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UDC 530.1

TWO-PARTICLE SYSTEM IN NONCOMMUTATIVE SPACE WITH PRESERVED ROTATIONAL SYMMETRY

We consider a system of two particles in the noncommutative space, which is rotationally invariant. It is shown that the coordinates of the center-of-mass position and the coordinates of relative motion satisfy a noncommutative algebra with corresponding effective tensors of noncommutativity. A hydrogen atom is studied as a two-particle system. We have found the corrections to the energy levels of the hydrogen atom up to the second order in the parameter of noncommutativity.

Keywords: noncommutative space, rotational symmetry, two-particle system, hydrogen atom.

1. Introduction

The idea of a noncommutative structure of the space has a long history. This idea was suggested by Heisenberg and later was formalized by Snyder in paper [1]. In recent years, the noncommutativity has received much attention. Such an interest is motivated by the development of String Theory and Quantum Gravity (see, e.g., [2, 3]).

The canonical version of a noncommutative space is realized with the help of the following commutation relations for coordinates and momenta:

$$[X_i, X_j] = i\hbar\theta_{ij}, \tag{1}$$

$$[X_i, P_j] = i\hbar\delta_{ij}, \tag{2}$$

$$[P_i, P_j] = 0, \tag{3}$$

where θ_{ij} is a constant antisymmetric matrix. A hydrogen atom (see, for example, [4–11]), the Landau problem (see, e.g., [12–16]), quantum mechanical systems in a central potential [17], a classical particle in a gravitational potential [18, 19], and many other problems have been studied in the noncommutative space with the canonical version of noncommutativity of the coordinates.

The many-particle problem is of great importance. Studies of this problem give possibility to analyze the properties of a wide class of physical systems in the noncommutative space. The classical problem of many particles in the noncommutative space-time was examined in [20]. There, the authors studied the set of N interacting harmonic oscillators and the system of N particles moving in the gravitational field. In [21], the two-body system of particles interacting through the harmonic oscillator potential was examined on a noncommutative plane. In [5], the problems of the noncommutative multiparticle quantum mechanics were studied. The authors considered the case where the particles with opposite charges feel opposite noncommutativities. The system of two charged quantum particles was considered in a space with the noncommutativity of coordinates in [22]. Also the two-particle system was considered in the context of noncommutative quantum mechanics characterized by the coordinate and momentum noncommutativities in [23]. The quantum model of many particles moving in the twisted N-enlarged Newton-Hooke space-time was proposed in [24]. As an example, the author examined the system of N particles moving in and interacting by the Coulomb potential.

A composite system was studied also in the deformed space with minimal length $[X, P] = i\hbar(1 + \beta P^2)$ in [25, 26]. The authors concluded that the coordinates of the center-of-mass position and the total momentum satisfy a noncommutative algebra with an effective parameter of deformation. Using this result, the condition for the recovering of the equivalence principle was found [26].

It is worth to note that a deformation of the algebra of creation and annihilation operators can be used to describe systems of particles possessing interaction and compositeness (see, e.g., recent papers [27, 28] and references therein).

In our previous papers [29, 30], we studied a composite system in the two-dimensional space with canonical noncommutativity of the coordinates $[X_1, X_2] = i\hbar\theta$, where θ is a constant. We showed that, in order to describe the motion of a composite system in the noncommutative space, we have to introduce an effective parameter of noncommutativity. The motion of the composite system was considered in the gravitational field, and the equivalence principle was studied. We proposed the condition to recover the equivalence principle in the noncommutative space with canonical noncommutativity of the coordinates [29].

In a two-dimensional space with canonical noncommutativity of the coordinates, the rotational symmetry is survived. Note, however, that, in a three-dimensional noncommutative space, we face the problem of breaking of the rotational symmetry [4, 31]. To preserve this symmetry, new classes of noncommutative algebras were explored (see, e.g., [32, 38] and references therein). Much attention was also paid to the problem of violation of the Lorentz invariance (see, e.g., [33–35]), studying the Lorentz symmetry deformations (see, e.g., [36, 37], and references therein).

In our paper [38], we proposed to construct a rotationally invariant noncommutative algebra with the help of a generalization of the matrix of noncommutativity to a tensor, which is constructed with the help of additional coordinates governed by a rotationally symmetric system. The energy levels of a hydrogen atom were studied in the rotationally invariant noncommutative space [38, 39].

In the general case, different particles may feel noncommutativity with different tensors of noncommutativity. Therefore, there is the problem of describing the motion of a system of particles in a noncommutative space. In the present paper, we study a system of two particles in the rotationally invariant noncommutative space proposed in [38]. We find the total momentum of the system and introduce the coordinates of the center-of-mass position. It is shown that the coordinates of the center-of-mass position and the relative coordinates satisfy a noncommutative algebra with corresponding effective tensors of noncommutativity. As the example of a two-particle system, we consider a hydrogen atom. We find the corrections to its energy levels caused by the noncommutativity of the coordinates.

The paper is organized as follows. In Section 2, a rotationally invariant noncommutative space is considered. In Section 3, we study a two-particle system in a rotationally invariant noncommutative space. The total momentum is found, and the corresponding coordinates of the center-of-mass position are introduced. In Section 4, we consider the hydrogen atom as a two-particle system and find the corrections to the energy levels up to the second order in the parameter of noncommutativity. Conclusions are presented in Section 5.

2. Rotational Symmetry in a Noncommutative Space

In order to solve the problem of rotational symmetry breaking in a noncommutative space, we proposed [38] a generalization of the constant antisymmetric matrix θ_{ij} to a tensor constructed with the help of additional coordinates. We considered the additional coordinates to be governed by a rotationally symmetric system. For simplicity, we supposed that the coordinates are governed by the harmonic oscillator [38]. Therefore, the following rotationally invariant noncommutative algebra was constructed:

$$[X_i, X_j] = i\hbar\theta_{ij}, \tag{4}$$

$$[X_i, P_j] = i\hbar\delta_{ij}, \tag{5}$$

$$[P_i, P_j] = 0, \tag{6}$$

where the tensor of noncommutativity θ_{ij} is defined as follows:

$$\theta_{ij} = \frac{\alpha l_P^2}{\hbar}\varepsilon_{ijk}\tilde{a}_k. \tag{7}$$

Here, α is a dimensionless constant, l_P is the Planck length, and $\tilde{a}_i$ are additional dimensionless coordinates governed by a harmonic oscillator

$$H_{\text{osc}} = \hbar\omega\left(\frac{(\tilde{p}^a)^2}{2} + \frac{\tilde{a}^2}{2}\right). \tag{8}$$

The frequency of the harmonic oscillator ω is considered to be very large [38]. Therefore, the distance between its energy levels is very large as well. So, the harmonic oscillator put into the ground state remains in it.

The coordinates $\tilde{a}_i$ and the momenta $\tilde{p}^a_i$ satisfy the commutation relations

$$[\tilde{a}_i, \tilde{a}_j] = 0, \quad (9)$$

$$[\tilde{a}_i, \tilde{p}^a_j] = i\delta_{ij}, \quad (10)$$

$$[\tilde{p}^a_i, \tilde{p}^a_j] = 0. \quad (11)$$

In addition, the coordinates $\tilde{a}_i$ commute with X_i and P_i. As a consequence, the tensor of noncommutativity (7) also commutes with X_i and P_i. So, X_i, P_i, and θ_{ij} satisfy the same commutation relations as in the case of the canonical version of noncommutativity. Moreover, algebra (4)-(6) is rotationally invariant.

We can represent the coordinates X_i and the momenta P_i by coordinates x_i and momenta p_i, which satisfy the ordinary commutation relations

$$[x_i, x_j] = 0, \quad (12)$$

$$[p_i, p_j] = 0, \quad (13)$$

$$[x_i, p_j] = i\hbar\delta_{ij}. \quad (14)$$

Namely, we can use the representation

$$X_i = x_i - \frac{1}{2}\theta_{ij}p_j, \quad (15)$$

$$P_i = p_i, \quad (16)$$

where θ_{ij} is given by (7).

The coordinates x_i and the momenta p_i commute with $\tilde{a}_i$, $\tilde{p}^a_i$, namely $[x_i, \tilde{a}_j] = 0$, $[x_i, \tilde{p}^a_j] = 0$, $[p_i, \tilde{a}_j] = 0$, $[p_i, \tilde{p}^a_j] = 0$.

The explicit representations for coordinates X_i (15) and momenta P_i (16) guarantee that the Jacobi identity is satisfied.

The noncommutative space characterized by the commutation relations (4)–(6), which remain the same after a rotation, is rotationally invariant. After the rotation $X'_i = U(\varphi)X_iU^+(\varphi)$, $\tilde{a}'_i = U(\varphi)\tilde{a}_i \times U^+(\varphi)$, $P'_i = U(\varphi)P_iU^+(\varphi)$, we have

$$[X'_i, X'_j] = i\alpha, \quad (17)$$

$$[X'_i, P'_j] = i\hbar\delta_{ij}, \quad (18)$$

$$[P'_i, P'_j] = 0. \quad (19)$$

Here, $U(\varphi) = e^{\frac{i}{\hbar}\varphi(\mathbf{n}\cdot\mathbf{L}^t)}$ is the rotation operator, with $\mathbf{L}^t$ being the total angular momentum,

$$\mathbf{L}^t = [\mathbf{x} \times \mathbf{p}] + \hbar[\tilde{\mathbf{a}} \times \tilde{\mathbf{p}}^a]. \quad (20)$$

In view of (7), (15), and (16), the total angular momentum $\mathbf{L^t}$ reads

$$\mathbf{L}^t = [\mathbf{X} \times \mathbf{P}] + \frac{\alpha l_P^2}{2\hbar}[\mathbf{P} \times [\tilde{\mathbf{a}} \times \mathbf{P}]] + \hbar[\tilde{\mathbf{a}} \times \tilde{\mathbf{p}}^a]. \quad (21)$$

Therefore, algebra (4)–(6) is rotationally invariant.

Note that $\mathbf{L}^t$ satisfies the commutation relations $[X_i, L^t_j] = i\hbar\varepsilon_{ijk}X_k$, $[P_i, L^t_j] = i\hbar\varepsilon_{ijk}P_k$, $[\tilde{a}_i, L^t_j] = i\hbar\varepsilon_{ijk}\tilde{a}_k$, $[\tilde{p}^a_i, L^t_j] = i\hbar\varepsilon_{ijk}\tilde{p}^a_k$, which are the same as in the ordinary space.

It is worth to mention that, by virtue of (15), the operators X_i depend on the momenta p_i and, therefore, on the mass m. It is clear that the operators X_i do not depend on the mass and can be considered as kinematic variables in the case where the tensor of noncommutativity (7) is proportional to $1/m$, namely where the following condition is satisfied:

$$\alpha = \tilde{\gamma}\frac{m_P}{m}. \quad (22)$$

Here, $\tilde{\gamma}$ is a dimensionless constant, which is the same for particles of different masses, and m_P is the Planck mass. It is worth noting that this condition is similar to the condition proposed in [29]. There, we considered a two-dimensional space with canonical noncommutativity of the coordinates $[X_1, X_2] = i\hbar\theta$, where θ is a constant, and found the condition

$$\theta = \frac{\gamma}{m}, \quad (23)$$

with θ being the parameter of noncommutativity, which corresponds to the particle of mass m, and γ is a constant, which takes the same value for all particles. The condition gives the possibility to solve an important problem in the two-dimensional noncommutative space, namely the problem of violation of the equivalence principle. In addition, expression (23) was derived from the condition of independence of the kinetic energy of a composite system of the composition [29].

3. Total Momentum and Coordinates of the Center-of-Mass of a Two-Particle System

Let us consider a system made of two particles of masses m_1 and m_2, which interact only with each other in a rotationally invariant noncommutative space. In general case, the coordinates of different particles may satisfy the noncommutative algebra (4)–(6) with different tensors of noncommutativity. It is also natural to suppose that the coordinates corresponding to different particles commute. Therefore, we have the algebra

$$[X_i^{(n)}, X_j^{(m)}] = i\hbar\theta_{ij}^{(n)}\delta_{nm}, \tag{24}$$

$$[X_i^{(n)}, P_j^{(m)}] = i\hbar\delta_{ij}\delta_{nm}, \tag{25}$$

$$[P_i^{(n)}, P_j^{(m)}] = 0. \tag{26}$$

where the indices n and m label the particles, and $\theta_{ij}^{(n)}$ is the tensor of noncommutativity, which corresponds to the particle of mass m_n. We would like to note here that different particles live in the same noncommutative space. The coordinates a_i are responsible for the noncommutativity of the space. Therefore, we suppose that they are the same for different particles. Nevertheless, according to condition (22), the particles with different masses feel noncommutativity with different tensors of noncommutativity, which depend on their masses. So, taking (22) into account, we can write

$$\theta_{ij}^{(n)} = \tilde{\gamma}\frac{l_P^2 m_P}{\hbar m_n}\varepsilon_{ijk}\tilde{a}_k. \tag{27}$$

We assume that the Hamiltonian in a noncommutative space has a similar form as in the ordinary space. Therefore, the Hamiltonian which corresponds to the two-particle system reads

$$H_s = \frac{(P^{(1)})^2}{2m_1} + \frac{(P^{(2)})^2}{2m_2} + V(|\mathbf{X}^{(1)} - \mathbf{X}^{(2)}|), \tag{28}$$

where $V(|\mathbf{X}^{(1)} - \mathbf{X}^{(2)}|)$ is the interaction potential energy of the two particles, which depends on the distance between them. Note that the coordinates $X_i^{(n)}$ satisfy the noncommutative algebra (24)-(26). We would like to mention that, in the ordinary case ($\theta_{ij} = 0$), the operator of distance corresponds to the ordinary distance. In the noncommutative space, we have the operator of distance $|\mathbf{X}^{(1)} - \mathbf{X}^{(2)}|$. Eigenvalues of this operator correspond to the measured distance between two particles in this space. A detailed consideration of this issue is worth to be a subject of separated studies.

In the rotationally invariant noncommutative space because of the definition of the tensor of noncommutativity (7), we have to consider the additional terms, which correspond to the harmonic oscillator. Therefore, we consider the total Hamiltonian

$$H = \frac{(P^{(1)})^2}{2m_1} + \frac{(P^{(2)})^2}{2m_2} + V(|\mathbf{X}^{(1)} - \mathbf{X}^{(2)}|) + \hbar\omega\left(\frac{(\tilde{p}^a)^2}{2} + \frac{\tilde{a}^2}{2}\right). \tag{29}$$

Let us introduce the total momentum of the two-particle system as an integral of motion and find the coordinates of the center-of-mass position as its conjugate variable. Note that the total momentum $\mathbf{P}^c$ defined in the traditional way

$$\mathbf{P}^c = \mathbf{P}^{(1)} + \mathbf{P}^{(2)} \tag{30}$$

satisfies the relation

$$[\mathbf{P}^c, H] = 0. \tag{31}$$

So, the total momentum (30) is an integral of motion in the rotationally invariant noncommutative space. The coordinate conjugate to the total momentum reads

$$\mathbf{X}^c = \frac{m_1\mathbf{X}^{(1)} + m_2\mathbf{X}^{(2)}}{m_1 + m_2}. \tag{32}$$

It is important to mention that the coordinates of the center-of-mass X_i^c satisfy a noncommutative algebra with effective tensor of noncommutativity $\tilde{\theta}_{ij}$. With regard for relations (24), (27), and (32), we have

$$[X_i^c, X_j^c] = i\hbar\tilde{\theta}_{ij}, \tag{33}$$

where the effective tensor of noncommutativity is defined as

$$\tilde{\theta}_{ij} = \frac{m_1^2\theta_{ij}^{(1)} + m_2^2\theta_{ij}^{(2)}}{M^2} = \tilde{\gamma}\frac{l_P^2 m_P}{\hbar M}\varepsilon_{ijk}\tilde{a}_k, \tag{34}$$

with $M = m_1 + m_2$.

The coordinates and the momenta, which describe the relative motion, can be also introduced in the traditional way:

$$\mathbf{X}^r = \mathbf{X}^{(2)} - \mathbf{X}^{(1)}, \tag{35}$$

$$\mathbf{P}^r = \mu_1\mathbf{P}^{(2)} - \mu_2\mathbf{P}^{(1)}, \tag{36}$$

with $\mu_i = m_i/M$.

The coordinates X_i^r and the momenta P_i^r satisfy the algebra

$$[X_i^r, X_j^r] = i\hbar(\theta_{ij}^{(1)} + \theta_{ij}^{(2)}) = i\hbar\theta_{ij}^{\mu}, \tag{37}$$

$$[X_i^r, P_j^r] = i\hbar\delta_{ij}, \tag{38}$$

$$[P_i^r, P_j^r] = 0. \tag{39}$$

So, the relative coordinates also satisfy the noncommutative algebra with the effective tensor of noncommutativity defined as

$$\theta^{\mu}_{ij} = \tilde{\gamma}\frac{l_P^2 m_P}{\hbar\mu}\varepsilon_{ijk}\tilde{a}_k, \quad (40)$$

with $\mu = m_1m_2/(m_1+m_2)$ being the reduced mass. It is worth noting that the coordinates of the center-of-mass position and the coordinates of the relative motion commute due to condition (22). We have

$$[X_i^c, X_j^r] = i\frac{\hbar}{M}\left(m_2\theta^{(2)}_{ij} - m_1\theta^{(1)}_{ij}\right) = 0. \quad (41)$$

The Hamiltonian H_s of the two-particle system becomes

$$H_s = \frac{(P^c)^2}{2M} + \frac{(P^r)^2}{2\mu} + V(|\mathbf{X}^r|). \quad (42)$$

In the next section, we consider a hydrogen atom as the example of a two-particle system.

4. Effect of the Noncommutativity of Coordinates on the Energy Levels of a Hydrogen Atom

Let us consider the hydrogen atom as a two-particle system in rotationally invariant noncommutative space (24)–(26). The total Hamiltonian reads

$$H = \frac{(P^{(1)})^2}{2m_p} + \frac{(P^{(2)})^2}{2m_e} - \frac{e^2}{|\mathbf{X}^{(1)} - \mathbf{X}^{(2)}|} + \hbar\omega\left(\frac{(\tilde{p}^a)^2}{2} + \frac{\tilde{a}^2}{2}\right), \quad (43)$$

where m_e and m_p are the masses of the electron and the proton, respectively. According to the results obtained in the previous section we can rewrite Hamiltonian (43) as follows:

$$H = \frac{(P^c)^2}{2M} + \frac{(P^r)^2}{2\mu} - \frac{e^2}{X^r} + \hbar\omega\left(\frac{(\tilde{p}^a)^2}{2} + \frac{\tilde{a}^2}{2}\right), \quad (44)$$

where $M = m_e + m_p$, $\mu = m_em_p/(m_e + m_p)$, and $X^r = |\mathbf{X}^r|$.

Let us find the corrections to the energy levels of the hydrogen atom up to the second order in the parameter of noncommutativity.

Using (15) and (16), it is easy to show that the coordinates X_i^r and the momenta P_i^r can be represented as

$$X_i^r = x_i^r + \frac{\tilde{\gamma}m_Pl_P^2}{2\hbar\mu}[\tilde{\mathbf{a}} \times \mathbf{p}^r]_i, \quad (45)$$

$$P_i^r = p_i^r, \quad (46)$$

where

$$x_i^r = x_i^{(2)} - x_i^{(1)}, \quad (47)$$

$$p_i^r = \mu_1 p_i^{(2)} - \mu_2 p_i^{(1)}. \quad (48)$$

The coordinates x_i^r and the momenta p_i^r satisfy the ordinary commutation relations:

$$[x_i^r, x_j^r] = 0, \quad (49)$$

$$[p_i^r, p_j^r] = 0, \quad (50)$$

$$[x_i^r, p_j^r] = i\hbar\delta_{ij}. \quad (51)$$

Using representation (45), (46) and taking (16) into account, we can rewrite Hamiltonian (44) as follows:

$$H = \frac{(p^c)^2}{2M} + \frac{(p^r)^2}{2\mu} - \frac{e^2}{\left|\mathbf{x}^r + \frac{\tilde{\gamma}m_Pl_P^2}{2\hbar\mu}[\tilde{\mathbf{a}} \times \mathbf{p}^r]\right|} + \hbar\omega\left(\frac{(\tilde{p}^a)^2}{2} + \frac{\tilde{a}^2}{2}\right), \quad (52)$$

where $\mathbf{p}^c = \mathbf{p}^{(1)} + \mathbf{p}^{(2)}$.

Let us write the expansion for H up to the second order in

$$\boldsymbol{\theta}^{\mu} = \tilde{\gamma}\frac{l_P^2 m_P}{\hbar\mu}\tilde{\mathbf{a}}. \quad (53)$$

To do this, let us find the expansion for X^r. We have

$$X^r = \left|\mathbf{x}^r + \frac{\tilde{\gamma}m_Pl_P^2}{2\hbar\mu}[\tilde{\mathbf{a}} \times \mathbf{p}^r]\right| = \sqrt{(x^r)^2 - (\boldsymbol{\theta}^{\mu}\cdot\mathbf{L}^r) + \frac{1}{4}[\boldsymbol{\theta}^{\mu} \times \mathbf{p}^r]^2}, \quad (54)$$

where $\mathbf{L}^r = [\mathbf{x}^r \times \mathbf{p}^r]$. It is worth noting that the operators under the square root do not commute. Therefore, we write the expansion for X^r in the following form with unknown function $f(\mathbf{x}^r)$:

$$X^r = x^r - \frac{1}{2x^r}(\boldsymbol{\theta}^{\mu}\cdot\mathbf{L}^r) - \frac{1}{8(x^r)^3}(\boldsymbol{\theta}^{\mu}\cdot\mathbf{L}^r)^2 + \frac{1}{16}\left(\frac{1}{x^r}[\boldsymbol{\theta}^{\mu} \times \mathbf{p}^r]^2 + [\boldsymbol{\theta}^{\mu} \times \mathbf{p}^r]^2\frac{1}{x^r} + (\theta^{\mu})^2 f(\mathbf{x}^r)\right). \quad (55)$$

The function $f(\mathbf{x}^r)$ can be found by squaring the left- and right-hand sides of Eq. (55). We have

$$\frac{\hbar^2}{(x^r)^4}[\boldsymbol{\theta}^{\mu} \times \mathbf{x}^r]^2 - x^r(\theta^{\mu})^2 f(\mathbf{x}^r) = 0. \quad (56)$$

So, from (56), we can write

$$(\theta^{\mu})^2 f(\mathbf{x}^r) = \frac{\hbar^2}{(x^r)^5}[\boldsymbol{\theta}^{\mu} \times \mathbf{x}^r]^2. \qquad (57)$$

On the basis of the obtained results (55) and (57), it is easy to find the expansion for $1/X^r$ and to write the expansion of Hamiltonian (52) up to the second order in $\boldsymbol{\theta}^{\mu}$

$$H = H_0 + V, \qquad (58)$$

$$H_0 = H_h^{(0)} + H_{\rm osc}. \qquad (59)$$

Here,

$$H_h^{(0)} = \frac{(p^c)^2}{2M} + \frac{(p^r)^2}{2\mu} - \frac{e^2}{x^r} \qquad (60)$$

is the Hamiltonian of a hydrogen atom in the ordinary space, and $H_{\rm osc}$ is given by (8). The perturbation caused by the noncommutativity of coordinates V reads

$$V = -\frac{e^2}{2(x^r)^3}(\boldsymbol{\theta}^{\mu}\cdot\mathbf{L}^r) - \frac{3e^2}{8(x^r)^5}(\boldsymbol{\theta}^{\mu}\cdot\mathbf{L}^r)^2 + \\ + \frac{e^2}{16}\left(\frac{1}{(x^r)^2}[\boldsymbol{\theta}^{\mu}\times\mathbf{p}^r]^2\frac{1}{x^r} + \frac{1}{x^r}[\boldsymbol{\theta}^{\mu}\times\mathbf{p}^r]^2\frac{1}{(x^r)^2} + \\ + \frac{\hbar^2}{(x^r)^7}[\boldsymbol{\theta}^{\mu}\times\mathbf{x}^r]^2\right). \qquad (61)$$

Let us find the corrections to the energy levels of the hydrogen atom using perturbation theory. Note that the total momentum $\mathbf{p}^c$ is an integral of motion. The momentum $\mathbf{p}^c$ commutes with $H_h^{(0)}$ and $H_{\rm osc}$ and may be replaced by its eigenvalue. So, we can consider the eigenvalue problem for the Hamiltonian

$$\tilde{H}_0 = \frac{(p^r)^2}{2\mu} - \frac{e^2}{x^r} + H_{\rm osc}. \qquad (62)$$

The Hamiltonian, which corresponds to the relative motion, $(p^r)^2/2\mu - e^2/x^r$ commutes with $H_{\rm osc}$. Therefore, we can write the eigenvalues and the eigenstates of $\tilde{H}_0$:

$$E^{(0)}_{n,\{n^a\}} = -\frac{e^2}{2a_B^* n^2} + \hbar\omega\left(n_1^a + n_2^a + n_3^a + \frac{3}{2}\right), \qquad (63)$$

$$\psi^{(0)}_{n,l,m,\{n^a\}} = \psi_{n,l,m}\psi^a_{n_1^a,n_2^a,n_3^a}, \qquad (64)$$

where $\psi_{n,l,m}$ are the well-known eigenfunctions of a hydrogen atom in the ordinary space, $\psi^a_{n_1^a,n_2^a,n_3^a}$, are the eigenfunctions of a three-dimensional harmonic oscillator $H_{\rm osc}$, and $a_B^* = \hbar^2/\mu c^2$ is the Bohr radius including the effect of reduced mass.

In the case where the oscillator is in the ground state, we have, according to perturbation theory,

$$\Delta E^{(1)}_{n,l} = \langle\psi^{(0)}_{n,l,m,\{0\}}|V|\psi^{(0)}_{n,l,m,\{0\}}\rangle = \\ = -\frac{\hbar^2 e^2\langle(\theta^{\mu})^2\rangle}{(a_B^*)^5 n^5}\left(\frac{1}{6l(l+1)(2l+1)} - \right. \\ - \frac{6n^2 - 2l(l+1)}{3l(l+1)(2l+1)(2l+3)(2l-1)} + \\ + \frac{5n^2 - 3l(l+1) + 1}{2(l+2)(2l+1)(2l+3)(l-1)(2l-1)} - \\ \left. - \frac{5}{6}\frac{5n^2 - 3l(l+1) + 1}{l(l+1)(l+2)(2l+1)(2l+3)(l-1)(2l-1)}\right), \qquad (65)$$

where $\langle(\theta^{\mu})^2\rangle$ is given by

$$\langle(\theta^{\mu})^2\rangle = \langle\psi^a_{0,0,0}|\theta^2_{\mu}|\psi^a_{0,0,0}\rangle = \frac{3}{2}\left(\frac{\tilde{\gamma} m_P l_P^2}{\hbar\mu}\right)^2. \qquad (66)$$

The details of calculations of the corresponding integrals can be found in [38]. In the second order of perturbation theory, we have

$$\Delta E^{(2)}_{n,l,m,\{0\}} = \\ = \sum_{n',l',m',\{n^a\}} \frac{\left|\left\langle\psi^{(0)}_{n',l',m',\{n^a\}}\right|V\left|\psi^{(0)}_{n,l,m,\{0\}}\right\rangle\right|^2}{E_n^{(0)} - E_{n'}^{(0)} - \hbar\omega(n_1^a + n_2^a + n_3^a)}, \qquad (67)$$

where the set of numbers n', l', m', $\{n^a\}$ does not coincide with the set n, l, m, $\{0\}$. We also use the following notation for the unperturbed energy of the hydrogen atom: $E_n^{(0)} = -e^2/(2a_B^* n^2)$. As $\omega \to \infty$, we have

$$\lim_{\omega\to\infty} \Delta E^{(2)}_{n,l,m,\{0\}} = 0. \qquad (68)$$

So, the corrections to the energy levels up to the second order in the parameter of noncommutativity are the following:

$$\Delta E_{n,l} = \Delta E^{(1)}_{n,l}. \qquad (69)$$

It is important to note that the obtained result (69) is divergent for the ns and np energy levels. It means that expansion of $1/X^r$ in the parameter of noncommutativity cannot be applied to the calculation of the corrections to the ns and np energy levels. Therefore, let us write the corrections to the ns energy levels in the following way:

$$\Delta E_{ns} = \left\langle\psi^{(0)}_{n,0,0,\{0\}}\left|\frac{e^2}{x^r} - \frac{e^2}{X^r}\right|\psi^{(0)}_{n,0,0,\{0\}}\right\rangle, \qquad (70)$$

where X^r is given by (54). Here, we do not use the expansion in the parameter of noncommutativity. Note that $(\boldsymbol{\theta}^\mu \cdot \mathbf{L}^r)$ commutes with $[\boldsymbol{\theta}^\mu \times \mathbf{p}^r]^2$ and $(x^r)^2$. Moreover, $(\boldsymbol{\theta}^\mu \cdot \mathbf{L}^r)\psi^{(0)}_{n,0,0,\{0\}} = 0$. Therefore, we can rewrite corrections (70) in the form

$$\Delta E_{ns} = \left\langle \psi^{(0)}_{n,0,0,\{0\}} \left| \frac{e^2}{x^r} - \frac{e^2}{\sqrt{(x^r)^2 + \frac{1}{4}[\boldsymbol{\theta}^\mu \times \mathbf{p}^r]^2}} \right| \psi^{(0)}_{n,0,0,\{0\}} \right\rangle = \frac{\chi^2 e^2}{a_B} I_{ns}(\chi), \tag{71}$$

where we use the notation

$$I_{ns}(\chi) = \int d\tilde{\mathbf{a}}\tilde{\psi}^a_{0,0,0}(\tilde{\mathbf{a}}) \int d(\tilde{\mathbf{x}}^r)\tilde{\psi}_{n,0,0}(\chi\tilde{\mathbf{x}}^r)\left(\frac{1}{\tilde{x}^r} - \frac{1}{\sqrt{(\tilde{x}^r)^2 + [\tilde{\mathbf{a}} \times \tilde{\mathbf{p}}^r]^2}}\right)\tilde{\psi}_{n,0,0}(\chi\tilde{\mathbf{x}}^r)\tilde{\psi}^a_{0,0,0}(\tilde{\mathbf{a}}), \tag{72}$$

with

$$\chi = \sqrt{\frac{\tilde{\gamma} m_P}{2\mu}\frac{l_P}{a^*_B}}. \tag{73}$$

Here, $\tilde{\psi}_{n,0,0}(\chi\tilde{\mathbf{x}}^r)$ are dimensionless eigenfunctions of the hydrogen atom

$$\tilde{\psi}_{n,0,0}(\chi\tilde{\mathbf{x}}^r) = \sqrt{\frac{1}{\pi n^5}} e^{-\frac{\chi\tilde{x}^r}{n}} L^1_{n-1}\left(\frac{2\chi\tilde{x}^r}{n}\right), \tag{74}$$

with $L^1_{n-1}\left(\frac{2\chi\tilde{x}^r}{n}\right)$ being the generalized Laguerre polynomials,

$$\tilde{\psi}^a_{0,0,0}(\tilde{\mathbf{a}}) = \pi^{-\frac{3}{4}} e^{-\frac{\tilde{a}^2}{2}} \tag{75}$$

are the dimensionless eigenfunctions corresponding to the harmonic oscillator,

$$\tilde{\mathbf{x}}^r = \sqrt{\frac{2\mu}{\tilde{\gamma} m_P l_P^2}}\mathbf{x}^r. \tag{76}$$

Integral (72) has a finite value in the case of $\chi = 0$. Consequently, as $\chi \to 0$, the asymptotics of ΔE_{ns} reads

$$\Delta E_{ns} = \frac{\chi^2 e^2}{a^*_B} I_{ns}(0) = \frac{\chi^2 e^2}{a^*_B}\langle I_{ns}(0, \tilde{\mathbf{a}})\rangle_{\tilde{\mathbf{a}}}, \tag{77}$$

where $\langle ...\rangle_{\tilde{\mathbf{a}}}$ denotes $\langle\tilde{\psi}^a_{0,0,0}(\tilde{\mathbf{a}})|...|\tilde{\psi}^a_{0,0,0}(\tilde{\mathbf{a}})\rangle$. Let us consider the integral

$$I_{ns}(\chi, \tilde{\mathbf{a}}) = \int d(\tilde{\mathbf{x}}^r)\tilde{\psi}_{n,0,0}(\chi\tilde{\mathbf{x}}^r) \times \left(\frac{1}{\tilde{x}^r} - \frac{1}{\sqrt{(\tilde{x}^r)^2 + [\tilde{\mathbf{a}} \times \tilde{\mathbf{p}}^r]^2}}\right)\tilde{\psi}_{n,0,0}(\chi\tilde{\mathbf{x}}^r). \tag{78}$$

For $\chi = 0$, we have

$$I_{ns}(0, \tilde{\mathbf{a}}) \simeq 1.72\frac{\pi\tilde{a}}{4n^3}, \tag{79}$$

with $\tilde{a} = |\tilde{\mathbf{a}}|$. The details of calculations of integral (78) at $\chi = 0$ are presented in our previous paper [39].

So, in view of (73), (77), (79), we have the following result for the leading term in the asymptotic expansion of the corrections to the ns energy levels in the small parameter of noncommutativity:

$$\Delta E_{ns} \simeq 1.72\frac{\hbar\langle\theta^\mu\rangle\pi e^2}{8(a^*_B)^3 n^3}, \tag{80}$$

where

$$\langle\theta^\mu\rangle = \langle\psi^a_{0,0,0}|\theta^\mu|\psi^a_{0,0,0}\rangle = \frac{2\tilde{\gamma} m_P l_P^2}{\sqrt{\pi}\hbar\mu}. \tag{81}$$

It is important to note that the ns energy levels are more sensitive to the noncommutativity of coordinates. The corrections to these levels are proportional to $\langle\theta^\mu\rangle$ (80). For the corrections to the energy levels with $l > 1$, we obtained that they are proportional to $\langle(\theta^\mu)^2\rangle$ (69). The corrections to the np energy levels are planned to be considered in a forthcoming publication.

5. Conclusions

In this paper, we have studied the two-particle problem in a noncommutative space with preserved rotational symmetry proposed in [38]. We have considered a rotationally invariant noncommutative algebra, which is constructed with the help of a generalization of the matrix of noncommutativity to a tensor constructed with the help of additional coordinates (7). We have studied a general case where the different particles satisfy a noncommutative algebra with different tensors of noncommutativity (24)–(26). In such a case, the total momentum has been found as an integral of motion, and the corresponding coordinates of the center-of-mass position have been introduced. We have concluded that the coordinates of the center-of-mass and the relative coordinates satisfy a noncommutative algebra with corresponding effective tensors of noncommutativity (33) and (37). It is important to note that, in the case where the masses of particles of the system are the same,

$m_1 = m_2$, $\theta_{ij}^{(1)} = \theta_{ij}^{(2)} = \theta_{ij}$ according to (34), we have $\tilde{\theta}_{ij} = \theta_{ij}/2$. So, there is a reduction of the effective tensor of noncommutativity with respect to the tensor of noncommutativity for the individual particles. In addition, from (40), we have $\theta_{ij}^{\mu} = 2\theta_{ij}$. So, the relative motion is more sensitive to the noncommutativity of the coordinates comparing to the motion of individual particles.

The hydrogen atom has been studied as a two-particle system. We have found the corrections to the energy levels of the atom caused by the noncommutativity of coordinates (69), (80). On the basis of the obtained results, we have concluded that the *ns* energy levels are more sensitive to the noncommutativity of coordinates.

This work was partly supported by Project FF-30F (No. 0116U001539) from the Ministry of Education and Science of Ukraine and by the grant from the State Fund For Fundamental Research of Ukraine, Competition F-64, Project "Classical and quantum systems beyond standard approaches".

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Received 12.11.15

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СИСТЕМА ДВОХ ЧАСТИНОК
У НЕКОМУТАТИВНОМУ ПРОСТОРІ
ЗІ ЗБЕРЕЖЕНОЮ СФЕРИЧНОЮ СИМЕТРІЄЮ

Р е з ю м е

Ми розглядаємо систему двох частинок у некомутативному просторі, який є сферично-симетричним. Показується, що координати центра мас та координати відносного руху задовольняють некомутативну алгебру з відповідними ефективними тензорами некомутативності. Атом водню вивчається як двочастинкова система. Ми знаходимо поправки до енергетичних рівнів атома з точністю до другого порядку за параметром некомутативності.