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ON THE CLASSICAL MAXWELL–LORENTZ ELECTRODYNAMICS, THE ELECTRON INERTIA PROBLEM, AND THE FEYNMAN PROPER TIME PARADIGM

The classical Maxwell electromagnetic field and the Lorentz-type force equations are rederived in the framework of the Feynman proper time paradigm and the related vacuum field theory approach. The classical Ampere law origin is rederived, and its relationship with the Feynman proper time paradigm is discussed. The electron inertia problem is analyzed in detail within the Lagrangian and Hamiltonian formalisms and the related pressure-energy compensation principle of stochastic electrodynamics. The modified Abraham–Lorentz damping radiation force is derived, and the electromagnetic electron mass origin is argued.

Keywords: classical Maxwell elctrodynamics, electron inertia problem, Feynman proper time paradigm, least action principle, Lagrangian and Hamiltonian formalisms, Lorentz-type force derivation, Ampere law, modified Abraham–Lorentz damping radiation force.

1. Classical Relativistic Electrodynamics Models Revisiting: Lagrangian and Hamiltonian Analysis

1.1. Introductory setting

Classical electrodynamics is nowadays considered [45, 51, 63, 72] as the most fundamental physical theory, largely owing to the depth of its theoretical foundations and wealth of experimental verifications. In the work, we describe a new approach to the classical Maxwell theory, based on a vacuum field medium model, and reanalyze some of the modern classical electrodynamics problems related to the description of the dynamics of a charged point particle in an external electromagnetic field. We remark here that, as usual, the term *"a charged point particle"* means an elementary material charged particle, whose internal spatial structure is assumed to be unimportant and is not taken into account, if the contrary is not specified.

We will discuss the important physical principles, characterizing the related electrodynamical vacuum field structure and based on the least action principle, for different charged point particle dynamics. In particular, we will obtain the main classical relativistic relations characterizing the charge point particle dynamics by means of the least action principle within Feynman's approach to the derivation of the Maxwell electromagnetic equations and the Lorentz-type force. Moreover, for each least action principle constructed in the work, we describe the corresponding Hamiltonian pictures and present the related energy conservation laws. Using the developed modified least action approach, the classical hadronic string model is analyzed in detail.

As the classical Lorentz force expression with respect to an arbitrary inertial reference frame is related to many theoretical and experimental controversies, such as the relativistic potential energy impact into the charged point particle mass, the Aharonov–Bohm effect [3] and the Abraham–Lorentz–Dirac radiation force [7, 45, 51] expression, the analysis of its structure subject to the assumed vacuum field medium structure is a very interesting important problem, which was discussed by many physicists including E. Fermi, G. Schott, R. Feynman, F. Dyson [22, 23, 26, 28, 33, 77] and many others. Trying to explain the latter, R. Feynman [28] wrote in his "Lectures on Physics":

"Now we would like to state the law that replaces the law $F = q\mathbf{v}\times\mathbf{B}$ for quantum mechanics. It will be the law that determines the behavior of quantum mechanical particles in an electromagnetic field. Since what happens is determined by amplitudes, the law must tell us how the magnetic influences affect the amplitudes; we are no longer dealing with the acceleration of the particle. The law is the following: the phase of the amplitude to arrive via any trajectory is changed by the presence of a magnetic field by an amount equal to the integral of the vector potential along the whole trajectory times the charge of the particle over Planck's constant. That is,

$$\text{Magnetic change in phase} = -\frac{q}{\hbar}\int \mathbf{A}\cdot d\mathbf{s}. \qquad (15.29)$$

If there were no magnetic field there would be a certain phase of arrival. If there is a magnetic field anywhere, the phase of the arriving wave is increased by the integral in Eq. (15.29). Although we will not need to use it for our present discussion, let us mention that the effect of an electrostatic field is to produce a phase change given by the negative of the time integral of the scalar potential :

$$\text{Electric change in phase} = -\frac{q}{\hbar}\int \phi\cdot dt.$$

These two expressions are correct not only for static fields, but together give the correct result for any electromagnetic field, static or dynamic. This is the law that replaces $F = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$."

To describe the essence of the electrodynamic problems related to the description of a charged point particle dynamics under an external electromagnetic field, let us begin with analyzing the classical Lorentz force expression

$$dp/dt = F_\xi := \xi E + \xi u \times B, \qquad (1.1)$$

where $\xi \in \mathbb{R}$ is a particle electric charge, $u \in T(\mathbb{R}^3)$ is its velocity [2, 9] vector expressed here in the light speed c units,

$$E := -\partial A/\partial t - \nabla\varphi \qquad (1.2)$$

is the corresponding external electric field, and

$$B := \nabla \times A \qquad (1.3)$$

is the corresponding external magnetic field acting on the charged particle, which can be expressed in terms of suitable vector $A : M^4 \to \mathbb{E}^3$ and scalar $\varphi : M^4 \to \mathbb{R}$ potentials. Here, "∇" is the standard gradient operator with respect to the spatial variable $r \in \mathbb{E}^3$, "$\times$" is the usual vector product in the three-dimensional Euclidean vector space $\mathbb{E}^3 := (\mathbb{R}^3, \langle\cdot,\cdot\rangle)$, which is naturally endowed with the classical scalar product $\langle\cdot,\cdot\rangle$. These potentials are defined on the Minkowski space $M^4 \simeq \mathbb{R}\times\mathbb{E}^3$, which models a chosen laboratory reference frame $\mathcal{K}_t$. Now, it is a well-known fact [28, 51, 63, 78] that the force expression (1.1) does not account for the dual influence of the charged particle on the electromagnetic field and should be considered valid only if the particle charge $\xi \to 0$. This also means that expression (1.1) cannot be used for studying the interaction between two different moving charged point particles, as was pedagogically demonstrated in classical manuals [28, 51]. Since the classical Lorentz force expression (1.1) is a natural consequence of the interaction of a charged point particle with an ambient electromagnetic field, its corresponding derivation based on the general principles of dynamics, was profoundly analyzed by R. Feynman and F. Dyson [22, 23, 28].

Taking this into account, it is natural to reanalyze this problem from the classical point of view, involving the Maxwell–Faraday wave theory aspect and specifying the corresponding vacuum field medium. Other questionable inferences from the classical electrodynamics theory, which strongly motivated the analysis in this work, are related both to an alternative interpretation of the well-known *Lorenz condition* imposed on the four-vector of electromagnetic observable potentials $(\varphi, A) : M^4 \to T^*(M^4)$ and the classical Lagrangian formulation [51] of the charged particle dynamics under the action of an external electromagnetic field. The problem of the mass of an elementary point charged particle, like an electron, was inspiring many physicists [46] from the past such as J.J. Thompson, G.G. Stokes, H.A. Lorentz, E. Mach, M. Abraham, P.A.M. Dirac, G.A. Schott, and others. Nevertheless, their studies have not given rise to a clear explanation of this phenomenon that stimulated new researchers to tackle it from different approaches based on new ideas stemming both from the classical Maxwell–Lorentz electromagnetic theory, as in [16,28–31,35,36,43,44,48,49,55,57,59,61,62,64,67, 70, 76, 79, 81, 84], and modern quantum field theories of Yang–Mills and Higgs types, as in [5, 37, 38, 83], and others, whose recent extensive review was done in [82].

In the present work, we will mostly concentrate on the detailed analysis and consequences of the Feynman proper time paradigm [22, 23, 28, 29] subject to deriving the electromagnetic Maxwell equations and the related Lorentz like force expression considered from the vacuum field theory approach developed in works [10,12–14] and further on its applications to the electromagnetic mass origin problem. Our treatment of this and related problems, based on the least action principle within the Feynman proper time paradigm [28], has allowed us to construct the respectively modified Lorentz-type equation for a charged point particle moving in space and radiating energy. Our analysis also elucidates, in particular, the computations of the self-interacting electron mass term in [55], where a not proper solution to the well-known classical Abraham–Lorentz [1, 52–54] and Dirac [20] electron electromagnetic "4/3-electron mass" problem was proposed. As a result of our scrutinized studying the classical electromagnetic mass problem, we have stated that it can be satisfactorily solved within the classical H. Lorentz and M. Abraham reasonings augmented with the additional electron stability condition, which was not taken before into account, but appeared to be very important for balancing the related electromagnetic field and mechanical electron momenta. The latter, following recent enough works [59,70] devoted to analyzing the electron charged shell model, can be realized within the suggested *pressure-energy compensation principle* suitably applied to the ambient electromagnetic energy fluctuations and the own electrostatic Coulomb electron energy.

In our investigation, we were in part inspired by works [18, 21, 43, 44, 70, 82–84] and especially by [30, 32, 42, 70] devoted to the classical problem of reconciling gravitational and electrodynamic charges within the Feynman proper time and zero energy point paradigms. First, we will revisit the classical Mach–Einstein relativistic electrodynamics of a moving charged point particle, and second, we study the resulting electrodynamic theories, by using the fundamental Lagrangian and Hamiltonian formalisms, which were specially devised in [13, 14, 69].

1.2. Maxwell equations, Lorenz constraint, and spatial energy flow

In a laboratory reference frame $\mathcal{K}_t$, let us consider the additional *Lorenz condition*

$$\partial\varphi/\partial t + \langle\nabla, A\rangle = 0, \tag{1.4}$$

the *a priori* assumed Lorentz invariant wave scalar field equation

$$\partial^2\varphi/\partial t^2 - \nabla^2\varphi = \rho, \tag{1.5}$$

and the charge continuity equation

$$\partial\rho/\partial t + \langle\nabla, J\rangle = 0, \tag{1.6}$$

where $\rho : M^4 \to \mathbb{R}$ and $J : M^4 \to \mathbb{E}^3$ are, respectively, the charge and current densities of the ambient matter. Then one can derive [14, 69] that the Lorentz-invariant wave equation

$$\partial^2 A/\partial t^2 - \nabla^2 A = J \tag{1.7}$$

and the classical electromagnetic Maxwell field equations [28, 45, 51, 63, 78]

$$\begin{aligned} &\nabla\times E + \partial B/\partial t = 0, \quad \langle\nabla, E\rangle = \rho,\\ &\nabla\times B - \partial E/\partial t = J, \quad \langle\nabla, B\rangle = 0 \end{aligned} \tag{1.8}$$

hold for all $(t, r) \in M^4$ with respect to the chosen laboratory reference frame $\mathcal{K}_t$.

Note that, conversely, Maxwell's equations (1.8) do not directly reduce, via definitions (1.2) and (1.3), to the wave field equations (1.5) and (1.7) without the Lorenz condition (1.4). This fact is very important and suggests that when it comes to a choice of governing equations, it may be reasonable to replace Maxwell's equations (1.8) with the Lorenz condition (1.4) and the charge continuity equation (1.6). To make the equivalence statement, claimed above, more transparent, we formulate it as the following proposition.

Proposition 1.1. *The Lorentz invariant wave equation (1.5) together with the Lorenz condition (1.4) for the observable potentials* $(\varphi, A) : M^4 \to T^*(M^4)$ *and the charge continuity relation (1.6) are completely equivalent to the Maxwell field equations (1.8).*

Proof. Substituting (1.4) into (1.5), one easily obtains

$$\partial^2\varphi/\partial t^2 = -\langle\nabla, \partial A/\partial t\rangle = \langle\nabla, \nabla\varphi\rangle + \rho, \qquad (1.9)$$

which implies the gradient expression

$$\langle\nabla, -\partial A/\partial t - \nabla\varphi\rangle = \rho. \qquad (1.10)$$

With regard for the electric field definition (1.2), expression (1.10) reduces to

$$\langle\nabla, E\rangle = \rho, \qquad (1.11)$$

which is the second of the first pair of Maxwell's equations (1.8).

Now applying $\nabla\times$ to definition (1.2), we find, owing to definition (1.3), that

$$\nabla \times E + \partial B/\partial t = 0, \qquad (1.12)$$

which is the first pair of the Maxwell equations (1.8). Having differentiated Eq. (1.5) with respect to the temporal variable $t \in \mathbb{R}$ and taking the charge continuity equation (1.6) into account, we find

$$\langle\nabla, \partial^2 A/\partial t^2 - \nabla^2 A - J\rangle = 0. \qquad (1.13)$$

The latter is equivalent to the wave equation (1.7), if we observe that the current vector $J : M^4 \to \mathbb{E}^3$ is defined by means of the charge continuity equation (1.6) up to a vector function $\nabla \times S : M^4 \to \mathbb{E}^3$. Applying now the operation $\nabla\times$ to relation (1.3) with regard for the wave equation (1.7), we obtain

$$\nabla \times B = \nabla \times (\nabla \times A) = \nabla\langle\nabla, A\rangle - \nabla^2 A =$$
$$= -\nabla(\partial\varphi/\partial t) - \partial^2 A/\partial t^2 + \left(\partial^2 A/\partial t^2 - \nabla^2 A\right) =$$
$$= \frac{\partial}{\partial t}(-\nabla\varphi - \partial A/\partial t) + J = \partial E/\partial t + J, \qquad (1.14)$$

which leads directly to

$$\nabla \times B = \partial E/\partial t + J.$$

This is the first of the second pair of the Maxwell equations (1.8). The final "*no magnetic charge*" equation

$$\langle\nabla, B\rangle = \langle\nabla,\, \nabla \times A\rangle = 0,$$

in (1.8) follows directly from the elementary identity $\langle\nabla, \nabla\times\rangle = 0$, thereby completing the proof. □

This proposition allows us to consider the observable potential functions $(\varphi, A) : M^4 \to T^*(M^4)$ as fundamental ingredients of the ambient *vacuum field medium*, by means of which we can try to describe the related physical behavior of charged point particles imbedded in the space-time M^4. The following observation provides a strong support for this approach:

Observation. *The Lorenz condition (1.4) actually means that the scalar potential field* $\varphi : M^4 \to \mathbb{R}$ *continuity relation, whose origin lies in some new field conservation law, characterizes the deep intrinsic structure of the vacuum field medium.*

To make this observation more transparent and precise, let us recall the definition [28, 51, 63, 78] of the electric current $J : M^4 \to \mathbb{E}^3$ in the dynamical form

$$J := \rho u, \qquad (1.15)$$

where the vector $u \in T(\mathbb{R}^3)$ is the corresponding charge velocity. Thus, the continuity relation

$$\partial\rho/\partial t + \langle\nabla, \rho u\rangle = 0 \qquad (1.16)$$

holds. It can easily be rewritten [56] as the integral conservation law

$$\frac{d}{dt}\int_{\Omega_t} \rho(t, r) d^3 r = 0 \qquad (1.17)$$

for the charge inside of any bounded domain $\Omega_t \subset \mathbb{E}^3$, moving in the space-time M^4 according to the natural evolution equation

$$dr/dt := u. \qquad (1.18)$$

Using the above reasoning, we obtain the following result.

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Proposition 1.2. *The Lorenz condition (1.4) is equivalent to the integral conservation law*

$$\frac{d}{dt}\int\limits_{\Omega_t}\varphi(t,r)d^3r=0, \tag{1.19}$$

where $\Omega_t\subset\mathbb{E}^3$ *is any bounded domain, moving with respect to the charged point particle* ξ *evolution equation*

$$dr/dt=u(t,r), \tag{1.20}$$

which represents the velocity vector of the related local potential field changes propagating in the Minkowski space-time M^4*. Moreover, for a particle with the distributed charge density* $\rho: M^4\to\mathbb{R}$*, the Umov-type local energy conservation relation*

$$\frac{d}{dt}\int\limits_{\Omega_t}\frac{\rho(t,r)\varphi(t,r)}{(1-|u(t,r)|^2)^{1/2}}d^3r=0 \tag{1.21}$$

holds for any $t\in\mathbb{R}$.

Proof. First, consider we the corresponding solutions to the potential field equations (1.5), taking condition (1.15) into account. Owing to the standard results from [28, 51], we find

$$A=\varphi u, \tag{1.22}$$

which gives rise to the following form of the Lorenz condition (1.4):

$$\partial\varphi/\partial t+\langle\nabla,\varphi u\rangle=0, \tag{1.23}$$

This can be rewritten obviously [56] as the integral conservation law (1.19), so expression (1.19) is stated.

To state the local energy conservation relation (1.21), it is necessary to combine conditions (1.16) and (1.23) and to find that

$$\partial(\rho\varphi)/\partial t+\langle u,\nabla(\rho\varphi)\rangle+2\rho\varphi\langle\nabla,u\rangle=0. \tag{1.24}$$

We note that the infinitesimal volume transformation $d^3r=\chi(t,r)d^3r_0$, where the Jacobian $\chi(t,r):=|\partial r(t;r_0)/\partial r_0|$ of the corresponding transformation $r:\Omega_{t_0}\to\Omega_t$ induced by the Cauchy problem for the differential relation (1.20) for any $t\in\mathbb{R}$ satisfies the evolution equation

$$d\chi/dt=\langle\nabla,u\rangle\chi, \tag{1.25}$$

easily following from (1.20). Applying the operator $\int_{\Omega_{t_0}}(...)\chi^2d^3r_0$ to equality (1.24), we obtain

$$0=\int\limits_{\Omega_{t_0}}\frac{d}{dt}\left(\rho\varphi\chi^2\right)d^3r_0=\frac{d}{dt}\int\limits_{\Omega_{t_0}}\left(\rho\varphi\chi\right)\chi d^3r_0=$$
$$=\frac{d}{dt}\int\limits_{\Omega_t}(\rho\varphi\chi)d^3r:=\frac{d}{dt}\mathcal{E}(\xi;\Omega_t). \tag{1.26}$$

Here, we denote the conserved charge $\xi:=\int_{\Omega_t}\rho(t,r)d^3r$ and the local energy conservation quantity $\mathcal{E}(\xi;\Omega_t):=\int_{\Omega_t}(\rho\varphi\chi)d^3r$. The latter quantity can be simplified, owing to the infinitesimal Lorentz invariance four-volume measure relation $d^3r(t,r_0)\wedge dt=d^3r_0\wedge dt_0$, where the variables $(t,r)\in\mathbb{R}_t\times\Omega_t\subset M^4$ are, within the present context, taken with respect to the moving reference frame $\mathcal{K}_t$ related to the infinitesimal charge quantity $d\xi(t,r):=\rho(t,r)d^3r$, and the variables $(t_0,r_0)\in\mathbb{R}_{t_0}\times\Omega_{t_0}\subset M^4$ are taken with respect to the laboratory reference frame $\mathcal{K}_{t_0}$ assigned to the infinitesimal charge quantity $d\xi(t_0,r_0)=\rho(t_0,r_0)d^3r_0$. The above-mentioned infinitesimal Lorentz invariance relations make it possible to calculate the local energy conservation quantity $\mathcal{E}(\xi;\Omega_0)$ as

$$\mathcal{E}(\xi;\Omega_0)=\int\limits_{\Omega_t}(\rho\varphi\chi)d^3r=\int\limits_{\Omega_t}\left(\rho\varphi\frac{d^3r}{d^3r_0}\right)d^3r=$$
$$=\int\limits_{\Omega_t}\left(\rho\varphi\frac{d^3r\wedge dt}{d^3r_0\wedge dt}\right)d^3r=\int_{\Omega_t}\left(\rho\varphi\frac{d^3r_0\wedge dt_0}{d^3r_0\wedge dt}\right)d^3r=$$
$$=\int\limits_{\Omega_t}\left(\rho\varphi\frac{dt_0}{dt}\right)d^3r=\int\limits_{\Omega_t}\frac{\rho\varphi d^3r}{(1-|u|^2)^{1/2}}, \tag{1.27}$$

where we took into account that $dt=dt_0(1-|u|^2)^{1/2}$. Thus, owing to (1.26) and (1.27), the local energy conservation relation (1.21) is satisfied, proving the proposition. □

The above-constructed local energy conservation quantity (1.27) can be rewritten as

$$\mathcal{E}(\xi;\Omega_t)=\int\limits_{\Omega_t}\frac{d\xi(t,r)\varphi(t,r)}{(1-|u|^2)^{1/2}}:=\int\limits_{\Omega_t}d\mathcal{E}(t,r), \tag{1.28}$$

where $d\mathcal{E}(t,r)=d\xi(t,r)\varphi(t,r)(1-|u|^2)^{-1/2}$ is the electromagnetic field energy density distributed in vacuum, which is related to the electric charge $d\xi(t,r)$

located at a point $(t, r) \in M^4$. It is worth to mention that the obtained quantity (1.28), owing to its conservation, can be interpreted in the case of a spatially structured charged particle ξ as the charged particle rest mass $m_0 := \mathcal{E}(\xi; \mathbb{R}^3)/c^2$ (in Gauss units). The latter appeared to be of decisive importance, when applying the Feynman proper time paradigm [12, 28] to the analysis of the inertial electron mass problem within the vacuum field theory approach based on the Lagrangian least action principle. The related Lagrangian approach is strongly dependent on Einstein's important notions of the laboratory $\mathcal{K}_t$ and the rest $\mathcal{K}_\tau$ reference frames and on the related least action principle. So, before explaining it in more details, we firstly reanalyze [67] the classical Maxwell electromagnetic theory from a strictly dynamical point of view based on the classical Ampere analysis of its origin.

1.3. The Ampere law in electrodynamics – the derivations of the classical and modified Lorentz forces

The classical ingenious Ampere analysis of two electric currents magnetically interacting with each other in thin conductors, as is well known, was based [28, 51, 63, 78] on the following experimental fact: the force between two electric currents depends on the distance between conductors, their mutual spatial orientation, and the quantitative values of currents. Having additionally accepted the infinitesimal superposition principle, A.-M. Ampere had derived a general analytical expression for the force between two infinitesimal elements of the currents under study:

$$df(r, r') = I\, I' \frac{(r - r')}{|r - r'|^2} \alpha(s, s'; n) dl dl', \tag{1.29}$$

where the vectors $r, r' \in \mathbb{E}^3$ point at infinitesimal currents $dr = sdl, dr' = s'dl'$ with the normalized orientation vectors $s, s' \in \mathbb{E}^3$ of two closed conductors l and l' carrying currents $I \in \mathbb{R}$ and $I' \in \mathbb{R}$, respectively, and the unit vector $n := (r - r')/|r - r'|$ fixing the spatial orientations of these infinitesimal elements, and the function $\alpha : (\mathbb{S}^2)^2 \times \mathbb{S}^2 \to \mathbb{R}$ being some real-valued smooth mapping. In view of the mutual symmetry between the infinitesimal elements of the currents dl and dl' belonging, respectively, to these two electric conductors, the infinitesimal force (1.29) was assumed by A.-M. Ampere *to satisfy locally the third Newton law*

$$df(r, r') = -df(r', r) \tag{1.30}$$

with the mapping

$$\alpha(s, s'; n) = \frac{\mu_0}{4\pi}(3k_1 \langle s, n\rangle \langle s', n\rangle + k_2 \langle s, s'\rangle), \tag{1.31}$$

where $\langle \cdot, \cdot \rangle$ is the natural scalar product in $\mathbb{E}^3$, and $k_1, k_2 \in \mathbb{R}$ are some still undetermined real and dimensionless parameters. The assumption (1.30) is evidently looking very restrictive and can be considered as reasonable only for a stationary system of conductors in the case where the principle of mutual action at a distance [28, 51] can be applied. Owing to himself, J.C. Maxwell [17]: "... we may draw the conclusions, first, that action and reaction are not always equal and opposite, and second, that apparatus may be constructed to generate any amount of work from its own resources. For let two oppositely electrified bodies A and B travel along the line joining them with equal velocities in the direction AB, then if either the potential or the attraction of the bodies at a given time is that due to their position at some former time (as these authors suppose), B, the foremost body, will attract A forwards more than B attracts A backwards. Now let A and B be kept asunder by a rigid rod. The combined system, if set in motion in the direction AB, will pull in that direction with a force which may either continually augment the velocity, or may be used as an inexhaustible source of energy."

Based on the fact that there is no possibility to measure the force between two infinitesimal current elements, A.-M. Ampere took into account (1.30), (1.31) and calculated the corresponding force exerted by the whole conductor l' on an infinitesimal current element of another conductor under regard:

$$\begin{aligned} dF(r) &:= \oint_{l'} df(r, r') = \\ &= \frac{I\, I' \mu_0}{4\pi} \oint_{l'} \frac{(r - r')}{|r - r'|^2} \left(3k_1 \left\langle dr, \frac{r - r'}{|r - r'|} \right\rangle \times \right. \\ &\left. \times \left\langle dr', \frac{r - r'}{|r - r'|} \right\rangle + k_2 \frac{r - r'}{|r - r'|} \langle dr, dr' \rangle \right) = \\ &= \frac{I\, I' \mu_0}{4\pi} \oint_{l'} \nabla_{r'} \left(\frac{1}{|r - r'|}\right) (3k_1 \langle dr, r - r'\rangle \times \\ &\times \langle dr', r - r'\rangle + k_2 \langle dr, dr'\rangle), \end{aligned} \tag{1.32}$$

which can be equivalently transformed as

$$dF(r) = \frac{I\,I'\mu_0}{4\pi} \oint_{l'} \nabla_{r'} \left(\frac{1}{|r-r'|}\right) \times$$
$$\times\,(3k_1\langle dr, r-r'\rangle\langle dr', r-r'\rangle + k_2\langle dr, dr'\rangle) =$$
$$= \frac{I\,I'\mu_0}{4\pi} \oint_{l'} \nabla_{r'} \left(\frac{1}{|r-r'|}\right) \times$$
$$\times\,[k_1\,(3\langle dr, r-r'\rangle\langle dr', r-r'\rangle -$$
$$-\,\langle dr, dr'\rangle) + (k_1+k_2)\langle dr, dr'\rangle] =$$
$$= -k_1\frac{\mu_0 I}{4\pi}\left\langle dr, \nabla \oint_{l'} \left(\frac{I'dr'}{|r-r'|}\right)\right\rangle -$$
$$-\,(k_1+k_2) < \nabla, \oint_{l'} \left\langle dr, \frac{I'dr'}{|r-r'|}\right\rangle, \qquad (1.33)$$

owing to the integral identity

$$\oint_{l'} \nabla_{r'} \left(\frac{1}{|r-r'|}\right) (3\langle dr, r-r'\rangle\langle dr', r-r'\rangle -$$
$$-\,\langle dr, dr'\rangle) = \langle dr, \nabla\rangle \oint_{l'} \frac{dr'}{|r-r'|}, \qquad (1.34)$$

which can be easily checked by means of the integration by parts. Let us introduce the vector potential

$$A(r) := \frac{\mu_0 I'}{4\pi} \oint_{l'} \frac{dr'}{|r-r'|}, \qquad (1.35)$$

generated by the conductor l' at the point $r \in \mathbb{E}^3$ belonging to the infinitesimal element dl of the conductor l. The resulting infinitesimal force (1.32) gives rise to the expression

$$dF(r) = k_1(-I < dr, \nabla)A(r) + I\nabla\langle dr, A(r)\rangle) -$$
$$-\,(2k_1+k_2)I\nabla\langle dr, A(r)\rangle =$$
$$= k_1 I dr \times (\nabla \times A(r)) - (2k_1+k_2)I\nabla\langle dr, A(r)\rangle =$$
$$= k_1 J(r)d^3r \times B(r) - (2k_1+k_2)\nabla\langle Jd^3r, A(r)\rangle, \qquad (1.36)$$

where we have accounted for the standard definition of a magnetic field

$$B(r) := \nabla \times A(r) \qquad (1.37)$$

and the corresponding relation for a current density:

$$J(r)d^3r := Idr. \qquad (1.38)$$

There are, evidently, many different possibilities to choose the dimensionless parameters $k_1, k_2 \in \mathbb{R}$. In his analysis, A.-M. Ampere had chosen the case where $k_1 = 1$, $k_2 = -2$ and obtained the well-known expression for a *magnetic force*:

$$dF(r) = J(r)d^3r \times B(r). \qquad (1.39)$$

It is easily reduced to the *classical Lorentz expression*

$$df_L(r) = \xi u \times B(r) \qquad (1.40)$$

for a force exerted by an external magnetic field on a point particle with an electric charge $\xi \in \mathbb{R}$ moving with a constant velocity $u \in T(\mathbb{R}^3)$.

If we take *an alternative choice* and put $k_1 = 1$, $k_2 = -1$, expression (1.36) yields *a modified magnetic Lorentz-type force* generated by a charged particle moving with a velocity $u' \in T(\mathbb{R}^3)$ and acting on a point particle, which is endowed with the electric charge $\xi \in \mathbb{R}$ and moves with a velocity $u \in T(\mathbb{R}^3)$:

$$dF_L(r) = J(r)d^3r \times B(r) - \nabla\langle J(r)d^3r, A(r)\rangle. \qquad (1.41)$$

This formula was before occasionally discussed in various works [60, 65, 71] and recently enough strongly obtained and analyzed in detail from the Lagrangian point of view in works [13, 14, 69] in the following infinitesimal form:

$$\delta f_L(r) = \xi u \times (\nabla \times \xi\delta A(r)) - \xi\nabla\langle u - u_f, \delta A(r)\rangle, \qquad (1.42)$$

where $\delta A(r) \in T^*(\mathbb{R}^3)$ denotes the magnetic potential generated by an external charged point particle moving with the velocity $u_f \in T(\mathbb{R}^3)$ and exerting the magnetic force $\delta f_L(r)$ on the charged particle located at a point $r \in \mathbb{R}^3$ and moving with the velocity $u \in T(\mathbb{R}^3)$ with respect to a common reference system $\mathcal{K}_t$. We also need to mention here that the modified Lorentz force (1.41) does not naturally account for the resulting pure electric force, as the conductors l and l' are considered to be electrically neutral. Simultaneously, we see that the magnetic potential has a physical significance in its own right [7, 13, 60, 71] and has meaning in a way that extends beyond the calculation of force fields.

Really, to obtain the Lorentz-type force (1.41) exerted by the external magnetic field generated by *the* *whole conductor* l' on an infinitesimal current element dl of the conductor l, it is necessary to integrate expression (1.42) along this conductor loop l':

$$dF_L(r) := \oint_{l'} \delta f_L(r) = J(r)dr \times$$
$$\times (\nabla \times \oint_{l'} \delta A(r)) - \nabla\langle J(r)dr, \oint_{l'} \delta A(r)\rangle +$$
$$+\nabla \oint_{l'} \langle u', \xi\delta A(r)\rangle = J(r)dr \times$$
$$\times (\nabla \times A(r)) - \nabla\langle J(r)dr, \oint_{l'} \delta A(r)\rangle +$$
$$+\nabla \oint_{l'} \langle dr', \xi\delta A(r)/dt\rangle = J(r)dr \times$$
$$\times B(r) - \nabla\langle J(r)dr, \oint_{l'} \delta A(r)\rangle +$$
$$+\nabla \int_{S(l')} \langle dS(l'), \nabla \times \xi\delta A(r)/dt\rangle = J(r)dr \times$$
$$\times B(r) - \nabla\langle J(r)dr, \oint_{l'} \delta A(r)\rangle +$$
$$+\nabla \oint_{l'} \langle dS(l'), \xi\delta B(r)/dt\rangle = J(r)dr \times$$
$$\times B(r) - \nabla\langle J(r)dr, \oint_{l'} \delta A(r)\rangle +$$
$$+\xi\nabla(d\Phi(r)/dt) = J(r)dr \times$$
$$\times B(r) - \nabla\langle J(r)dr, A(r)\rangle - \rho(r)d^3r\nabla\overline{W} =$$
$$= J(r)dr \times B(r) - \nabla\langle J(r)dr, \oint_{l'} \delta A(r)\rangle +$$
$$+\rho(r)d^3r(-\nabla\overline{W} - \partial A(r)/\partial t) = J(r)dr \times B(r) -$$
$$-\nabla\langle J(r)dr, \oint_{l'} \delta A(r)\rangle + \rho(r)d^3rE(r), \quad (1.43)$$

that is the equality

$$dF(r) = \rho(r)d^3rE(r) + J(r)d^3r \times$$
$$\times B(r) - \nabla\langle J(r)d^3r, A(r)\rangle, \quad (1.44)$$

where, by definition, the electric field $E(r) := -\nabla\varphi - \partial A(r)/\partial t$ with $\varphi \in \mathbb{R}$, being the corresponding scalar potential generated by the conductors under regard. Now, relation (1.44) easily yields the searched expression (1.42) for a *Lorentz-type force*, if we take into account that the whole electric field $E(r) \simeq 0$ due to the neutrality of the conductors. Concerning the latter, it is worth mentioning the following D. Kastler's [47] remark:

"It is true that Ampere's formula is no more admissible today, because it is based on the Newtonian idea of instantaneous action at a distance and it leads notably to the strange consequence that two consecutive elements of the same current should repel each other. Ampere presumed to have demonstrated experimentally this repulsion force, but on this point he was wrong. The modern method, the more rational in order to establish the existence of electrodynamic forces and to determine their value, consists in starting from the electrostatic interaction law of Coulomb between two charges (two electrons), whose one of them is at rest in the adopted frame of reference and studying how the interaction forces transform when one goes, thanks to the Lorentz–Einstein relations, to a system of coordinates in which both charges are in motion. One sees the appearance of additional forces proportional to e^2/c^2, e being the electrostatic charge and c the light velocity, hence one sees that not only the spin but also the magnetic moment of the electron are of relativistic origin – as Dirac has shown – but that the whole of electromagnetic forces has such an origin."

The above-presented analysis of Ampere's derivation of expression (1.36) for a magnetic force, as well as its consequences (1.41) and (1.42), makes it possible to suppose that the missed modified Lorentz-type force expression (1.43) could also be embedded into the classical relativistic Lagrangian and related Hamiltonian formalisms, giving rise to eventually new aspects and interpretations of many looking "*strange*" [7] experimental phenomena observed during the past centuries.

2. Electrodynamic Equations of the Vacuum Field Theory: Lagrangian Analysis

We proceed to describing a charged point particle ξ moving in the space-time with a velocity vector $u \in T(\mathbb{R}^3)$ and interacting with another external charged point particle ξ_f, moving with a velocity vector $u_f \in T(\mathbb{R}^3)$ with respect to a common laboratory reference frame $\mathcal{K}_t$. As was shown in [14, 69], the respectively modified dynamical equation for the vacuum potential field function $\bar{W}' : M^4 \rightarrow \mathbb{R}$ in the shifted reference frame $\mathcal{K}'$, moving with respect to the laboratory reference frame $\mathcal{K}_t$ with velocity $u_f \in T(\mathbb{R}^3)$, is as follows:

$$\frac{d}{dt'}[-\bar{W}'(u' - u'_f)] = -\nabla \bar{W}'. \tag{2.1}$$

Here, as before, the velocity vectors $u' := dr/dt'$ and $u'_f := dr_f/dt' \in T(\mathbb{R}^3)$ are calculated with respect to the shifted reference frame $\mathcal{K}'$. Since the external charged particle ξ_f moves in the space-time M^4, it generates the related magnetic field $B := \nabla \times A$, whose magnetic vector potentials $A : M^4 \rightarrow \mathbb{E}^3$ and $A' : M^4 \rightarrow \mathbb{E}^3$ are defined, owing to the results of [14, 69, 71], as

$$\xi A := \bar{W} u_f, \quad \xi A' := \bar{W}' u'_f, \tag{2.2}$$

From whence, taking into account that the field potential

$$\bar{W} = \bar{W}' \left(1 - |u_f|^2\right)^{-1/2} \tag{2.3}$$

and the particle momentum $p' = -\bar{W}' u' = -\bar{W} u$, equality (2.1) becomes equivalent to

$$\frac{d}{dt'}(p' + \xi A') = -\nabla \bar{W}', \tag{2.4}$$

if considered with respect to the shifted reference frame $\mathcal{K}'$, or to the Lorentz-type force equality

$$\frac{d}{dt}(p + \xi A) = -\nabla \bar{W} \left(1 - |u_f|^2\right), \tag{2.5}$$

if considered with respect to the laboratory reference frame $\mathcal{K}_t$, owing to the classical Lorentz invariance relation (2.3), as the corresponding magnetic vector potential generated by the external charged point test particle ξ_f with respect to the shifted reference frame $\mathcal{K}'$, is identically equal to zero. To imbed the dynamical equation (2.5) into the classical Lagrangian formalism, we start from the action functional

$$S^{(\tau)} := -\int_{\tau_1}^{\tau_2} \bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2}\, d\tau \tag{2.6}$$

based on the Lagrangian function calculated with respect to the shifted reference frame $\mathcal{K}'$. Here, as before, $\bar{W}'$ is the respectively calculated vacuum field potential $\bar{W}$ in the shifted reference frame $\mathcal{K}'$, $\dot{r} = u' dt'/d\tau, \dot{r}_f = u'_f dt'/d\tau$, $d\tau = dt'(1 - |u' - u'_f|^2)^{1/2}$, which accounts for the relative velocity of the charged point particle ξ in the reference frame $\mathcal{K}'$ specified by the Euclidean coordinates $(t', r - r_f) \in \mathbb{R}^4$, and moving simultaneously with a velocity vector $u_f \in T(\mathbb{R}^3)$ with respect to the laboratory reference frame $\mathcal{K}_t$ specified by the Minkowski coordinates $(t, r) \in M^4$ and related to those of the reference frame $\mathcal{K}'$ and $\mathcal{K}_\tau$ by means of the following infinitesimal relations:

$$dt^2 = (dt')^2 + |dr_f|^2, \quad (dt')^2 = d\tau^2 + |dr - dr_f|^2. \tag{2.7}$$

So, it is clear in this case that our charged point particle ξ moves with the velocity vector $u' - u'_f \in T(\mathbb{R}^3)$ with respect to the reference frame $\mathcal{K}'$, in which the external charged particle ξ_f is at rest. Thereby, we have reduced the problem of deriving the dynamical equation for a charged point particle ξ to that for a charged particle moving under the action of the electrical field of an external charged particle ξ_f persisting to be at rest with respect to the laboratory reference frame $\mathcal{K}_t$.

Now, we can compute the least action variational condition $\delta S^{(\tau)} = 0$, taking into account that, owing to (2.6), the corresponding Lagrangian function with respect to the rest reference frame $\mathcal{K}_\tau$ is given as

$$\mathcal{L}^{(\tau)} := -\bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2}. \tag{2.8}$$

As a result of simple calculations, the generalized momentum of the charged particle ξ equals

$$\begin{aligned} P &:= \partial \mathcal{L}^{(\tau)}/\partial \dot{r} = -\bar{W}'(\dot{r} - \dot{r}_f)(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} = \\ &= -\bar{W}'\dot{r}(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} + \\ &+ \bar{W}'\dot{r}_f(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} = \\ &= m'u' + \xi A' := p' + \xi A' = p + \xi A, \end{aligned} \tag{2.9}$$

where, owing to (2.3), the vectors $p' := -\bar{W}'u' = -\bar{W}u = p \in \mathbb{E}^3$, $A' = \bar{W}'u'_f = \bar{W}u_f = A \in \mathbb{E}^3$, and the dynamical equality

$$\frac{d}{d\tau}(p' + \xi A') = -\nabla \bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2} \tag{2.10}$$

holds with respect to the rest reference frame $\mathcal{K}_\tau$. As $dt' = d\tau(1 + |\dot{r} - \dot{r}_f|^2)^{1/2}$ and $(1 + |\dot{r} - \dot{r}_f|^2)^{1/2} = (1 - |u' - u'_f|^2)^{-1/2}$, relation (2.10) yields the equality

$$\frac{d}{dt'}(p' + \xi A') = -\nabla \bar{W}' \tag{2.11}$$

exactly coinciding with equality (2.4) in the moving reference frame $\mathcal{K}'$. Now, making use of expressions (2.7) and (2.3), one can rewrite (2.11) as that with respect to the laboratory reference frame $\mathcal{K}_t$:

$$\frac{d}{dt'}(p' + \xi A') = -\nabla \bar{W}' \Rightarrow$$

$$\Rightarrow \frac{d}{dt'}\left(\frac{-\bar{W}u'}{(1+|u'_f|^2)^{1/2}} + \frac{\xi \bar{W} u'_f}{(1+|u'_f|^2)^{1/2}}\right) =$$

$$= -\frac{\nabla \bar{W}}{(1+|u'_f|^2)^{1/2}} \Rightarrow \frac{d}{dt'}\left(\frac{-\bar{W}dr}{(1+|u'_f|^2)^{1/2}dt'} +\right.$$

$$\left. + \frac{\xi \bar{W} dr_f/}{(1+|u'_f|^2)^{1/2}}\right) = -\frac{\nabla \bar{W}}{(1+|u'_f|^2)^{1/2}} \Rightarrow$$

$$\Rightarrow \frac{d}{dt}\left(-\bar{W}\frac{dr}{dt} + \xi \bar{W}\frac{dr_f}{dt}\right) = -\nabla \bar{W}(1-|u_f|^2), \quad (2.12)$$

exactly coinciding with (2.5):

$$\frac{d}{dt}(p + \xi A) = -\nabla \bar{W}\left(1 - |u_f|^2\right). \quad (2.13)$$

Remark 2.1. Equation (2.13) allows one to infer the following important physically reasonable phenomenon: if the velocity $u_f \in T(\mathbb{R}^3)$ of a test charged point particle tends to the light velocity $c = 1$, the corresponding acceleration force $F_{\text{ac}} := := -\nabla \bar{W}(1 - |u_f|^2)$ is vanishing. Thereby, the electromagnetic fields generated by such rapidly moving charged point particles have no influence on the dynamics of charged objects, if they are observed with respect to an arbitrarily chosen laboratory reference frame $\mathcal{K}_t$.

Equation (2.13) can be easily rewritten as

$$dp/dt = -\nabla \bar{W} - \xi dA/dt + \nabla \bar{W}|u_f|^2 =$$

$$= \xi(-\xi^{-1}\nabla \bar{W} - \partial A/\partial t) - \xi\langle u, \nabla\rangle A + \xi \nabla \langle A, u_f\rangle \quad (2.14)$$

or, in the standard Lorentz-type form,

$$dp/dt = \xi E + \xi u \times B - \nabla\langle \xi A, u - u_f\rangle, \quad (2.15)$$

where the point particle momentum $p := mu$, and

$$m := -\bar{W} \quad (2.16)$$

is the corresponding inertial particle mass.

Result (2.15), being before found and written down with respect to the shifted reference frame $\mathcal{K}'$ in [14, 69, 71] makes it possible to formulate the next important proposition.

Proposition 2.2. *The alternative classical relativistic electrodynamic model (2.4) allows one to formulate the least action based on the action functional (2.6) with respect to the rest reference frame $\mathcal{K}_\tau$, where the Lagrangian function is given by expression (2.8). The resulting Lorentz-type force is given by expression (2.15), being modified by the additional force component $F_c := -\nabla\langle \xi A, u - u_f\rangle$, important for the explanation [3, 15, 80] of the well-known Aharonov–Bohm effect.*

2.1. An alternative relativistic electrodynamic model

It is easy to see that the action functional (2.6) is written, by utilizing the classical Galilean transformations of reference frames. Let us consider the action functional for a charged point particle moving with respect the reference frame $\mathcal{K}_t$, as well as its interaction with an external magnetic field generated by the vector 4-potential $(\varphi, A) : M^4 \to T^*(M)$. It can be naturally generalized as the relativistically invariant expression

$$S^{(t)} := \int_{t_1}^{t_2} (-\varphi dt + \xi\langle A, dr\rangle) =$$

$$= \int_{\tau_1}^{\tau_2} [-\bar{W}(1 + |\dot{r}|^2)^{1/2} + \xi\langle A, \dot{r}\rangle] d\tau, \quad (2.17)$$

where $d\tau = dt(1 - |u|^2)^{1/2}$ and $\bar{W} := \xi\varphi$. As a result, we obtain that, with respect to the the action functional, $S^{(\tau)} = \int_{\tau_1}^{\tau_2} [-\bar{W}(1+|\dot{r}|^2)^{1/2} + \xi\langle A, \dot{r}\rangle] d\tau$. Thus, the corresponding common particle-field momentum takes the form

$$P := \partial \mathcal{L}^{(\tau)}/\partial \dot{r} = -\bar{W}\dot{r}(1 + |\dot{r}|^2)^{-1/2} + \xi A =$$

$$= mu + \xi A := p + \xi A \quad (2.18)$$

and satisfies the relations

$$\dot{P} := dP/d\tau = \partial \mathcal{L}^{(\tau)}/\partial r = -\nabla \bar{W}(1 + |\dot{r}|^2)^{1/2} +$$

$$+ \xi \nabla\langle A, \dot{r}\rangle = -\nabla \bar{W}(1 - |u|^2)^{-1/2} +$$

$$+ \xi \nabla \langle A, u\rangle(1 - |u|^2)^{-1/2}, \quad (2.19)$$

where

$$\mathcal{L}^{(\tau)} := -\bar{W}(1 + |\dot{r}|^2)^{1/2} + \xi\langle A, \dot{r}\rangle \quad (2.20)$$

is the corresponding Lagrangian function. Since $d\tau = dt(1-|u|^2)^{1/2}$, we finds easily from (2.19) that

$$dP/dt = -\nabla\bar{W} + \xi\nabla\langle A, u\rangle. \quad (2.21)$$

Substituting (2.18) into (2.21), we obtain the classical expression for the Lorentz force F acting on a moving charged point particle ξ :

$$dp/dt := F = \xi E + \xi u \times B, \quad (2.22)$$

where, by definition,

$$E := -\xi^{-1}\nabla\bar{W} - \partial A/\partial t \quad (2.23)$$

is its associated electric field, and

$$B := \nabla \times A \quad (2.24)$$

is the corresponding magnetic field. This result can be summarized as follows:

Proposition 2.3. *The classical relativistic Lorentz force (2.22) allows one to formulate the least action (2.17) with respect to the rest reference frame variables, where the Lagrangian function is given by formula (2.20). Its electrodynamics described by the Lorentz force (2.22) is not equivalent to the classical relativistic point particle electrodynamics described [51] by the Lorentz force expression of the same form as (2.22), but with the momentum $p \in \mathbb{E}^3$ not coinciding with that entering (2.22).*

Expressions (2.22) and (2.15) are equal up to the gradient term $F_c := -\xi\nabla\langle A, u-u_f\rangle$, which reconciles the Lorentz forces acting on a charged moving particle ξ with respect to different reference frames. This fact is important for our vacuum field theory approach, since it uses no special geometry and makes it possible to analyze both electromagnetic and gravitational fields simultaneously by employing the new definition of the dynamical mass by means of expression (2.16).

2.2. The electrodynamic equations of the vacuum field theory: Hamiltonian analysis

Any Lagrangian theory has an equivalent canonical Hamiltonian representation via the classical Legendre transformation [2, 6, 9, 78]. Since we have already formulated our vacuum field theory of a moving charged particle ξ in Lagrangian form, we proceed now to its Hamiltonian analysis making use of the action functional (2.8).

The dynamical field equation (2.4) describing the motion of a charged particle ξ in an external electromagnetic field looks in the canonical Hamiltonian form as

$$\dot{r} := dr/d\tau = \partial H/\partial P, \ \dot{P} := dP/d\tau = -\partial H/\partial r, \quad (2.25)$$

where, by definition,

$$\begin{aligned} H &:= \langle P, \dot{r}\rangle - \mathcal{L}^{(\tau)} = \\ &= \left\langle P, \dot{r}_f - P\bar{W}'^{,-1}(1-|P|^2/\bar{W}'^{,2})^{-1/2}\right\rangle + \\ &+ \bar{W}'[\bar{W}'^{,2}(\bar{W}'^{,2}-|P|^2)^{-1}]^{1/2} = \\ &= \langle P, \dot{r}_f\rangle + |P|^2(\bar{W}'^{,2}-|P|^2)^{-1/2} - \\ &- \bar{W}'^{,2}(\bar{W}'^{,2}-|P|^2)^{-1/2} = \\ &= -(\bar{W}'^{,2}-|P|^2)(\bar{W}'^{,2}-|P|^2)^{-1/2} + \langle P, \dot{r}_f\rangle = \\ &= -(\bar{W}'^{,2}-|P|^2)^{1/2} - \xi\langle A', P\rangle(\bar{W}'^{,2}-|P|^2)^{-1/2} = \\ &= -(\bar{W}^2-|\xi A|^2-|P|^2)^{1/2} - \\ &- \xi\langle A, P\rangle(\bar{W}^2-|\xi A|^2-|P|^2)^{-1/2}, \end{aligned} \quad (2.26)$$

being rewritten with respect to the laboratory reference frame $\mathcal{K}_t$. Here, we took into account that, owing to definitions (2.2), (2.3), and (3.17),

$$\begin{aligned} \xi A' &:= \bar{W}'u'_f = \bar{W}'dr_f/dt' = \xi A = \\ &= \bar{W}'\frac{dr_f}{d\tau}\cdot\frac{d\tau}{dt'} = \bar{W}'\dot{r}_f(1-|u-u_f|)^{1/2} = \\ &= \bar{W}'\dot{r}_f(1+|\dot{r}-\dot{r}_f|^2)^{-1/2} = \\ &= -\bar{W}'\dot{r}_f(\bar{W}'^{,2}-|P|^2)^{1/2}\bar{W}'^{,-1} = \\ &= -\dot{r}_f(\bar{W}'^{,2}-|P|^2)^{1/2}, \end{aligned} \quad (2.27)$$

and, in particular,

$$\begin{aligned} \dot{r}_f &= -\xi A(\bar{W}'^{,2}-|P|^2)^{-1/2}, \\ \bar{W} &= \bar{W}'(1-|u_f|^2)^{-1/2}, \end{aligned} \quad (2.28)$$

where $A : M^4 \to \mathbb{R}^3$ is the related magnetic vector potential generated by a moving external charged particle ξ_f. Equations (2.25) can be rewritten with respect to the laboratory reference frame $\mathcal{K}_t$ in the form

$$\begin{aligned} &dr/dt = u, \\ &dp/dt = \xi E + \xi u \times B - \xi\nabla\langle A, u-u_f\rangle, \end{aligned} \quad (2.29)$$

which coincides with result (2.15).

From whence, we see that the Hamiltonian function (2.26) satisfies the energy conservation conditions

$$dH/d\tau = dH/dt' = dH/dt = 0, \tag{2.30}$$

for all τ, t' and $t \in \mathbb{R}$, and the suitable energy expression is

$$\mathcal{E} = (\bar{W}^2 - \xi^2|A|^2 - |P|^2)^{1/2} +$$

$$+\,\xi\langle A, P\rangle(\bar{W}^2 - \xi^2|A|^2 - |P|^2)^{-1/2}, \tag{2.31}$$

where the generalized momentum $P = p + \xi A$. Result (2.31) evidently differs essentially from that obtained in [51], which makes use of the *a priori* relativistic Lagrangian function for a moving charged point particle ξ in an external electromagnetic field. Thus, we obtained the following proposition:

Proposition 2.4. *The alternative classical relativistic electrodynamic model (2.29), which is intrinsically compatible with the classical Maxwell equations (1.8), allows the Hamiltonian formulation (2.25) with respect to the rest reference frame variables, where the Hamiltonian function is given by expression (2.26).*

The inference above is a natural candidate for the experimental validation of our theory. It is strongly motivated by the following remark.

Remark 2.5. It is necessary to mention here that the Lorentz force expression (2.29) uses the particle momentum $p = mu$, where the dynamical "mass" $m := -\bar{W}$ satisfies condition (2.31). This gives rise to the following crucial relation between the particle energy $\mathcal{E}_0$ and its rest mass $m_0 = -\bar{W}_0$ (for the velocity $u = 0$ at the initial time moment $t = 0$):

$$\mathcal{E}_0 = m_0 \frac{(1 - |\xi A_0/m_0|^2)}{(1 - 2|\xi A_0/m_0|^2)^{1/2}}, \tag{2.32}$$

or, equivalently, under the condition $|\xi A_0/m_0|^2 < 1/2$,

$$m_0 = \mathcal{E}_0 \left(\frac{1}{2} + |\xi A_0/\mathcal{E}_0|^2 \pm \frac{1}{2}\sqrt{1 - 4|\xi A_0/\mathcal{E}_0|^2}\right)^{1/2}, \tag{2.33}$$

where $A_0 := A|_{t=0} \in \mathbb{E}^3$, which differs markedly from the classical [51] expression $m_0 = \mathcal{E}_0 - \xi\varphi_0$ and does not *a priori* satisfy the canonical Einsteinian rest mass relation $\mathcal{E}_0 = m_0$. We note that the quantity $|\xi A_0/\mathcal{E}_0| \to 0$, as the energy modulus $|\mathcal{E}_0| \to \infty$. Then the following asymptotic mass values follow from (2.33):

$$\bar{m}_0 \simeq \mathcal{E}_0, \quad m_0^{(\pm)} \simeq \pm\sqrt{2}|\xi A_0|. \tag{2.34}$$

The first mass value $\bar{m}_0 \simeq \mathcal{E}_0$ is looking from the relativistic physics standard, yet the second mass values $m_0^{(\pm)} \simeq \pm\sqrt{2}|\xi A_0|$ give rise to the existence of charged particle excitations of the vacuo with both positive and negative mass values at large enough energies.

3. The Maxwell and Lorentz Force Equations: the Analysis of the Electron Inertial Mass Problem

3.1. Short historical notes

The problem of the mass of an elementary point charged particle, like an electron, was inspiring many physicists [46] from the past: J.J. Thompson, G.G. Stokes, H.A. Lorentz, E. Mach, M. Abraham, P.A.M. Dirac, G.A. Schott, and others. Nonetheless, their studies had not given rise to a clear explanation of this phenomenon, which stimulated new researchers to tackle it from different approaches based on new ideas stemming both from the classical Maxwell–Lorentz electromagnetic theory, as in [16, 28–31, 35, 36, 43, 44, 48, 49, 55, 57, 59, 61, 62, 64, 67, 70, 76, 79, 81, 84], and modern quantum field theories of the Yang–Mills- and Higgs-type, as, e.g., in [5, 37, 38, 83], whose recent extensive review was done in [82].

In the present work, we will mostly concentrate on the detailed analysis and consequences of the Feynman proper time paradigm [22, 23, 28, 29] with the purpose to derive the electromagnetic Maxwell equations and the expression for a related Lorentz-like force considered within the vacuum field theory approach developed in works [10, 12–14], as well as on its applications to the problem of the electromagnetic mass origin. Our treatment of this and related problems, based on the least action principle within the Feynman proper time paradigm [28], has allowed us to construct the respectively modified Lorentz-type equation for a charged point particle moving in space and radiating energy. Our analysis also elucidates, in particular, the computations of the self-interacting electron mass term in [55], where a not proper solution to the well-known classical Abraham–Lorentz [1, 52–54] and Dirac [20] electron electromagnetic "4/3-electron mass" problem was proposed. As a result of our scrutinized study of the classical electromagnetic mass problem, we have stated that it can be satisfactorily solved within the classical reasonings by H. Lorentz and M. Abraham augmented with the additional electron stability condition, which was not taken before into account, but appeared to be very important for balancing the related electromagnetic field and mechanical electron momenta. The latter, following recent works [59, 70] devoted to the analysis of the electron charged shell model, can be realized within the suggested *pressure-energy compensation principle* suitably applied to the ambient electromagnetic energy fluctuations and the own electrostatic Coulomb electron energy.

3.2. The Feynman proper time paradigm: geometric analysis

In this section, we will develop further the vacuum field theory approach within the Feynman proper time paradigm devised before in [12, 14] to the electromagnetic electron theories by J.C. Maxwell and H. Lorentz and will show that they should be suitably modified: namely, the basic equations for the Lorentz force should be generalized, by following the Landau–Lifshitz least action recipe [51] and considering the pure electromagnetic field impact. When the devised vacuum field theory approach is applied to the classical electron shell model, the resulting Lorentz force expression appears to satisfactorily explain the electron inertial mass term exactly coinciding with the electron relativistic mass, thus confirming the well-known assumption [45, 72] by M. Abraham and H. Lorentz.

As was reported by F. Dyson [22, 23], the original derivation of the electromagnetic Maxwell equations within the Feynman approach was based on an *a priori* general form of the classical Newton-type force acting on a charged point particle moving in the three-dimensional space $\mathbb{R}^3$ endowed with the canonical Poisson brackets on the phase variables defined on the associated tangent space $T(\mathbb{R}^3)$. As a result of this approach, only the first part of the Maxwell equations was derived, whereas the second part, according to F. Dyson [22], is related to the charged matter nature, which appears to be hidden. Trying to complete this Feynman approach to the derivation of Maxwell's equations more systematically, we have observed [12] that the original Feynman's calculations based on the analysis of Poisson brackets were performed on the *tangent space* $T(\mathbb{R}^3)$ which is, concerning the problem posed, not physically proper. The true Poisson brackets can be correctly defined only on the *coadjoint phase space* $T^*(\mathbb{R}^3)$, as seen from the classical Lagrangian equations and the related Legendre transformation [2, 6, 9, 34] from $T(\mathbb{R}^3)$ to $T^*(\mathbb{R}^3)$. Moreover, within this observation, the corresponding dynamical Lorentz-type equation for a charged point particle should be written for the particle momentum, not for the particle velocity, whose value is well defined only with respect to the proper relativistic reference frame associated with the charged point particle, owing to the fact that the Maxwell equations are Lorentz-invariant.

Thus, from the very beginning, we shall reanalyze the structure of the Lorentz force exerted on a moving charged point particle with a charge $\xi \in \mathbb{R}$ by another point charged particle with a charge $\xi_f \in \mathbb{R}$, making use of the classical Lagrangian approach, and rederive the corresponding electromagnetic Maxwell equations. The latter appear to be strongly related to the charged point mass structure of the electromagnetic origin, as was suggested by R. Feynman and F. Dyson.

Consider now a charged point particle moving in an electromagnetic field. For its description, it is convenient to introduce a trivial fiber bundle structure $\pi\colon \mathcal{M} \to \mathbb{R}^3, \mathcal{M} = \mathbb{R}^3 \times G$, with the Abelian structure group $G := \mathbb{R}\backslash\{0\}$, equivariantly acting on the canonically symplectic coadjoint space $T^*(\mathcal{M})$ endowed both with the canonical symplectic structure

$$\omega^{(2)}(p, y; r, g) := d\, pr^*\alpha^{(1)}(r, g) = \langle dp, \wedge dr\rangle + \\ + \langle dy, \wedge g^{-1}dg\rangle_{\mathcal{G}} + \langle ydg^{-1}, \wedge dg\rangle_{\mathcal{G}} \tag{3.1}$$

for all $(p, y; r, g) \in T^*(\mathcal{M})$, where $\alpha^{(1)}(r, g) := \langle p, dr\rangle + \langle y, g^{-1}dg\rangle_{\mathcal{G}} \in T^*(\mathcal{M})$ is the corresponding Liouville form on $\mathcal{M}$, and with a connection one-form $\mathcal{A} : M \to T^*(M) \times \mathcal{G}$ as

$$\mathcal{A}(r, g) := g^{-1}\langle \xi A(r), dr\rangle g + g^{-1}dg, \tag{3.2}$$

with $\xi \in \mathcal{G}^*, (r, g) \in \mathbb{R}^3 \times G$, and $\langle\cdot,\cdot\rangle$ being the scalar product in $\mathbb{E}^3$. The corresponding curvature 2-form $\Sigma^{(2)} \in \Lambda^2(\mathbb{R}^3) \otimes \mathcal{G}$ is

$$\Sigma^{(2)}(r) := d\mathcal{A}(r, g) + \mathcal{A}(r, g) \wedge \mathcal{A}(r, g) = \\ = \xi \sum_{i,j=1}^{3} F_{ij}(r)dr^i \wedge dr^j, \tag{3.3}$$

where

$$F_{ij}(r) := \frac{\partial A_j}{\partial r^i} - \frac{\partial A_i}{\partial r^j} \tag{3.4}$$

for $i, j = \overline{1,3}$ is the electromagnetic tensor with respect to the reference frame $\mathcal{K}_t$ characterized by the phase space coordinates $(r, p) \in T^*(\mathbb{R}^3)$. Since an element $\xi \in \mathcal{G}^*$ is yet not fixed, it is natural to apply the standard [2, 6, 9] invariant Marsden–Weinstein–Meyer reduction to the orbit factor space $\tilde{P}_\xi := P_\xi / G_\xi$ subject to the related momentum mapping $l : T^*(\mathcal{M}) \to \to \mathcal{G}^*$ constructed with respect to the canonical symplectic structure (3.1) on $T^*(\mathcal{M})$, where, by definition, $\xi \in \mathcal{G}^*$ is constant, $P_\xi := l^{-1}(\xi) \subset T^*(\mathcal{M})$ and $G_\xi = \{g \in G : Ad^*_G \xi\}$ is the isotropy group of the element $\xi \in \mathcal{G}^*$.

As a result of the Marsden–Weinstein–Meyer reduction, we find that $G_\xi \simeq G$, the factor-space $\tilde{P}_\xi \simeq T^*(\mathbb{R}^3)$ is endowed with a suitably reduced symplectic structure $\bar{\omega}^{(2)}_\xi \in T^*(\tilde{P}_\xi)$, and the corresponding Poisson brackets on the reduced manifold $\tilde{P}_\xi$ are

$$\begin{aligned} &\{r^i, r^j\}_\xi = 0, \quad \{p_j, r^i\}_\xi = \delta^i_j, \\ &\{p_i, p_j\}_\xi = \xi F_{ij}(r) \end{aligned} \tag{3.5}$$

for $i, j = \overline{1,3}$ considered with respect to the reference frame $\mathcal{K}_t$. Introducing a new momentum variable

$$\tilde{\pi} := p + \xi A(r) \tag{3.6}$$

on $\tilde{P}_\xi$, it is easy to verify that $\bar{\omega}^{(2)}_\xi \to \tilde{\omega}^{(2)}_\xi := \langle d\tilde{\pi}, \wedge dr \rangle$, giving rise to the following "*minimal interaction*" canonical Poisson brackets:

$$\begin{aligned} &\{r^i, r^j\}_{\tilde{\omega}^{(2)}_\xi} = 0, \\ &\{\tilde{\pi}_j, r^i\}_{\tilde{\omega}^{(2)}_\xi} = \delta^i_j, \quad \{\tilde{\pi}_i, \tilde{\pi}_j\}_{\tilde{\omega}^{(2)}_\xi} = 0 \end{aligned} \tag{3.7}$$

for $i, j = \overline{1,3}$ with respect to some new reference frame $\tilde{\mathcal{K}}_{t'}$ characterized by the phase space coordinates $(r, \tilde{\pi}) \in \tilde{P}_\xi$ and a new evolution parameter $t' \in \mathbb{R}$, if the Maxwell field compatibility equations

$$\partial F_{ij}/\partial r_k + \partial F_{jk}/\partial r_i + \partial F_{ki}/\partial r_j = 0 \tag{3.8}$$

are satisfied on $\mathbb{R}^3$ for all $i, j, k = \overline{1,3}$ with the curvature tensor (3.4).

Now, we proceed to a dynamic description of the interaction between two moving charged point particles ξ and ξ_f moving, respectively, with the velocities $u := dr/dt$ and $u_f := dr_f/dt$ in the reference frame $\mathcal{K}_t$. Unfortunately, there is a fundamental problem in correctly formulating a physically suitable action functional and the related least action condition. There are clearly possibilities such as

$$S^{(t)}_p := \int_{t_1}^{t_2} dt \mathcal{L}^{(t)}_p [r; dr/dt] \tag{3.9}$$

on a temporal interval $[t_1, t_2] \subset \mathbb{R}$ with respect to the laboratory reference frame $\mathcal{K}_t$,

$$S^{(t')}_p := \int_{t'_1}^{t'_2} dt' \mathcal{L}^{(t')}_p [r; dr/dt'] \tag{3.10}$$

on a temporal interval $[t'_1, t'_2] \subset \mathbb{R}$ with respect to the moving reference frame $\mathcal{K}_{t'}$, and

$$S^{(\tau)}_p := \int_{\tau_1}^{\tau_2} d\tau \mathcal{L}^{(\tau)}_p [r; dr/d\tau] \tag{3.11}$$

on a temporal interval $[\tau_1, \tau_2] \subset \mathbb{R}$ with respect to the proper time reference frame $\mathcal{K}_\tau$ naturally related to the moving charged point particle ξ.

It was first observed by Poincaré and Minkowski [63] that the temporal differential $d\tau$ is not a closed differential one-form, which physically means that a particle can traverse many different paths in space $\mathbb{R}^3$ with respect to the reference frame $\mathcal{K}_t$ during a given proper time interval $d\tau$, *naturally* related to its motion. This fact was stressed [24, 25, 58, 63, 66] by Einstein, Minkowski, and Poincaré, and later exhaustively analyzed by R. Feynman, who argued [28] that the dynamical equation of a moving point charged particle is physically sensible only with respect to its proper time reference frame, and the corresponding Lagrangian functional should be initially calculated with respect to the laboratory reference frame $\mathcal{K}_t$. This is Feynman's proper time reference frame paradigm, which was recently further elaborated and applied to the electromagnetic Maxwell equations in [30–32] and to the Lorentz-type equation for a moving charged point particle under the action of an external electromagnetic field in [9, 12–14]. As was there argued from the physical point of view, the least action principle should be applied only to expression (3.11) written with respect to the proper time reference frame $\mathcal{K}_\tau$, whose temporal parameter $\tau \in \mathbb{R}$ is independent of an observer and is a closed differential one-form. Consequently, this action functional is also mathematically sensible, which reflects partially Poincaré's and Minkowski's observation that the infinitesimal quadratic interval

$$d\tau^2 = (dt')^2 - |dr - dr_f|^2 \tag{3.12}$$

relating the reference frames $\mathcal{K}_{t'}$ and $\mathcal{K}_\tau$ can be invariantly used for the four-dimensional relativistic geometry. The most natural way to contend with this problem is to consider firstly the quasirelativistic dynamics of a charged point particle ξ with respect to the moving reference frame $\mathcal{K}_{t'}$, relative to which the charged point particle ξ_f is at rest. Therefore, it is possible to write down a suitable action functional (3.10), up to $O(1/c^4)$, as the light velocity $c \to \infty$, where the quasiclassical Lagrangian function $\mathcal{L}_p^{(t')}[r; dr/dt']$ can be naturally chosen as

$$\mathcal{L}_p^{(t')}[r; dr/dt'] := \\ := m'(r)\,|dr/dt' - dr_f/dt'|^2/2 - \xi\varphi'(r). \tag{3.13}$$

Here, $m'(r) \in \mathbb{R}_+$ is the charged particle ξ inertial mass parameter, and $\varphi'(r)$ is the potential function generated by the charged particle ξ_f at a point $r \in \mathbb{R}^3$ with respect to the reference frame $\mathcal{K}_{t'}$. The standard temporal relations between the reference frames $\mathcal{K}_t$ and $\mathcal{K}_{t'}$ read

$$dt' = dt\left(1 - |dr_f/dt'|^2\right)^{1/2}, \tag{3.14}$$

as well as between the reference frames $\mathcal{K}_{t'}$ and $\mathcal{K}_\tau$,

$$d\tau = dt'\left(1 - |dr/dt' - dr_f/dt'|^2\right)^{1/2}, \tag{3.15}$$

give rise, up to $O(1/c^2)$, as $c \to \infty$, to $dt' \simeq dt$ and $d\tau \simeq dt'$, respectively. Then it is easy to verify that the least action condition $\delta S_p^{(t')} = 0$ is equivalent to the dynamical equation

$$d\pi/dt = \nabla\mathcal{L}_p^{(t')}[r; dr/dt] = \\ = \left(\frac{1}{2}\,|dr/dt - dr_f/dt|^2\right)\nabla m - \xi\nabla\varphi(r), \tag{3.16}$$

where we have defined the generalized canonical momentum as

$$\pi := \partial\mathcal{L}_p^{(t')}[r; dr/dt]/\partial(dr/dt) = \\ = m(dr/dt - dr_f/dt), \tag{3.17}$$

with the dash signs dropped. By "∇," we denote the usual gradient operator in $\mathbb{E}^3$. Equating the canonical momentum expression (3.17) with respect to the reference frame $\mathcal{K}_{t'}$ to that of (3.6) with respect to the canonical reference frame $\tilde{\mathcal{K}}_{t'}$, and identifying the reference frame $\tilde{\mathcal{K}}_{t'}$ with $\mathcal{K}_{t'}$, we obtain

$$m(dr/dt - dr_f/dt) = m\,dr/dt - \xi A(r), \tag{3.18}$$

giving rise to the important expression determining the inertial particle mass

$$m \ = -\xi\varphi(r), \tag{3.19}$$

which follows directly from the relation

$$\varphi(r)dr_f/dt = A(r). \tag{3.20}$$

The latter is well known in the classical electromagnetic theory [45, 51] for potentials $(\varphi, A) \in T^*(M^4)$ satisfying the Lorentz condition

$$\partial\varphi(r)/\partial t + \langle\nabla, A(r)\rangle = 0. \tag{3.21}$$

Expression (3.19) looks very nontrivial, by relating the "*inertial*" mass of the charged point particle ξ to the electric potential, being both generated by the ambient charged point particles ξ_f. As was argued in [12, 13, 68], the above mass phenomenon is closely related to the classical electromagnetic mass problem from a physical perspective.

Before the further analysis of the motion of a completely relativistic charge ξ under consideration, we substitute the mass expression (3.19) into the quasirelativistic action functional (3.10) with Lagrangian (3.13). As a result, we obtain two possible action functional expressions with regard for two choices of main temporal parameters,

$$S_p^{(t')} = -\int_{t'_1}^{t'_2} \xi\varphi'(r)\left(1 + \frac{1}{2}\,|dr/dt' - dr_f/dt'|^2\right)dt', \tag{3.22}$$

on an interval $[t'_1, t'_2] \subset \mathbb{R}$ or

$$S_p^{(\tau)} = -\int_{\tau_1}^{\tau_2} \xi\varphi'(r)\left(1 + \frac{1}{2}\,|dr/d\tau - dr_f/d\tau|^2\right)d\tau \tag{3.23}$$

on an $[\tau_1, \tau_2] \subset \mathbb{R}$. The direct relativistic transformations of (3.23) entail that

$$S_p^{(\tau)} = -\int_{\tau_1}^{\tau_2} \xi\varphi'(r)\left(1 + \frac{1}{2}\,|dr/d\tau - dr_f/d\tau|^2\right)d\tau \simeq$$

$$\simeq -\int_{\tau_1}^{\tau_2} \xi\varphi'(r)\left(1+|dr/d\tau - dr_f/d\tau|^2\right)^{1/2} d\tau =$$
$$= -\int_{\tau_1}^{\tau_2} \xi\varphi'(r)\left(1-|dr/dt' - dr_f/dt'|\right)^{-1/2} d\tau =$$
$$= -\int_{t_1'}^{t_2'} \xi\varphi'(r)dt', \qquad (3.24)$$

giving rise to the correct, from the physical point of view, relativistic action functional (3.10) suitably transformed to the proper time reference frame representation (3.11) via the Feynman proper time paradigm. Thus, we have shown that the true action functional procedure consists in a physically motivated choice of either the action functional expression (3.9) or (3.10). Then it is transformed to the proper time action functional representation form (3.11) within the Feynman paradigm, and the least action principle is applied.

Concerning the above-discussed problem of describing the motion of a charged point particle ξ in the electromagnetic field generated by another moving charged point particle ξ_f, it must be mentioned that we have chosen the quasirelativistic functional expression (3.13) in form (3.10) with respect to the moving reference frame $\mathcal{K}_{t'}$, because its form is physically reasonable and acceptable, since the charged point particle ξ_f is then at rest, by generating no magnetic field.

Based on the above relativistic action functional

$$S_p^{(\tau)} := -\int_{\tau_1}^{\tau_2} \xi\varphi'(r)\left(1+|dr/d\tau - dr_f/d\tau|^2\right)^{1/2} d\tau \quad (3.25)$$

written with respect to the proper reference frame $\mathcal{K}_\tau$, one finds the evolution equation

$$d\pi_p/d\tau = -\xi\nabla\varphi'(r)\left(1+|dr/d\tau - dr_f/d\tau|^2\right)^{1/2}, \quad (3.26)$$

where the generalized momentum is given exactly by relation (3.17):

$$\pi_p = m(dr/dt - dr_f/dt). \qquad (3.27)$$

Making use of the relativistic transformation (3.14) and the next one (3.15), Eq. (3.26) can be easily transformed to

$$\frac{d}{dt}(p+\xi A) = -\nabla\varphi(r)\left(1-|u_f|^2\right), \qquad (3.28)$$

where we considered the related definitions: (3.19) for the charged particle ξ mass, (3.20) for the magnetic vector potential, and $\varphi(r) = \varphi'(r)/\left(1-|u_f|^2\right)^{1/2}$ for the scalar electric potential with respect to the laboratory reference frame $\mathcal{K}_t$. Equation (3.28) can be further transformed, by using the elementary vector algebra, to the classical Lorentz-type form

$$dp/dt = \xi E + \xi u \times B - \xi\nabla\langle u - u_f, A\rangle, \qquad (3.29)$$

where $E := -\partial A/\partial t - \nabla\varphi$ is the related electric field, and $B := \nabla\times A$ is the related magnetic field acting by the moving charged point particle ξ_f on the charged point particle ξ with respect to the laboratory reference frame $\mathcal{K}_t$. The above-presented result, as it was demonstrated in Section 2, follows also in part [73, 74] from Ampere's classical works on constructing the magnetic force between two neutral conductors with stationary currents.

Recall now that the dynamical pair of the Maxwell equations (1.8) reads as

$$\nabla\times B = \partial E/\partial t + J, \quad \nabla\times E = \partial B/\partial t. \qquad (3.30)$$

It is worth to mention now that the system of equations (3.30) can be represented by means of the least action principle $\delta S_{f-p}^{(t)} = 0$, where the action functional

$$S_{f-p}^{(t)} := \int_{t_1}^{t_2} dt\mathcal{L}_{f-p}^{(t)} \qquad (3.31)$$

is defined on an interval $[t_1, t_2] \subset \mathbb{R}$ by the Landau–Lifshitz-type [51] Lagrangian function

$$\mathcal{L}_{f-p}^{(t)} = \int_{\mathbb{R}^3} d^3r\left((|E|^2 - |B|^2)/2 + \langle J, A\rangle - \rho\varphi\right) \quad (3.32)$$

with respect to the laboratory reference frame $\mathcal{K}_t$, which is unique and physically reasonable. From (3.32), we deduce that the generalized field momentum

$$\pi_f := \partial\mathcal{L}_{f-p}^{(t)}/\partial(\partial A/\partial t) = -E, \qquad (3.33)$$

and its evolution is given as

$$\partial\pi_f/\partial t := \delta\mathcal{L}_{f-p}^{(t)}/\delta A = J - \nabla\times B, \qquad (3.34)$$

which is equivalent to the first Maxwell equation of (3.30). As the Maxwell equations allow the least action representation, it is easy to derive [2, 6, 9, 13, 68] their dual Hamiltonian formulation with the Hamiltonian function

$$H_{f-p} := \int_{\mathbb{R}^3} d^3r \langle \pi_f, \partial A/\partial t\rangle - \mathcal{L}^{(t)}_{f-p} = \int_{\mathbb{R}^3} d^3r \left((|E|^2 - |B|^2)/2 - \langle J, A\rangle\right), \tag{3.35}$$

satisfying the invariant condition

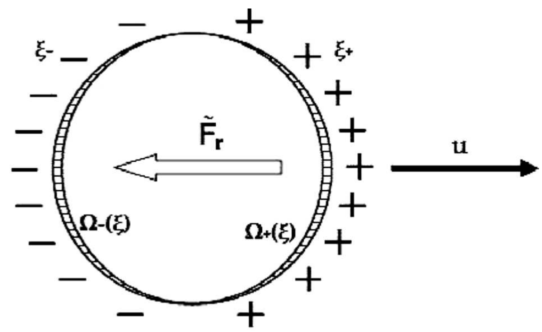
$$dH_{f-p}/dt = 0 \tag{3.36}$$

for all $t \in \mathbb{R}$.

It is worth noting here that the Maxwell equations were derived under the important condition that the charged system $(\rho, J) \in T(M^4)$ exerts no influence on the ambient electromagnetic field potentials $(\varphi, A) \in T^*(M^4)$. As this is not actually the case owing to the damping radiation reaction on accelerated charged particles, one can try to describe this self-interacting influence by means of the modified least action principle, making use of Lagrangian (3.32) recalculated with respect to the separately chosen charged particle ξ endowed with the uniform shell model geometric structure and generating this electromagnetic field.

Following the slightly modified well-known approach from [51] and reasonings from [8, 59], this Landau–Lifshitz type Lagrangian (3.32) can be recast (further in the Gauss units) as

$$\begin{aligned}
\mathcal{L}^{(t)}_{f-p} &= \int_{\mathbb{R}^3} d^3r((|E|^2 - |B|^2)/2 + \\
&+ \int_{\mathbb{R}^3} d^3r \left(\frac{1}{c}\langle J, A\rangle - \rho\varphi\right) - \langle k(t), dr/dt\rangle = \\
&= \int_{\mathbb{R}^3} d^3r \left(\frac{1}{2}\left\langle -\nabla\varphi - \frac{1}{c}\partial A/\partial t, -\nabla\varphi - \frac{1}{c}\partial A/\partial t\right\rangle - \right.\\
&\left. - \frac{1}{2}\langle \nabla \times (\nabla \times A), A\rangle\right) + \\
&+ \int_{\mathbb{R}^3} d^3r \left(\frac{1}{c}\langle J, A\rangle - \rho\varphi\right) - \langle k(t), dr/dt\rangle = \\
&= \int_{\mathbb{R}^3} d^3r \left(\frac{1}{2}\langle -\nabla\varphi, E\rangle - \frac{1}{2c}\langle \partial A/\partial t, E\rangle - \frac{1}{2}\langle A, \nabla \times B\rangle\right) + \\
&+ \int_{\mathbb{R}^3} \left(\frac{1}{c}\langle J, A\rangle - \rho\varphi\right) - \langle k(t), dr/dt\rangle = \\
&= \int_{\mathbb{R}^3} d^3r \left(\frac{1}{2}\varphi\langle \nabla, E\rangle + \frac{1}{2c}\langle A, \partial E/\partial t\rangle - \right.\\
&\left. - \frac{1}{2c}\langle A,\ J + \partial E/\partial t\rangle\right) + \int_{\mathbb{R}^3} \left(\frac{1}{c}\langle J, A\rangle - \rho\varphi\right) - \\
&- \frac{1}{2c}\frac{d}{dt}\int_{\mathbb{R}^3} d^3r\langle A, E\rangle - \frac{1}{2}\lim_{r\to\infty}\int_{\mathbb{S}^2_r} \langle \varphi E + A \times B, dS^2_r\rangle - \\
&- \langle k(t), dr/dt\rangle = -\frac{1}{2}\int_{\Omega_+(\xi)} d^3r \left(\frac{1}{c}\langle J, A\rangle - \rho\varphi\right) + \\
&+ \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} \left(\frac{1}{c}\langle J, A\rangle - \rho\varphi\right) - \langle k(t), dr/dt\rangle - \\
&- \frac{1}{2c}\frac{d}{dt}\int_{\mathbb{R}^3} d^3r\langle A, E\rangle - \frac{1}{2}\lim_{r\to\infty}\int_{\mathbb{S}^2_r} \langle \varphi E + A \times B, dS^2_r\rangle = \\
&= -\frac{1}{2}\int_{\Omega_+(\xi)} d^3r \left(\frac{1}{c}\langle J, A\rangle - \rho\varphi\right) - \frac{1}{2}\int_{\Omega_-(\xi)} d^3r \times \\
&\times \left(\frac{1}{c}\langle J, A\rangle - \rho\varphi\right) + \frac{1}{2}\int_{\Omega_-(\xi)} d^3r \left(\frac{1}{c}\langle J, A\rangle - \rho\varphi\right) + \\
&+ \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} \left(\frac{1}{c}\langle J, A\rangle - \rho\varphi\right) - \langle k(t), dr/dt\rangle - \\
&- \frac{1}{2c}\frac{d}{dt}\int_{\mathbb{R}^3} d^3r\langle A, E\rangle - \frac{1}{2}\lim_{r\to\infty}\int_{\mathbb{S}^2_r} \langle \varphi E + A \times B, dS^2_r\rangle = \\
&= \frac{1}{2}\int_{\Omega_-(\xi)} d^3r \left(\frac{1}{c}\langle J, A\rangle - \rho\varphi\right) - \\
&- \frac{1}{2}\int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r \left(\frac{1}{c}\langle J, A\rangle - \rho\varphi\right) + \\
&+ \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} \left(\frac{1}{c}\langle J, A\rangle - \rho\varphi\right) - \langle k(t), dr/dt\rangle - \\
&- \frac{1}{2c}\frac{d}{dt}\int_{\mathbb{R}^3} d^3r\langle A, E\rangle - \frac{1}{2}\lim_{r\to\infty}\int_{\mathbb{S}^2_r} \langle \varphi E + A \times B, dS^2_r\rangle = \\
&= \frac{1}{2}\int_{\Omega_-(\xi)} d^3r \left(\frac{1}{c}\langle J, A\rangle - \rho\varphi\right) + \\
&+ \frac{1}{2}\int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r \left(\frac{1}{c}\langle J, A\rangle - \rho\varphi\right) - \\
&- \frac{1}{2c}\frac{d}{dt}\int_{\mathbb{R}^3} d^3r\langle A, E\rangle - \frac{1}{2}\lim_{r\to\infty}\int_{\mathbb{S}^2_r} \langle \varphi E + A \times B, dS^2_r\rangle,
\end{aligned} \tag{3.37}$$



The courtesy picture from [59]

where we have introduced a radiation damping momentum $k(t) \in \mathbb{E}^3$ yet not determined, have denoted, by $\Omega_+(\xi) := \operatorname{supp} \xi_+ \subset \mathbb{R}^3$ and $\Omega_-(\xi) := \operatorname{supp} \xi_- \subset \mathbb{R}^3$, the corresponding charge ξ supports located on the electromagnetic field shadowed rear and electromagnetic field exerted front semispheres (see Figure) of the electron shell, respectively to its motion with a fixed velocity $u(t) \in \mathbb{E}^3$, as well as we denoted, by $\mathbb{S}^2_r$, a two-dimensional sphere of radius $r \to \infty$.

Having naturally assumed that the radiated charged particle energy is negligible at infinity, Lagrangian (3.37) becomes equivalent to

$$\mathcal{L}^{(t)}_{f-p} = \frac{1}{2} \int\limits_{\Omega_-(\xi)} d^3r \left(\frac{1}{c}\langle J, A\rangle - \rho\varphi\right) + \\ + \frac{1}{2c} \int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)} (\langle J, A\rangle - \rho\varphi) - \langle k(t), dr/dt\rangle, \quad (3.38)$$

which is needed to be additionally recalculated with regard for that the electromagnetic potentials $(\varphi, A) \in T^*(M^4)$ are retarded, generated by only the front part of the electron shell, and given, as $1/c^2 \to 0$, in the following form expanded in Lienard–Wiechert series:

$$\varphi = \int\limits_{\mathbb{R}^3} d^3r' \frac{\rho(t', r')}{|r - r'|}\bigg|_{t'=t-|r-r'|/c} = \\ = \lim_{\varepsilon\downarrow 0} \int\limits_{\mathbb{R}^3} d^3r' \frac{\rho(t-\varepsilon, r')}{|r - r'|} + \\ + \lim_{\varepsilon\downarrow 0} \frac{1}{2c^2} \int\limits_{\mathbb{R}^3} d^3r' |r - r'| \partial^2\rho(t-\varepsilon, r')/\partial t^2 + \\ + \lim_{\varepsilon\downarrow 0} \frac{1}{6c^3} \int\limits_{\mathbb{R}^3} d^3r' |r-r'|^2 \partial\rho(t-\varepsilon, r')/\partial t + O(1/c^4) = \\ = \int\limits_{\Omega_+(\xi)} d^3r' \frac{\rho(t, r')}{|r-r'|} + \frac{1}{2c^2} \int\limits_{\Omega_+(\xi)} d^3r' |r-r'| \partial^2\rho(t, r')/\partial t^2 + \\ + \frac{1}{6c^3} \int\limits_{\Omega_+(\xi)} d^3r' |r - r'|^2 \partial\rho(t, r')/\partial t + O(1/c^4), \quad (3.39)$$

$$A = \frac{1}{c} \int\limits_{\mathbb{R}^3} d^3r' \frac{J(t', r')}{|r - r'|}\bigg|_{t'=t-|r-r'|/c} = \\ = \lim_{\varepsilon\downarrow 0} \frac{1}{c} \int\limits_{\mathbb{R}^3} d^3r' \frac{J(t-\varepsilon, r')}{|r - r'|} - \\ - \lim_{\varepsilon\downarrow 0} \frac{1}{c^2} \int\limits_{\mathbb{R}^3} d^3r' \partial J(t-\varepsilon, r')/\partial t + \\ + \lim_{\varepsilon\downarrow 0} \frac{1}{2c^3} \int\limits_{\mathbb{R}^3} d^3r' |r - r'| \partial^2 J(t-\varepsilon, r')/\partial t^2 + O(1/c^4) = \\ = \frac{1}{c} \int\limits_{\Omega_+(\xi)} d^3r' \frac{J(t, r')}{|r - r'|} - \frac{1}{c^2} \int\limits_{\Omega_+(\xi)} d^3r' \partial J(t, r')/\partial t + \\ + \frac{1}{2c^3} \int\limits_{\Omega_+(\xi)} d^3r' |r - r'| \partial^2 J(t, r')/\partial t^2 + O(1/c^4),$$

where the current density $J(t, r) = \rho(t, r) dr/dt$ for all $t \in \mathbb{R}$ and $r \in \Omega(\xi) := \Omega_+(\xi) \cup \Omega_+(\xi) \simeq \simeq \mathbb{S}^2 := \operatorname{supp} \rho(t; r) \subset \mathbb{R}^3$, being the spherical compact support of the charged particle density distribution, and the limit $\lim_{\varepsilon\downarrow 0}$ *was treated physically, i.e., by considering the assumed shell model of the charged particle* ξ *and its corresponding charge density self-interaction.* Moreover, potentials (3.39) are both considered to be retarded, nonsingular, and moving in space with the velocity $u \in T(\mathbb{R}^3)$ relative to the laboratory reference frame $\mathcal{K}_t$. As a result of simple enough calculations like those in [45], making use of expressions (3.39), we obtain that Lagranfian (3.38) brings about

$$\mathcal{L}^{(t)}_{f-p} = \frac{\mathcal{E}_{\mathrm{es}}}{2c^2}|u|^2 - \langle k(t), dr/dt\rangle, \quad (3.40)$$

where we took into account that, owing to the reasonings from [8,59], the only the front half of the electric charge interacts with the whole virtually identical charge ξ, as well as made use of the following limiting integral expressions up to $O(1/c^4)$:

$$\int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r \int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r'\rho(t,r')\rho(t,r') := \xi^2,$$

$$\frac{1}{2}\int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r \int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r' \frac{\rho(t,r')\rho(t,r')}{|r-r'|} := \mathcal{E}_{\rm es},$$

$$\int\limits_{\Omega_+(\xi)} d^3r\rho(t,r) \int\limits_{\Omega_+(\xi)} d^3r' \frac{\rho(t;r')}{|r'-r|} = \frac{1}{2}\mathcal{E}_{\rm es},$$

$$\int\limits_{\Omega_-(\xi)} d^3r\rho(t,r) \int\limits_{\Omega_-(\xi)} d^3r' \frac{\rho(t;r')}{|r'-r|} = \frac{1}{2}\mathcal{E}_{\rm es}, \qquad (3.41)$$

$$\int\limits_{\Omega_-(\xi)} d^3r\rho(t,r) \int\limits_{\Omega_+(\xi)} d^3r' \frac{\rho(t;r')}{|r-r'|}\left|\frac{\langle r'-r,u\rangle}{|r'-r|}\right|^2\Bigg\rangle := \frac{\mathcal{E}_{\rm es}}{6}|u|^2,$$

$$\int\limits_{\Omega_+(\xi)} d^3r\rho(t,r) \int\limits_{\Omega_+(\xi)} d^3r' \frac{\rho(t;r')}{|r-r'|}\left|\frac{\langle r'-r,u\rangle}{|r'-r|}\right|^2\Bigg\rangle := \frac{\mathcal{E}_{\rm es}}{6}|u|^2.$$

To obtain the corresponding evolution equation for our charged particle ξ, we need, within the Feynman proper time paradigm, to transform Lagrangian (3.40) to one with respect to the proper time reference frame $\mathcal{K}_\tau$:

$$\mathcal{L}^{(\tau)}_{f-p} = (m_{\rm es}/2)|\dot r|^2 - \langle k(t), \dot r\rangle, \qquad (3.42)$$

where, for brevity, we have denoted, by $\dot r := dr/d\tau$, the charged particle velocity with respect to the proper reference frame $\mathcal{K}_\tau$ and, by definition, its so-called electrostatic mass $m_{\rm es} := \mathcal{E}_{\rm es}/c^2$ with respect to the laboratory refrence frame $\mathcal{K}_t$.

Thus, the generalized charged particle ξ momentum (up to $O(1/c^4)$) equals

$$\pi_p = \partial\mathcal{L}^{(\tau)}_{f-p}/\partial\dot r = \bar m_{\rm es}\dot r - k(t) = m_{\rm es}u - k(t), \qquad (3.43)$$

where we denoted, as before, the charged particle ξ velocity with respect to the laboratory reference frame $\mathcal{K}_t$ by $u := dr/dt$ and put, by definition,

$$\bar m_{\rm es} := m_{\rm es}(1-|u|^2)^{1/2} \qquad (3.44)$$

as its mass parameter $\bar m_{\rm es} \in \mathbb{R}_+$ with respect to the proper reference frame $\mathcal{K}_\tau$.

With respect to the proper reference frame $\mathcal{K}_\tau$, the generalized momentum (3.44) satisfies the evolution equation

$$d\pi_p/d\tau := \partial\mathcal{L}^{(\tau)}_{f-p}/\partial r = 0, \qquad (3.45)$$

being equivalent, with respect to the laboratory reference frame $\mathcal{K}_t$, to the Lorentz-type equation

$$\frac{d}{dt}(m_{\rm es}u) = dk(t)/dt. \qquad (3.46)$$

The evolution equation (3.46) allows one to formulate the corresponding canonical Hamiltonian on the phase space $T^*(\mathbb{R}^3)$ with the Hamiltonian function

$$H_{f-p} := \langle\pi_p, \dot r\rangle - \mathcal{L}^{(\tau)}_{f-p} = \langle m_{\rm es}u - k(t), \dot r\rangle - $$
$$- (\bar m_{\rm es}/2)|\dot r|^2 - \langle k(t), \dot r\rangle + \langle k(t), \dot r\rangle =$$
$$= \frac{m_{\rm es}|u|^2}{2}\left(1 + \frac{1}{2}|u|^2/c^2\right), \qquad (3.47)$$

naturally looking and satisfying, up to $O(1/c^4)$ for all τ and $t \in \mathbb{R}$, the conservation conditions

$$\frac{d}{d\tau}H_{f-p} = 0 = \frac{d}{dt}H_{f-p}. \qquad (3.48)$$

Looking at Eqs. (3.46) and (3.47), we can state that the mass parameter of a physically observable inertial charged particle ξ is as follows:

$$m_{\rm phys} := \bar m_{\rm es}, \qquad (3.49)$$

being exactly equal to the electromagnetic mass of a relativistic charged particle ξ, as it was assumed by H. Lorentz and M. Abraham.

To determine the damping radiation momentum $k(t) \in \mathbb{E}^3$, we can make use of the Lorentz-type force expression (3.38) and obtain, similarly to [45], the resulting self-interecting Abraham–Lorentz-type force expression up to $O(1/c^4)$ accuracy. Thus, owing to the zero net force condition, we have that

$$d\tilde\pi_p/dt + F_s = 0, \qquad (3.50)$$

where the Lorentz force

$$F_s = -\frac{1}{2c}\int\limits_{\Omega_-(\xi)} d^3r\rho(t,r)\frac{d}{dt}A(t,r) -$$
$$-\frac{1}{2c}\int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r\rho(t,r)\frac{d}{dt}A(t,r) -$$
$$-\frac{1}{2}\int\limits_{\Omega_-(\xi)} d^3r\rho(t,r)\nabla\varphi(t,r)(1-|u/c|^2) -$$
$$-\frac{1}{2}\int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r\rho(t,r)\nabla\varphi(t,r)(1-|u/c|^2). \qquad (3.51)$$

This expression easily follows from the least action condition $\delta S^{(t)}=0$, where $S^{(t)}:=\int_{t_1}^{t_2}\mathcal{L}_{f-p}^{(t)}dt$ with the Lagrangian function given by the above-derived Landau–Lifshitz-type expression (3.41) and with the potentials $(\varphi, A)\in T^*(M^4)$ given by the Lienard–Wiechert expressions (3.39). Followed by calculations similar to those in [19, 45], we can obtain, within the assumed before uniform shell electron model in view of (3.51) and (3.39) for small $|u/c|\ll 1$ and slow enough acceleration, that

$$F_s=\sum_{n\in\mathbb{Z}_+}\frac{(-1)^{n+1}}{2n!c^n}(1-|u/c|^2)\times$$
$$\times\left[\int\limits_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)+\int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\right]\times$$
$$\times\int\limits_{\Omega_+(\xi)}d^3r'\frac{\partial^n}{\partial t^n}\rho(t,r')\nabla|r-r'|^{n-1}+\sum_{n\in\mathbb{Z}_+}\frac{(-1)^{n+1}}{2n!c^{n+2}}\times$$
$$\times\left[\int\limits_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)+\int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\right]\times$$
$$\times\int\limits_{\Omega_+(\xi)}d^3r'|r-r'|^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}J(t,r']=$$
$$=\sum_{n\in\mathbb{Z}_+}\frac{(-1)^{n+1}}{2n!c^{n+2}}(1-|u/c|^2)\times$$
$$\times\left[\int\limits_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)+\int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\right]\times$$
$$\times\int\limits_{\Omega_+(\xi)}d^3r'\frac{\partial^{n=2}}{\partial t^{n+2}}\rho(t,r')\nabla|r-r'|^{n+1}+\sum_{n\in\mathbb{Z}_+}\frac{(-1)^{n+1}}{2n!c^{n+2}}\times$$
$$\times\left[\int\limits_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)+\int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\right]\times$$
$$\times\int\limits_{\Omega_+(\xi)}d^3r'|r-r'|^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}J(t,r'). \qquad (3.52)$$

Owing to the charge continuity equation (1.6), the relation above gives rise to the radiation force expression

$$F_s=\sum_{n\in\mathbb{Z}_+}\frac{(-1)^n}{2n!c^{n+2}}(1-|u/c|^2)\times$$
$$\times\left[\int\limits_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)+\int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\right]\times$$
$$\times\int\limits_{\Omega_+(\xi)}d^3r'|r-r'|^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}\times$$
$$\times\left(\frac{J(t,r')}{n+2}+\frac{n-1}{n+2}\frac{\langle r-r',J(t,r')\rangle(r-r')}{|r-r'|^2}\right)+$$
$$+\sum_{n\in\mathbb{Z}_+}\frac{(-1)^{n+1}}{2n!c^{n+2}}\times$$
$$\times\left[\int\limits_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)+\int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\right]\times$$
$$\times\int\limits_{\Omega_+(\xi)}d^3r'|r-r'|^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}J(t,r')=$$
$$=\sum_{n\in\mathbb{Z}_+}\frac{(-1)^{n+1}}{2n!c^{n+2}}(1-|u/c|^2)\times$$
$$\times\left[\int\limits_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)+\int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\right]\times$$
$$\times\int\limits_{\Omega_+(\xi)}d^3r'|r-r'|^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}\times$$
$$\times\left(\frac{J(t,r')}{n+2}+\frac{n-1}{n+2}\frac{|r-r',u|^2J(t,r')}{|r-r'|^2|u|^2}\right)+$$
$$+\sum_{n\in\mathbb{Z}_+}\frac{(-1)^{n+1}}{2n!c^{n+2}}\times$$
$$\times\left[\int\limits_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)+\int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\right]\times$$
$$\times\int\limits_{\Omega_+(\xi)}d^3r'|r-r'|^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}J(t,r'). \qquad (3.53)$$

Now, having applied the rotational symmetry property to (3.53) for the calculation of the internal integrals, we easily obtain within the uniform shell model of a charged particle ξ that

$$F_s=\sum_{n\in\mathbb{Z}_+}\frac{(-1)^n}{2n!c^{n+2}}(1-|u/c|^2)\times$$

$$\times\left[\int\limits_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)+\int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\right]\times$$
$$\times\int\limits_{\Omega_+(\xi)}d^3r'|r-r'|^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}\times$$
$$\times\left(\frac{J(t,r')}{n+2}+\frac{(n-1)J(t,r')}{3(n+2)}\right)+\sum_{n\in\mathbb{Z}_+}\frac{(-1)^{n+1}}{2n!c^n}\times$$
$$\times\left[\int\limits_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)+\int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\right]\times$$
$$\times\int\limits_{\Omega_+(\xi)}d^3r'\frac{|r-r'|^{n+1}}{c^2}\frac{\partial^{n+1}}{\partial t^{n+1}}J(t,r')=$$
$$=\frac{d}{dt}\left[\sum_{n\in\mathbb{Z}_+}\frac{(-1)^{n+1}}{6n!c^{n+2}}\times\right.$$
$$\times\left[\int\limits_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)+\int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\right]\times$$
$$\times\int\limits_{\Omega_+(\xi)}d^3r'|r-r'|^{n-1}\frac{\partial^n}{\partial t^n}J(t,r')-\sum_{n\in\mathbb{Z}_+}\frac{(-1)^n|u|^2}{6n!c^{n+4}}\times$$
$$\times\left[\int\limits_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)+\int\limits_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\right]\times$$
$$\left.\times\int\limits_{\Omega_+(\xi)}d^3r'|r-r'|^{n-1}\frac{\partial^n}{\partial t^n}J(t,r')\right]. \tag{3.54}$$

Now, with regard for the integral expressions (3.41), we find from (3.54) that, up to the $O(1/c^4)$ accuracy, the following expression for a radiation reaction force holds:

$$F_s=-\frac{d}{dt}\left(\frac{\mathcal{E}_{\rm es}}{c^2}u\right)+\frac{d}{dt}\left(\frac{\mathcal{E}_{\rm es}}{2c^2}\,|u/c|^2u(t)\right)+\frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2}+$$
$$+O(1/c^4)=-\frac{d}{dt}\,(m_{\rm es}u)+\frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2}+O(1/c^4)=$$
$$=-\frac{d}{dt}\,(m_{\rm es}u)+\frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2}+O(1/c^4)=$$
$$=-\frac{d}{dt}\left(m_{\rm es}u-\frac{2\xi^2}{3c^3}\frac{du}{dt}\right)+O(1/c^4). \tag{3.55}$$

We mention that, following the reasonings from [8, 59, 70], the above expressions involve the additional hidden electrostatic Coulomb surface self-force directed along the velocity $u\in T(\mathbb{R}^3)$ and acting only on the *front half* of the spherical electron shell. As a result, from (3.50), (3.51) and relation (3.43), we obtain that the electron momentum

$$\pi_p:=m_{\rm es}u-\frac{2\xi^2}{3c^3}\frac{du}{dt}=m_{\rm es}u-k(t), \tag{3.56}$$

thereby defining both the radiation reaction momentum $k(t)=\frac{2\xi^2}{3c^3}\frac{du}{dt}$ and the corresponding radiation reaction force

$$F_r=\frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2}+O(1/c^4), \tag{3.57}$$

which coincides exactly with the classical Abraham–Lorentz–Dirac expression. Moreover, it also follows that the observable electron inertial mass within the physical shell model,

$$m_{\rm ph}=m_{\rm es}:=\mathcal{E}_{\rm es}/c^2, \tag{3.58}$$

being completely of the electromagnetic origin, gives rise to the final force expression

$$\frac{d}{dt}(m_{\rm ph}u)=\frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2}+O(1/c^4). \tag{3.59}$$

This means, in particular, that the real physically observed "inertial" mass $m_{\rm ph}$ of an electron within the uniform shell model is strongly determined by its electromagnetic self-interaction energy $\mathcal{E}_{\rm es}$. A similar statement was recently demonstrated, by using completely different approaches in [59, 70], based on the vacuum Casimir effect considerations. Moreover, the above-assumed boundedness of the electrostatic self-energy $\mathcal{E}_{\rm es}$ appears to be completely equivalent to the existence of the so-called intrinsic Poincaré-type "*tensions*" analyzed in [8, 59] and to the existence of a special compensating Coulomb "*pressure*" suggested in [70], guaranteeing the observable electron stability.

Some years ago, a "solution" to the above-mentioned "4/3-electron mass" problem expressed by the physical mass relation (3.58) and formulated more than one hundred years ago by H. Lorentz and M. Abraham was suggested in work [55]. Unfortunately, the above-mentioned "solution" appeared to be false, which can be easily observed from the main not correct assumptions, on which work [55] was based: the first one is about the particle-field momentum conservation taken in the form

$$\frac{d}{dt}(p+\xi A)=0, \tag{3.60}$$

and the second one is a speculation about the 1/2-coefficient imbedded into the calculation of the Lorentz-type self-interaction force

$$F := -\frac{1}{2c}\int_{\mathbb{R}^3} d^3r\rho(t;r)\partial A(t;r)/\partial t, \qquad (3.61)$$

being not correctly argued by the reasoning that expression (3.61) represents "... the interaction of a given element of charge with all other parts, otherwise we count twice that reciprocal action" (cited from [55], p. 2710). This claim is false, as there was not taken the important fact into account that the interaction in integral (3.61) is, in reality, retarded, and its impact into it should be considered as that calculated for two virtually different charged particles, as it has been done in the classical works of H. Lorentz and M. Abraham. As for the first assumption (3.60), it is enough to recall that the vector of the field momentum $\xi A \in \mathbb{E}^3$ is not independent and is, within the considered charged particle model, strongly related to the local flow of the electromagnetic potential energy in the Lorentz constraint form:

$$\partial\varphi/\partial t + \langle \nabla, A\ \rangle = 0, \qquad (3.62)$$

under which the Lienard–Wiechert expressions (3.38) for potentials exploited in work [55] for the calculation of integral (3.61) hold. Thus, Eq. (3.60) should be replaced, following the classical Newton second law, by

$$\frac{d}{dt'}(p' + \xi A') = -\nabla(\xi\varphi'), \qquad (3.63)$$

written with respect to the reference frame $\mathcal{K}'(t';r)$, relative to which the charged particle ξ is at rest. Taking into account that, with respect to the laboratory reference frame $\mathcal{K}_t$, the relativistic relations $dt = dt'(1 - |u|^2/c^2)^{1/2}$ and $\varphi' = \varphi(1 - |u|^2/c^2)^{1/2}$ hold, we easily obtain from (3.63) that

$$\frac{d}{dt}(p + \xi A) = -\xi\nabla\varphi(1 - |u|^2/c^2) =$$
$$= -\xi\nabla\varphi + \frac{\xi}{c}\nabla\langle u, u\varphi/c\rangle = -\xi\nabla\varphi + \frac{\xi}{c}\nabla\langle u, A\rangle. \quad (3.64)$$

Here, we made use of the well-known relation $A = u\varphi c$ for the vector potential generated by this charged particle ξ moving in space with the velocity $u \in T(\mathbb{R}^3)$ with respect to the laboratory reference frame $\mathcal{K}_t$. Based now on Eq. (3.64), we can derive the final expression for the evolution of the momentum of a charged particle ξ:

$$dp/dt = -\xi\nabla\varphi - \frac{\xi}{c}dA/dt + \frac{\xi}{c}\nabla\langle u, A\rangle =$$
$$= -\xi\nabla\varphi - \frac{\xi}{c}\partial A/\partial t - \frac{\xi}{c}\langle u, \nabla\rangle A + \frac{\xi}{c}\nabla\langle u, A\rangle =$$
$$= \xi E + \frac{\xi}{c}u \times (\nabla \times A) = \xi E + \frac{\xi}{c}u \times B, \qquad (3.65)$$

that is exactly the well-known Lorentz force expression, used in the works of H. Lorentz and M. Abraham.

Recently enough, there appeared other interesting works devoted to this "4/3-electron mass" and related problems, amongst which we would like to mention [40–42, 59, 70], whose argumentations are close to each other and based on the charged shell electron model, within which a virtual stochastic electrodynamic interaction of the electron with the ambient "dark" radiation energy is assumed. The latter was first clearly demonstrated in [70], where a suitable compensation mechanism of the related singular electrostatic Coulomb electron energy and the deficiency of wide-band vacuum electromagnetic radiation energy fluctuations inside the electron shell were shown to be harmonically realized as the electron shell radius $a \to 0$. Moreover, this compensation happens exactly, when the induced outward directed electrostatic Coulomb pressure on the whole electron coincides, up to the sign, with that induced by the above-mentioned vacuum electromagnetic energy fluctuations outside the electron shell, since there was manifested their absence inside the electron shell.

Really, the outward directed electrostatic spatial Coulomb pressure on the electron equals

$$\eta_{\text{coul}} := \lim_{a\to 0} \frac{\varepsilon_0|E|^2}{2}\bigg|_{r=a} = \lim_{a\to 0}\frac{\xi^2}{32\varepsilon_0\pi^2 a^4}, \qquad (3.66)$$

where $E = \frac{\xi r}{4\pi\varepsilon_0|r|^3} \in \mathbb{E}^3$ is the electrostatic field at the point $r \in \mathbb{R}$, whereas the electron center is at the point $r = 0 \in \mathbb{R}$. The related inward directed vacuum electromagnetic fluctuations spatial pressure equals

$$\eta_{\text{vac}} := \lim_{\Omega\to\infty}\frac{1}{3}\int_0^\Omega d\mathcal{E}(\omega), \qquad (3.67)$$

where $d\mathcal{E}(\omega)$ is the density of electromagnetic energy fluctuations at a frequency $\omega \in \mathbb{R}_+$, and $\Omega \in \mathbb{R}_+$

is the corresponding electromagnetic frequency cut-off. Integral (3.67) can be calculated, if we consider the quantum statistical recipe [11, 27, 39] that

$$d\mathcal{E}(\omega) := \hbar\omega \frac{d^3 p(\omega)}{h^3}, \tag{3.68}$$

where the Plank constant $h := 2\pi\hbar$, and the electromagnetic wave momentum $p(\omega) \in \mathbb{E}^3$ satisfies the relativistic relation

$$|p(\omega)| = \hbar\omega/c. \tag{3.69}$$

Substituting (3.69) into (3.68), we obtain

$$d\mathcal{E}(\omega) = \frac{\hbar\omega^3}{2\pi^2 c^3} d\omega, \tag{3.70}$$

which entails, owing to (3.67), the following spatial pressure of vacuum electromagnetic energy fluctuations

$$\eta_{\mathrm{vac}} = \lim_{\Omega\to\infty} \frac{\hbar\Omega^4}{24\pi^2 c^3}. \tag{3.71}$$

For the charged electron shell model to be stable at rest, it is necessary to equate the inward (3.71) and outward 3.66) spatial pressures:

$$\lim_{\Omega\to\infty} \frac{\hbar\Omega^4}{24\pi^2 c^3} = \lim_{a\to 0} \frac{\xi^2}{32\varepsilon_0 \pi^2 a^4}, \tag{3.72}$$

giving rise to the balance limiting condition, as the electron shell radius $a_b \to 0$:

$$a_b = \lim_{\Omega\to\infty} \left[\Omega^{-1} \left(\frac{3\xi^2 c^2}{2\hbar} \right)^{1/4} \right]. \tag{3.73}$$

Simultaneously, we can calculate the corresponding energy deficiency of Coulomb and electromagnetic fluctuations inside the electron shell:

$$\Delta W_b := \frac{1}{2}\int_{a_b}^{\infty} \varepsilon_0 |E|^2 d^3 r - \int_0^{a_b} d^3 r \int_0^{\Omega} d\mathcal{E}(\omega) = \\ = \frac{\xi^2}{8\pi\varepsilon_0 a_b} - \frac{\hbar\Omega^4 a_b^3}{6\pi c^3} = 0, \tag{3.74}$$

additionally ensuring the electron shell model stability.

Another important consequence from this pressure-energy compensation mechanism can be derived, by concerning the electron inertial mass $m_{\mathrm{ph}} \in \mathbb{R}_+$ entering the momentum expression (3.56) in the case of the electron slow enough movement. Namely, following the reasonings from [59], we can observe that, during the electron movement, there arises an additional hidden electrostatic Coulomb surface self-pressure, which is not compensated, directed along the velocity $u \in T(\mathbb{R}^3)$, acting only on the *front half* of the electron shell, and equal to

$$\eta_{\mathrm{surf}} := \frac{|E\xi|}{4\pi a_b^2}\frac{1}{2} = \frac{\xi^2}{32\pi\varepsilon_0 a_b^4}. \tag{3.75}$$

It coincides, evidently, with the already compensated outward directed electrostatic Coulomb spatial pressure (3.66). Since, during the electron motion in space, its surface electric current energy flow is not vanishing [59], one ensues that the electron momentum gains an additional mechanical impact, which can be expressed as

$$\pi_\xi := -\eta_{\mathrm{surf}} \frac{4\pi a_b^3}{3c^2} u = -\frac{1}{3}\frac{\xi^2}{8\pi\varepsilon_0 a_b c^2} u = -\frac{1}{3} m_{\mathrm{es}} u, \tag{3.76}$$

where we took into account that, within this electron shell model, the corresponding electrostatic electron mass equals its electrostatic energy

$$m_{\mathrm{es}} = \frac{\xi^2}{8\pi\varepsilon_0 a_b c^2}. \tag{3.77}$$

Thus, one can claim that, owing to the structural stability of the electron shell model, its generalized self-interaction momentum $\pi_p \in T^*(\mathbb{R}^3)$ gains, during the movement with a velocity $u = dr/dt \in T(\mathbb{R}^3)$, the additional backward directed hidden impact (3.76), which can be identified with the back-directed momentum component

$$\pi_\xi = -\frac{1}{3} m_{\mathrm{es}} u \tag{3.78}$$

complementing the classical [19, 45] momentum expression

$$\pi_p = \frac{4}{3} m_{\mathrm{es}} u, \tag{3.79}$$

which can be easily obtained from the Lagrangian, if one does not consider the shading property of the moving uniform shell electron model. Then, owing to the additional momentum (3.78), the full momentum becomes

$$\pi_p = \pi_\xi + \frac{4}{3} m_{\mathrm{es}} u = \left(-\frac{1}{3} m_{\mathrm{es}} + \frac{4}{3} m_{\mathrm{es}} \right) u = m_{\mathrm{es}} u, \tag{3.80}$$

which coincides in modulus with the radiation reaction momentum $k(t) = \frac{2\xi^2}{3c^3}\frac{du}{dt}$ (3.43), by strongly supporting the electromagnetic energy origin of the electron inertion mass conceived, for the first time, by H. Lorentz and M. Abraham.

4. Comments

The electromagnetic mass origin problem has been reanalyzed in detail within the Feynman proper time paradigm and related vacuum field theory approach by means of the fundamental least action principle and the Lagrangian and Hamiltonian formalisms. The resulting electron inertia appeared to coincide in part, in the quasirelativistic limit, with the momentum expression obtained more than one hundred years ago by M. Abraham and H. Lorentz [1, 52–54], and it is proved to contain an additional hidden impact owing to the imposed electron stability constraint, which was accounted for in the original action functional as some preliminarily undetermined constant component. As was demonstrated in [59,70], this stability constraint can be successfully realized within the charged shell model of electron at rest, if one consider the existing ambient electromagnetic "dark" energy fluctuations, whose inward directed spatial pressure on the electron shell is compensated by the related outward directed electrostatic Coulomb spatial pressure, as the electron shell radius satisfies some limiting compatibility condition. The latter also allows one to compensate simultaneously the deficiency of corresponding electromagnetic energy fluctuations inside the electron shell, thereby forbidding the external energy to flow into the electron. In contrary to the lack of an energy flow inside the electron shell, the corresponding internal momentum flow is not vanishing during the electron movement, owing to the nonvanishing hidden electron momentum flow caused by the surface pressure flow and compensated by the suitably generated surface electric current flow. As was shown, this backward directed hidden momentum flow makes it possible to justify the corresponding expression for the self-interaction electron mass and to state, within the electron shell model, the fully electromagnetic electron mass origin, as it has been conceived by H. Lorentz and M. Abraham and strongly supported by R. Feynman in his Lectures [28]. This consequence is also independently supported by means of the least action approach based on the Feynman proper time paradigm and the suitably calculated regularized retarded electric potential impact into the charged particle Lagrangian function.

The charged particle radiation problem revisited above allows us to conceive the explanation of the charged particle mass as that of a compact stable object, which should be exerted by a vacuum field self-interaction energy. The latter can be satisfied iff expressions (3.41) hold, thereby imposing some nontrivial geometrical constraints on the intrinsic charged particle structure [57]. Moreover, as follows from the physically observed particle mass expressions (3.58), the electrostatic potential energy, being of the self-interaction origin, contributes into the inertial mass as its main relativistic mass component.

There exist different relativistic generalizations of the force expression (3.59), which suffer from the common physical inconsistency related to the no radiation effect of a charged particle in a uniform motion.

Another problem profoundly related to the radiation reaction force analyzed above is the search for an explanation to the Wheeler–Feynman reaction radiation mechanism, which is called the absorption radiation theory and strongly based on the Mach-type interaction of a charged particle with the ambient vacuum electromagnetic medium. Concerning this problem, one can also observe some of its relationships with the one devised here within the vacuum field theory approach, but this question needs a more detailed and extended analysis.

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1. M. Abraham, Nachr. von der Gesellsch. der Wissensch. zu Göttingen. Math.-Phys. Kl., S. 20 (1902).
2. R. Abraham, and J.E. Marsden, *Foundations of Mechanics* (Benjamin, New York, 1978).

3. Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).
4. Y. Aharonov and D. Rohrlich, *Quantum Paradoxes: Quantum Theory for the Perplexed* (VCH, Weinheim, 2005).
5. A. Annila, Int. J. of Theor. Math. Phys. **2**(4), 67 (2012).
6. V.I. Arnold, *Mathematical Methods of Classical Mechanics.* (Springer, New York, 1978).
7. T.W. Barrett, in: A. Lakhtakia (Ed.), *Essays on the Formal Aspects of Maxwell Theory* (World Scientific, Singapore, 1993), p. 8.
8. R. Becker, *Theorie der Elektrizitat. Bd. II, Elektronentheorie* (Teubner, Berlin, 1933).
9. D. Blackmore, A.K. Prykarpatsky, and V.Hr. Samoylenko, *Nonlinear Dynamical Systems of Mathematical Physics: Spectral and Differential-Geometrical Integrability Analysis* (World Scientific, Singapore, 2011).
10. D. Blackmore, A.K. Prykarpatski, and N.N. Bogolubov, (jr.), Univ. J. of Phys. and Appl. **1(2)**, 160 (2013).
11. N.N. Bogolubov and N.N. Bogolubov, (jr.) *Introduction into Quantum Statistical Mechanics* (World Scientific, Singapore, 1986).
12. N.N. Bogolubov (jr.) and A.K. Prykarpatsky, Int. J. Theor. Phys. **5**, 237 (2012).
13. N.N. Bogolubov (jr.) and A.K. Prykarpatsky, Found. Phys. **40**, 469 (2010).
14. N.N. Bogolubov (jr.), A.K. Prykarpatsky, U. Taneri, and Y.A. Prykarpatsky, J. Phys. A: Math. Theor. **42**, 165401 (2009).
15. T.H. Boyer, Phys. Rev. D **8**, 1679 (1973).
16. L. Brillouin, *Relativity Reexamined*, (Academic Press, New York, 1970).
17. O. Darrigol, *Les Équations de Maxwell: De McCullagh á Lorentz* (Belin, Paris, 2005).
18. S. Deser and R. Jackiw, Time travel? arXiv: hep-th/9206094 (1992).
19. B. Di Bartolo, *Classical Theory of Electromagnetism* (World Scientific, Singapore, 2004).
20. P.A.M. Dirac, Proc. R. Soc. Lond. Ser. A **167**, 148 (1938).
21. G. Dunner and R. Jackiw, "Peierles substitution" and Chern-Simons quantum mechanics, arXiv: hep-th/92004057 (1992).
22. F.J. Dyson, Am. J. Phys. **58**, 209 (1990).
23. F.J. Dyson, Phys. Today, **42**(2), 32 (1989).
24. A. Einstein, Jahrbuch Radioaktivitat, V 422 (1907).
25. Hans G. Dehmelt in his Nobel Lecture, December 8, (1989), quoted Einstein's own words: "You know, it would be sufficient to really understand the electron".
26. E. Fermi, Nuovo Cim. **25**, 159 (1923).
27. R.P. Feynman, *Statistical Mechanics* (Benjamin, New York, 1998).
28. R.P. Feynman, R.B. Leighton, and M. Sands, *The Feynman Lectures on Physics. Electrodynamics* (Addison-Wesley, Reading, MA, 1964), v. 2.
29. R.P. Feynman, R.B. Leighton, and M. Sands, *The Feynman lectures on physics. The modern science on the Nature. Mechanics. Space, Time, Motion* (Addison-Wesley, Reading, MA, 1963), v. 1.
30. T.L. Gill and W.W. Zachary, *Two mathematically equivalent versions of Maxwell equations*, Preprint (Univ. of Maryland, 2008).
31. T.L. Gill and W.W. Zachary, Found. Phys. **4**, 99 (2011).
32. T.L. Gill, W.W. Zachary, and J. Lindsey, Found. Phys. **31**, 1299 (2001).
33. R.J. Glauber, *Quantum Theory of Optical Coherence. Selected Papers and Lectures* (Wiley-VCH, Weinheim, 2007).
34. C. Godbillon, *Geometrie Differentielle et Mecanique Analytique* (Hermann, Paris, 1969).
35. R.T. Hammond, Found. Phys. **43**, 201 (2013).
36. R.T. Hammond, Electr. J. of Theor. Phys., No. 23, 221 (2010).
37. P. Higgs, Phys. Rev. Lett. **13**, 508 (1964); Phys. Rev. **145**, 1156 (1964).
38. G. t'Hooft, Nucl. Phys. B **35**, 167 (1971).
39. K. Huang, *Statistical Mechanics* (Wiley, New York, 1987).
40. M. Ibison, *Found. of Phys. Lett.* **16**, No. 1, 83 (2003).
41. M. Ibison, Massless electrodynamics, arXiv: physics/0106046v5 [physics.gen-ph] 4 Feb. (2004).
42. M. Ibison, J. Phys. A: Math. Gen. **34**, 3421 (2001).
43. R. Jackiw, Lorentz violation in a diffeomorphism-invariant theory, arXiv: hep-th/0709.2348 (2007).
44. R. Jackiw and A.P. Polychronakos, Dynamical Poincaré symmetry realized by field-dependent diffeomorphisms, arXiv: hep-th/9809123 (1998).
45. J.D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1999).
46. M. Jammer, *Concepts of Mass in Contemporary Physics and Philosophy* (Princeton Univ. Press, Princeton, 2009).
47. A. Kastler, Rev. Hist. Sei. **72**, 1193 (1977).
48. B.P. Kosyakov, Sov. Phys. Usp. **35**(2), 135 (1992).
49. B.P. Kosyakov, *Introduction to the Classical theory of Particles and Fields* (Springer, Berlin, 2007).
50. B.A. Kupershmidt, Diff. Geom. & Appl. **2**, 275 (1992).
51. L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1983), v. 2.
52. H.A. Lorentz, *Proceed. of the Royal Nether. Acad. of Arts and Sci.* **6**, 809 (1904).
53. H.A. Lorentz, Arch. néerland. des Sci. exact. et natur. **25**, 363 (1892); *The Theory of Electrons: and Its Applications to the Phenomena of Light and Radiant Heat* (Teubner, Leipzig, 1916).
54. H.A. Lorentz, *The Theory of Electrons: and Its Applications to the Phenomena of Light and Radiant Heat* (Dover, New York, 2011).
55. A.A. Martins and M.J. Pinheiro, Int. J. Theor. Phys. **47**, 2706 (2008).
56. A.J. Chorin and J.E. Marsden, *A Mathematical Introduction to Liquid Mechanics* (Springer, New York, 1993).
57. R. Medina, J. Phys. A: Math. Gen. 3801 (2006).
58. H. Minkowski, Physik. Z. **10**, 104 (1909).
59. V.B. Morozov, Physics Uspekhi **181**(4), 389 (2011).
60. G.V. Nikolaev, *Modern Electrodynamics and Reasons for Its Paradoxicity. Prospects of Construction of Not In-*

consistent Electrodynamics. Theories, Experiments, Paradoxes (Tomsk Polytech. Univ., Tomsk, 2003) (in Russian).
61. L. Page and N.I. Adams (jr.), Amer. J. Phys. **13**, 141 (1945).
62. P.T. Pappas, Il Nuovo Cimento B **76**(2), 189 (1983).
63. W. Pauli, *Theory of Relativity* (Dover, New York, 1981).
64. D.T. Pegg, Rep. Prog. Phys. **38**, 1339 (1975).
65. Y. Pierseaux and G. Rousseaux, Les structures fines de l'electromagnetisme classique et de la relativite restreinte, arXiv: physics/0601023v1, 5 Jan. (2006).
66. H.C.R. Poincaré, Acad. Sci. (Paris) **140**, 1504 (1905).
67. A.K. Prykarpatsky, Univ. J. of Phys. and Appl. **2**(8), 381 (2014), DOI: 10.13189/ujpa.2014.020804.
68. A.K. Prykarpatsky, N.N. Bogolubov (jr.), and U. Taneri, Int. J. Theor. Phys. **49**, 798 (2010).
69. A.K. Prykarpatsky, N.N. Bogolubov (jr.), and U. Taneri, Theor. Math. Phys. **160**, 1079 (2009); arXiv lanl: 0807.3691v.8 [gr-gc] 24.08.2008.
70. H.E. Puthoff, Int. J. Theor. Phys. **46**, 3005 (2007).
71. O. Repchenko, *Field Physics* (Moscow, Galeria Publ., 2005).
72. F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, 1965).
73. G. Rousseaux, The gauge non-invariance of classical electromagnetism, arXiv: physics/0506203v1, 28 Jun. (2005).
74. G. Rousseaux, On the physical meaning of the gauge conditions of Classical Electromagnetism: the hydrodynamics analogue viewpoint, arXiv: physics/0511047v1 [physics.class-ph] 5 Nov. (2005).
75. J.J. Slawianowski, *Geometry of Phase Spaces* (Wiley, New York, 1991).
76. V.M. Simulik, in: *What is the Electron?* edited by V.M. Simulik (Apeiron, Montreal, 2005), p. 109.
77. G.A. Schott, *Electromagnetic Radiation* (Cambridge Univ., Cambridge, 1912).
78. W. Thirring, *Classical Mathematical Physics* (Springer, Berlin, 1992).
79. C. Teitelboim, Phys. Rev. D **1**, 1512 (1970).
80. G.T. Trammel, Phys. Rev. **134** (5B), 1183 (1964).
81. J.B. Wheeler and R.P. Feynman, Rev. Mod. Phys. **17**, Nos. 2–3, 157 (1945).
82. F. Wilczek, Origins of Mass, MIT-CTP/4379, arXiv: 1206.7114v2 [hep-ph] 22 Aug. (2012).
83. F. Wilczek, Ann. Henri Poincaré **4**, 211 (2003).
84. Y. Yaremko and V. Tretyak, *Radiation Reaction in Classical Field Theory* (LAMBERT Acad. Publ., Saarbrücken, 2012).

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ПРО КЛАСИЧНУ ЕЛЕКТРОДИНАМІКУ МАКСВЕЛЛА–ЛОРЕНЦА, ПРОБЛЕМУ ІНЕРЦІЇ ЕЛЕКТРОНА ТА ФЕЙНМАНІВСЬКУ ПАРАДИГМУ ВЛАСНОГО ЧАСУ

Р е з ю м е

Класичні рівняння електромагнітного поля Максвелла та сили Лоренца виводяться в рамках фейнманівської парадигми власного часу та відповідного вакуумно-польового підходу. Дається наново виведення класичний закон Ампера, обговорюється його зв'язок із фейнманівською парадигмою власного часу. Проблема інерції електрона аналізується як на основі лагранжевого та гамільтонового формалізмів, так і відповідного компенсаційного принципу енергія–тиск стохастичної електродинаміки. Отримана модифікована сила Абрагама–Лоренца для радіаційного гальмування, приведена аргументація щодо електромагнітної природи маси електрона.

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О КЛАССИЧЕСКОЙ ЭЛЕКТРОДИНАМИКЕ МАКСВЕЛЛА–ЛОРЕНЦА, ПРОБЛЕМЕ ИНЕРЦИИ ЭЛЕКТРОНА И ФЕЙНМАНОВСКОЙ ПАРАДИГМЕ СОБСТВЕННОГО ВРЕМЕНИ

Р е з ю м е

Классические уравнения электромагнитного поля Максвелла и силы Лоренца выводятся в рамках фейнмановской парадигмы собственного времени и соответствующего вакуумно-полевого подхода. Дан наново вывод классического закона Ампера, обсуждается его связь с фейнмановской парадигмой собственного времени. Проблема инерции электрона анализируется как на основе лагранжевого и гамильтонова формализмов, так и соответствующего компенсационного принципа энергия–давление стохастической электродинамики. Получена модифицированная сила Абрахама–Лоренца для радиационного торможения, приведена аргументация относительно электромагнитной природы массы электрона.