}.$$

Now, we have

$$u_{1,2} = \frac{1}{2}\left(1 \pm \frac{1-2\sigma^2}{\sqrt{1-4\sigma^2}}\right), \quad \tilde{u}_{1,2} = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{1-4\sigma^2}}\right). \tag{C3}$$

An important identity follows from Eqs. (C2), (C3):

$$q_n = p_{n-1}.$$

This allows us to write ω_n as

$$\omega_n = \frac{p_n}{p_{n-1}} = \frac{q_{n+1}}{q_n}. \tag{C4}$$

At this point, we define new variables for the evaluation of the continuum limit. Following the definition of ω_n, we have

$$\omega_n = \frac{q_{n+1}}{q_n} = \frac{\tilde{u}_1\rho_1^{n+1} + \tilde{u}_2\rho_2^{n+1}}{\tilde{u}_1\rho_1^n + \tilde{u}_2\rho_2^n} =$$

$$= \sigma\frac{\left(\frac{\rho_1}{\sigma}\right)^{n+1} + \frac{\tilde{u}_2}{\tilde{u}_1}\left(\frac{\rho_2}{\sigma}\right)^{n+1}}{\left(\frac{\rho_1}{\sigma}\right)^{n} + \frac{\tilde{u}_2}{\tilde{u}_1}\left(\frac{\rho_2}{\sigma}\right)^{n}} = \sigma\frac{Q_{n+1}}{Q_n}. \tag{C5}$$

In the following evaluations, we will use the definition of the variable Q_n:

$$Q_n = \left(\frac{\rho_1}{\sigma}\right)^n + \frac{\tilde{u}_2}{\tilde{u}_1}\left(\frac{\rho_2}{\sigma}\right)^n. \tag{C6}$$

Compared to q_n, the variables Q_n are finite in the continuum limit. The primary variable in our calculation is ω_n , which is finite in the continuum limit and is more convenient from the point of view of the following evaluation to express it as the proportion of Q_n. For complexity, we define the variable $\tilde{Q}_n$:

$$\tilde{Q}_n = \left(\frac{\rho_1}{\sigma}\right)^n - \frac{\tilde{u}_2}{\tilde{u}_1}\left(\frac{\rho_2}{\sigma}\right)^n. \tag{C7}$$

To evaluate relation (C1), we need to do the key calculation:

$$\frac{1}{\omega_{N-2}} = \frac{q_{N-1}}{q_{N-2}} =$$

$$= 2\left(1 - \frac{\sqrt{1-4\sigma^2}(\tilde{u}_1\rho_1^{N-2} - \tilde{u}_2\rho_2^{N-2})}{(\tilde{u}_1\rho_1^{N-2} + \tilde{u}_2\rho_2^{N-2}) + \sqrt{1-4\sigma^2}(\tilde{u}_1\rho_1^{N-2} - \tilde{u}_2\rho_2^{N-2})}\right).$$

Inserting into Eq. (C1), we find the exponent in the form:

$$-a\triangle\varphi_N^4 - \varphi_N^2\left(\frac{c}{2\triangle} + b\triangle - \frac{c}{2\triangle(1+b\triangle^2/c)}\right) -$$

$$-\left(\frac{\sqrt{1-4\sigma^2}(\tilde{u}_1\rho_1^{N-2} - \tilde{u}_2\rho_2^{N-2})}{(\tilde{u}_1\rho_1^{N-2} + \tilde{u}_2\rho_2^{N-2}) + \sqrt{1-4\sigma^2}(\tilde{u}_1\rho_1^{N-2} - \tilde{u}_2\rho_2^{N-2})}\right) \times$$

$$\times \frac{c}{2\triangle\left(1 + \frac{b\triangle^2}{c}\right)}\varphi_N^2. \tag{C8}$$

In the continuum limit $\lim N \to \infty$ for the terms in the line, we take advantage of some useful identities:

$$\sqrt{1-4\sigma^2} = \frac{\triangle\sqrt{2b/c + b^2\triangle^2/c^2}}{1 + b\triangle^2/c},$$

$$\lim_{N\to\infty}\left(\frac{\tilde{u}_2}{\tilde{u}_1}\right) = \lim_{N\to\infty}\left(\frac{\sqrt{1-4\sigma^2}-1}{\sqrt{1-4\sigma^2}+1}\right) = -1,$$

and

$$\lim_{N\to\infty}(2\rho_{1,2})^{N-2} = \lim_{N\to\infty}(1 \pm \sqrt{1-4\sigma^2})^N =$$

$$= \lim_{N\to\infty}\left(1 \pm \frac{\triangle\sqrt{2b/c + b^2\triangle^2/c^2}}{1 + b\triangle^2/c}\right)^N = \exp(\pm\gamma t).$$

So, considering $\lim N \to \infty$, we have got the result for Eq. (C8):

$$-\frac{c\gamma}{2}\coth(\gamma t)\varphi_N^2. \tag{C9}$$

Recall that

$$\triangle = t/N\ ,\ \gamma = \sqrt{2b/c}.$$

APPENDIX D
Algebra of Integrals

For the product of integrals with three indices, we can write

$$I_{\alpha,\beta,\gamma}(\tau)I_{\underbrace{a,\ldots,a}_{n-3}}(\tau) = \sum_{j=1}^{n-2}\sum_{k=j+1}^{n-1}\sum_{l=k+1}^{n} \times$$

$$\times I_{\underbrace{a,\ldots,a,\alpha_j,a,\ldots,a,\beta_k,a,\ldots,a,\gamma_l,a,\ldots,a}_{n}}(\tau). \tag{D1}$$

To evaluate the anharmonic correction (21), we need the following terms characterized by different algebraic factors in the sum on the right-hand side:

$$I_{\alpha,\beta,a}(\tau)I_{\underbrace{a,\ldots,a}_{n-3}}(\tau) =$$

$$= \sum_{j=1}^{n-1}\sum_{k=j+1}^{n}(n-k)I_{\underbrace{a,\ldots,a,\alpha_j,a,\ldots,a,\beta_k,a,\ldots,a}_{n}}(\tau). \tag{D2}$$

The expansion of the summations over summation indices (j up to $n-1$ and k up to n) was done by adding the zero terms due to the factor $(n-k)$.

In the case of two indices a, we have

$$I_{\alpha,a,a}(\tau)I_{\underbrace{a,\ldots,a}_{n-3}}(\tau) =$$

$$= \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} I_{\underbrace{a,\ldots,a,\alpha_i,a,\ldots,a}_{n}}(\tau). \tag{D3}$$

The integrals $I_{\underbrace{a,\ldots,a,\alpha_i,a,\ldots,a}_{n}}(\tau)$ are independent of the summations indices j,k. Consequently, we have

$$I_{\alpha,a,a}(\tau)I_{\underbrace{a,\ldots,a}_{n-3}}(\tau) =$$

$$= \sum_{i=1}^{n} \binom{n-i}{2} I_{\underbrace{a,\dots,a,\alpha_i,a,\dots,a}_{n}}(\tau). \quad \text{(D4)}$$

We are free to do the summation for $i = n-1, i = n$, because the added terms are zero. In the case where the index a is sandwiched between α and β, we have

$$I_{\alpha,a,\beta}(\tau) I_{\underbrace{a,\dots,a}_{n-3}}(\tau) =$$

$$= \sum_{j=1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{l=k+1}^{n} I_{\underbrace{a,\dots,a,\alpha_j,a,\dots,a,\beta_l,a,\dots,a}_{n}}(\tau). \quad \text{(D5)}$$

Due to the k-independence of the integral, we may write

$$I_{\alpha,a,\beta}(\tau) I_{\underbrace{a,\dots,a}_{n-3}}(\tau) = \sum_{j=1}^{n-2} \sum_{l=j+2}^{n} (l-1-j)\times$$

$$\times I_{\underbrace{a,\dots,a,\alpha_j,a,\dots,a,\beta_l,a,\dots,a}_{n}}(\tau). \quad \text{(D6)}$$

Thanks to the factor $(l-1-j)$, we can expand the summations up to $j = n-1$ and $l = j+1$:

$$I_{\alpha,a,\beta}(\tau) I_{\underbrace{a,\dots,a}_{n-3}}(\tau) = \sum_{j=1}^{n-1} \sum_{l=j+1}^{n} (l-1-j)\times$$

$$\times I_{\underbrace{a,\dots,a,\alpha_j,a,\dots,a,\beta_l,a,\dots,a}_{n}}(\tau). \quad \text{(D7)}$$

Now, let us look at the product of integrals with four Greek indices:

$$I_{\alpha,\beta,\gamma,\delta}(\tau) I_{\underbrace{a,\dots,a}_{n-4}}(\tau) = \sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{l=k+1}^{n} \times$$

$$\times I_{\underbrace{a,\dots,a,\alpha_i,a,\dots,a,\beta_j,a,\dots,a,\gamma_k,a,\dots,a,\delta_l,a,\dots,a}_{n}}(\tau). \quad \text{(D8)}$$

For the sake of the current evaluations, the case where two Greek indices are equal to a is of interest:

$$I_{\alpha,a,\gamma,a}(\tau) I_{\underbrace{a,\dots,a}_{n-4}}(\tau) = \sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{l=k+1}^{n} \times$$

$$\times I_{\underbrace{a,\dots,a,\alpha_i,a,\dots,a,\gamma_k,a,\dots,a}_{n}}(\tau). \quad \text{(D9)}$$

Because the integrals $I_{\underbrace{a,\dots,a,\alpha_i,a,\dots,a,\gamma_k,a,\dots,a}_{n}}(\tau)$ are independent of the summation indices j, l, we have:

$$I_{\alpha,a,\gamma,a}(\tau) I_{\underbrace{a,\dots,a}_{n-4}}(\tau) = \sum_{i=1}^{n-3} \sum_{k=i+2}^{n-1} \times$$

$$\times (n-k)(k-i-1) I_{\underbrace{a,\dots,a,\alpha_i,a,\dots,a,\gamma_k,a,\dots,a}_{n}}(\tau). \quad \text{(D10)}$$

Due to the factor $(n-k)(k-i-1)$, we can include $k = i+1$ and $k = n$ as well. Thanks to the identity

$$k - i - 1 = (n - i - 2) - (n - k - 1)$$

and the new algebraic factors allowing us to extend the sum over the index i, we have

$$I_{\alpha,a,\gamma,a}(\tau) I_{\underbrace{a,\dots,a}_{n-4}}(\tau) = \sum_{i=1}^{n-1} \sum_{k=i+1}^{n} (n-k)(n-i-2)\times$$

$$\times I_{\underbrace{a,\dots,a,\alpha_i,a,\dots,a,\gamma_k,a,\dots,a}_{n}}(\tau) - \sum_{i=1}^{n-1} \sum_{k=i+1}^{n} \times$$

$$\times (n-k)(n-k-1) I_{\underbrace{a,\dots,a,\alpha_i,a,\dots,a,\gamma_k,a,\dots,a}_{n}}(\tau).$$

As the last example, we evaluate the product

$$I_{\alpha,\beta,a,a}(\tau) I_{\underbrace{a,\dots,a}_{n-4}}(\tau) = \sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{l=k+1}^{n} \times$$

$$\times I_{\underbrace{a,\dots,a,\alpha_i,a,\dots,a,\beta_j,a,\dots,a}_{n}}(\tau). \quad \text{(D11)}$$

The integrals $I_{\underbrace{a,\dots,a,\alpha_i,a,\dots,a,\beta_j,a,\dots,a}_{n}}(\tau)$ are independent of the summation indices k and l, and the factor $\binom{n-j}{2}$ appears. This factor allows us to extend the summation over the indices i and j, so we can write

$$I_{\alpha,\beta,a,a}(\tau) I_{\underbrace{a,\dots,a}_{n-4}}(\tau) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \times$$

$$\times \binom{n-j}{2} I_{\underbrace{a,\dots,a,\alpha_i,a,\dots,a,\beta_j,a,\dots,a}_{n}}(\tau). \quad \text{(D12)}$$

The evaluation of other identities is not necessary for the purposes of this paper.

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ДО ФОРМУЛИ МЕХЛЕРА ДЛЯ ПРОПАГАТОРА
ГАРМОНІЙНОГО ОСЦИЛЯТОРА

Р е з ю м е

Розглянуто можливість непертурбативної ангармонічної поправки до формули Мехлера для пропагатора гармонійного осцилятора. Функціональний інтеграл по умовній мірі Вігнера з членом четвертого порядку в експоненті оцінено в рамках методу альтернативного звичайному пертурбативному підходу. На відміну від звичайної теорії збурень, ми розкладаємо член в експоненті лінійний по змінній інтеграції в степеневий ряд. Обговорено випадок, коли початкова точка пропагатора дорівнює нулю. Результати дано в аналітичному вигляді як для позитивних, так і негативних частот.

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НЕПЕРТУРБАТИВНАЯ
АНГАРМОНИЧЕСКАЯ ПОПРАВКА
К ФОРМУЛЕ МЕХЛЕРА ДЛЯ ПРОПАГАТОРА
ГАРМОНИЧЕСКОГО ОСЦИЛЛЯТОРА

Р е з ю м е

Рассмотрена возможность непертурбативной ангармонической поправки к формуле Мехлера для пропагатора гармонического осциллятора. Функциональный интеграл по условной мере Вигнера с членом четвертого порядка в экспоненте оценен в рамках метода альтернативного обычному пертурбативному подходу. В отличие от обычной теории возмущений, мы раскладываем член в экспоненте линейный по переменной интегрирования в степенной ряд. Обсужден случай, когда начальная точка пропагатора равна нулю. Результаты даны в аналитическом виде как для положительных, так и отрицательных частот.