ZURA KAKUSHADZE [1,2]

[1] Quantigic® Solutions LLC
(1127, High Ridge Road #135, Stanford, CT 06905)
[2] Free University of Tbilisi, Business School & School of Physics
(240, David Agmashenebeli Alley, Tbilisi 0159, Georgia; e-mail: zura@quantigic.com)

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QUANTIZATION RULES FOR DYNAMICAL SYSTEMS

We discuss a manifestly covariant way of arriving at the quantization rules based on the causality, with no reference to Poisson or Peierls brackets of any kind.

Keywords: quantization rules, Poisson brackets, Peierls brackets.

1. Introduction

The canonical quantization of dynamical systems replaces classical dynamical variables by operators, and classical Poisson brackets by commutators such that the Correspondence Principle is satisfied. This procedure comes with some "drawbacks". On the one hand, it is not manifestly covariant, which is unappealing in relativistic theories. On the other hand, it masks the importance of measurements in quantum theory. These "shortcomings" can be circumvented via Peierls brackets [1], a manifestly covariant generalization of Poisson brackets. In this approach, the roles of elementary and complete measurements in quantum theory are prominent [2].

In this note, we discuss a way of arriving at the quantization rules based on the Causality Principle, with no reference to the Poisson or Peierls brackets of any kind. We use DeWitt's condensed notations [2]. We focus on bosonic theories. A generalization to superclassical systems with Grassmann valued variables is straightforward.

2. Classical Fields

Let $S[\phi]$ be a real local action functional for a classical dynamical system described by a set of real variables ϕ^i. The classical dynamical equations of motion read

$$S_{,i}[\phi] = 0. \quad (1)$$

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Here, i is a generic index, which combines a discrete label for the field components and a continuous label for the space-time points, which the field ϕ^i depends on. The left-hand side of (1) is the first functional derivative of $S[\phi]$. Repeated indices imply summation and integration. We will omit the arguments of classical functionals; thus, S stands for $S[\phi]$, where ϕ^i is an arbitrary solution of (1) so long as it is not a singular point of the functional S.

Consider the case where the continuous matrix $S_{,ij}$ is nonsingular, i.e., there is no constraint and, therefore, the action does not possess any infinite-dimensional invariance group. Here, the following remarks are in order. First, the notion of a constraint is understood in the context of a "gauge algebra" [3]. Second, while the nonsingularity for a finite matrix implies that it has no null eigenvalue, the notion of eigenvectors and eigenvalues is subtle for continuous matrices. Thus, the equation

$$S_{,ij}\ f^i = 0 \quad (2)$$

has nontrivial solutions even if $S_{,ij}$ is nonsingular. As a general rule, a continuous matrix can be considered nonsingular if it has no eigenvector with a null eigenvalue, either vanishing outside a limited region of the space-time or quadratically integrable. In (2), f^i satisfies neither of these conditions.

Since $S_{,ij}$ is a nonsingular matrix, it can be inverted. The inversion depends on boundary conditions. For example, the advanced and retarded Green functions G^{+ij} and G^{-ij} satisfy the following equations and boundary conditions:

$$S_{,ik}\ G^{+kj} = -\delta_i{}^j; \quad G^{+ij} = 0, \quad i > j, \tag{3}$$

$$S_{,ik}\ G^{-kj} = -\delta_i{}^j; \quad G^{-ij} = 0, \quad j > i, \tag{4}$$

$$G^{+ij} = G^{-ji}. \tag{5}$$

Here, the delta-symbol $\delta_i{}^j$ is understood to contain a space-time delta-function. The symbol “$>$” means “is in the future with respect to".

3. Operators

Quantization amounts to replacing the classical real-valued variables ϕ^i by Hermitian operators Φ^i, which, in general, do not commute. Therefore, ambiguities arise in the quantum dynamical equations of motion

$$S_{,i}[\Phi] = 0, \tag{6}$$

which need not have the classical form. These ambiguities in products of operators must be resolved by means of their symmetrization. In other words, a real functional Z^{ij} must exist such that

$$\left[\Phi^i, \Phi^j\right] = i\ Z^{ij}[\Phi]. \tag{7}$$

Consider a linear theory described by the action

$$\Sigma = \frac{1}{2}\ S_{,ij}\ \Phi^i\ \Phi^j \tag{8}$$

and the commutation relations

$$\left[\Phi^i, \Phi^j\right] = i\ \Omega^{ij}[\Phi]. \tag{9}$$

In this theory, there is no ambiguity in the quantum dynamical equations of motion, as they are linear:

$$S_{,ij}\ \Phi^j = 0. \tag{10}$$

From (9) and (10), we get

$$S_{,ik}\ \Omega^{kj}[\Phi] = 0. \tag{11}$$

Then it follows that the functional Ω^{ij} does not depend on Φ^i or else Eq. (11) would be a constraint, which would contradict our prior assumptions. So, we have

$$S_{,ik}\ \Omega^{kj} = 0, \quad \Omega^{ij} = -\Omega^{ji}, \tag{12}$$

where Ω^{ij} must be constructed solely from $S_{,ij}$ and/or its inverse operators, and we conclude that it is a linear combination of the real Green functions of $S_{,ij}$.

4. Action Variations

Consider an infinitesimal variation in the functional form of the action:

$$S \to S + \delta S, \tag{13}$$

where δS vanishes outside a limited region of the space-time. Such a variation can be thought of as describing a measurement process in the “quantum system + macroapparatus” (see [2] for details). Then the new dynamical equations of motion

$$S_{,ij}\ \delta\Phi^j = -\delta S_{,ij}\ \Phi^j \tag{14}$$

must be solved, by assuming the *retarded* boundary conditions in accordance with the Causality Principle, i.e.,

$$\delta\Phi^i = G^{-ij}\ \delta S_{,jk}\ \Phi^k. \tag{15}$$

Therefore,

$$\delta\Omega^{ij} = -i\left(\left[\delta\Phi^i, \Phi^j\right] + \left[\Phi^i, \delta\Phi^j\right]\right) =$$

$$= G^{-ik}\ \delta S_{,kl}\ \Omega^{lj} + \Omega^{ik}\ \delta S_{,kl}\ G^{+lj}, \tag{16}$$

where we have used (5).

Equation (16) implies that Ω^{ij} has a definite transformation property under the action variations (13). On the one hand, we concluded in the previous section that Ω^{ij} is a linear combination of the real Green functions. On the other hand, there are only two real inverse matrices, namely, $G^{\pm ij}$, with definite transformation properties determined by their kinematics (3) and (4):

$$\delta G^{\pm ij} = G^{\pm ik}\ \delta S_{,kl}\ G^{\pm lj}. \tag{17}$$

Therefore, Ω^{ij} must be a linear combination of $G^{\pm ij}$. With regard for (12), (16), and (17), we get

$$\Omega^{ij} = \alpha\left(G^{+ij} - G^{-ij}\right), \tag{18}$$

where α is a constant and does not depend on the functional form of S. So, we have the following commutation relations:

$$\left[\Phi^i, \Phi^j\right] = i\ \alpha\left(G^{+ij} - G^{-ij}\right). \tag{19}$$

To match the experimental data, α must be Planck's constant $\hbar$.

5. Concluding Remarks

The above argument, which employs neither Poisson nor Peierls brackets, can be generalized to constrained systems along the lines of [2] and also to interacting nonlinear systems along the lines of [4]. For a recent discussion on the quantization of non-Lagrangian systems, see, e.g., [5] and references therein.

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З. Какушадзе

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