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FRIEDMANN COSMOLOGICAL MODELS WITH VARIOUS EQUATIONS OF STATE OF MATTER

The Friedmann models of the universe filled with various types of matter are considered. For these models, the cosmological parameters are constructed. Some alternative types of matter with negative pressure are considered instead of cosmological constant. It is shown that the model with dust and domain walls leads to different types of the universe evolution depending on the ratio between constants of the model. The case where the resulting model is consistent with the cosmological observations is represented.

Keywords: Friedmann models, acceleration of the universe, domain walls, radiation, negative pressure.

1. Introduction

The discovery of the accelerated expansion of the universe based on the SN Ia observational data became a crucial point in cosmology of the 20^{th} century [2]. It turned out that approximately 70% of all matter consists of some sort of the unknown entity now called the *dark energy* that drives the dynamics of the universe. From astronomical observations, it follows that the energy density of this matter is homogeneous and constant in time, and its pressure is negative. The most popular candidate for this entity is the cosmological constant [1], but there are also some other possible types of matter, e.g., quintessence [5], domain walls [6–9], spatial curvature variations [10] etc. In this work, the evolution of the universe is studied on the basis of the Friedmann models for different equations of state of matter against the background of dark energy represented by the cosmological constant and other sources with negative pressure [3, 4].

The paper is organized as follows. First, we present the general points of Friedmann models with different sources and cosmological constant. Then we consider a two-component model with radiation and cosmological constant and analyze the case where matter with negative pressure, but not cosmological constant, is taken into account for the two-component model with dust.

2. Equations of State of Matter and the Cosmological Constant

In order to describe the evolution of a homogeneous isotropic universe, one usually employs the scale factor a, Hubble constant H, and deceleration parameter q. In this paper, we present a qualitative analysis of the deceleration parameter for various models of the Friedmann universe. Let us consider the homogeneous isotropic universe with the following metric [11]:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left[\mathrm{d}\chi^2 + F^2(\chi)\,\mathrm{d}\sigma^2\right], \tag{1}$$

where $a(t)$ is the scale factor, which describes the size of the universe; furthermore, $F^2(\chi) = \sin^2\chi$ corresponds to the closed universe, $F^2(\chi) = \sinh^2\chi$ describes the Lobachevsky space, $F^2(\chi) = \chi^2$ corresponds to the flat space, and, finally, $d\sigma^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$ is the interval on a 2-sphere.

For a given model of the universe, we can easily write the Friedmann equation in a comoving frame and the energy conservation law as follows:

$$\varepsilon(t) = \frac{\dot{a}^2(t)}{a^2(t)} + \frac{k}{a^2(t)}, \tag{2}$$

$$\dot{\varepsilon}(t) = -3\,\frac{\dot{a}(t)}{a(t)}\,[\varepsilon(t) + p(t)], \tag{3}$$

where the dot denotes the derivative with respect to ct (d/cdt); k is the curvature parameter, so that $k = -1$, 0, 1 correspond to the open, flat, and closed models of the universe, respectively; $\varepsilon(t)$ is the energy density (including the factor of $8\pi G/c^4$), and $p(t)$ is the pressure of matter.

Assuming the equation of state in the form $p(t) = n\,\varepsilon(t)$, we can find from (3) that

$$\frac{\dot{\varepsilon}(t)}{\varepsilon(t)} = -3\,(1+n)\,\frac{\dot{a}(t)}{a(t)} \tag{4}$$

and, hence,

$$\varepsilon(t) = \frac{C}{[a(t)]^{3(1+n)}}, \tag{5}$$

where C is the constant of integration. For the further purposes, we rewrite it as $C = a_n^{3(1+n)-2}$ with a_n being an unknown constant with the dimension of length and n determining the type of matter.

If the universe contains several types of matter, one should write the sum over all contributing sources in (2) in place of $\varepsilon(t)$. If we denote $\varepsilon_i(t)$, the energy density for the i^{th} type of matter the system (3) becomes:

$$\sum_i \dot{\varepsilon}_i(t) = -3\,\frac{\dot{a}(t)}{a(t)}\,\sum_i [\varepsilon_i(t) + p_i(t)], \tag{6}$$

Table 1. **Energy density ε_i for various types of matter**

Type of matter	β_i	n_i	ε_i
Vacuum	0	-1	$1/a_\Lambda^2$
Domain walls	1	$-2/3$	$1/(a_{\text{dw}}\,a)$
Cosmic strings	2	$-1/3$	a_{str}/a^2
Dust	3	0	a_{dust}/a^3
Radiation	4	1/3	a_{rad}^2/a^4

where $p_i(t)$ is the pressure of the i^{th} type of matter. If different types of matter do not interact, the above equation turns into the following system of equations:

$$\dot{\varepsilon}_i(t) = -3\,\frac{\dot{a}(t)}{a(t)}\,[\varepsilon_i(t) + p_i(t)]. \tag{7}$$

Thus, system (2)–(3) takes the form

$$\sum_i \varepsilon_i(t) = \frac{\dot{a}^2(t)}{a^2(t)} + \frac{k}{a^2(t)}, \tag{8}$$

$$\dot{\varepsilon}_i(t) = -3\,\frac{\dot{a}(t)}{a(t)}\,[\varepsilon_i(t) + p_i(t)]. \tag{9}$$

Let us find the solution of system (8)–(9), by assuming

$$p_i = n_i\,\varepsilon_i. \tag{10}$$

If this is the case, for the i^{th} type of matter, we obtain

$$\varepsilon_i = \frac{a_i^{\beta_i-2}}{a^{\beta_i}}, \tag{11}$$

where $\beta_i = 3\,(n_i+1)$, and a_i is a constant with the dimension of length. The strong energy condition for the i^{th} type of matter reads

$$\varepsilon_i + p_i \geq 0, \tag{12}$$

$$|\varepsilon_i| \geq |p_i|, \tag{13}$$

from which we get

$$0 \leq \beta_i \leq 6. \tag{14}$$

System (8)–(9) can be solved for some particular values of β_i [12], which are summarized in Table 1.

Using (9) and (11), we can rewrite metric (1) into an alternative form:

$$ds^2 = \frac{da^2}{\sum_i \left(\frac{a_i}{a}\right)^{\beta_i-2} - k} - a^2\left[d\chi^2 + F^2(\chi)\,d\sigma^2\right], \tag{15}$$

where the scale factor a parametrizes the new time. If the universe is filled with only one (i^{th}) type of matter and the cosmological constant, it is possible to find an exact solution by choosing the metric (15) for the flat space ($k = 0$). From (8), we obtain

$$\frac{1}{c}\,\frac{da}{dt} = \sqrt{\left(\frac{a}{a_\Lambda}\right)^2 + \left(\frac{a_i}{a}\right)^{\beta_i-2}}. \tag{16}$$

After changing the variables by defining

$$a_i^{\beta_i-2}\,a_\Lambda^2\,a^{-\beta_i} \equiv \sinh^2\alpha, \tag{17}$$

we have

$$-\frac{2a_\Lambda}{\beta_i}\,\frac{\mathrm{d}\alpha}{\sinh\alpha} = \pm c\,\mathrm{d}t. \tag{18}$$

Upon integrating, we readily obtain

$$-\frac{2a_\Lambda}{\beta_i}\,\ln\tanh\frac{\alpha}{2} = \pm c\,(t - t_0). \tag{19}$$

Choosing the constant of integration $t_0 = 0$, we finally get:

$$a(t) = a_i\left[\frac{a_\Lambda}{a_i}\,\sinh\left(\frac{\beta_i}{2a_\Lambda}\,ct\right)\right]^{2/\beta_i}. \tag{20}$$

In this case, we can find the cosmological parameters q (deceleration parameter), Ω_i, and Ω_Λ (energy density parameter for the i^{th} type of matter and the cosmological constant, respectively):

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{\beta_i - 1 - \cosh\left(\frac{\beta_i}{a_\Lambda}\,ct\right)}{1 + \cosh\left(\frac{\beta_i}{a_\Lambda}\,ct\right)}, \tag{21}$$

$$\Omega_i = \frac{\varepsilon_i}{\varepsilon_{\text{cr}}} = \frac{1}{\cosh^2\left(\frac{\beta_i}{a_\Lambda}\,ct\right)}, \tag{22}$$

$$\Omega_\Lambda = \frac{\varepsilon_\Lambda}{\varepsilon_{\text{cr}}} = \tanh^2\left(\frac{\beta_i}{a_\Lambda}\,ct\right), \tag{23}$$

where $\varepsilon_{\text{cr}} = \frac{3H^2c^2}{8\pi G}$ is the critical energy density with the Hubble constant $H = \dot{a}/a$.

For the universe with nonzero curvature parameter $k = \pm 1$, the cosmological parameters read:

$$q = \frac{\frac{\beta_i - 2}{2}\left(\frac{a_i}{a}\right)^{\beta_i - 2} - \left(\frac{a}{a_\Lambda}\right)^2}{\left(\frac{a_i}{a}\right)^{\beta_i - 2} + \left(\frac{a}{a_\Lambda}\right)^2 - k}, \tag{24}$$

$$\Omega_i = \frac{\left(\frac{a_i}{a}\right)^{\beta_i - 2}}{\left(\frac{a_i}{a}\right)^{\beta_i - 2} + \left(\frac{a}{a_\Lambda}\right)^2 - k}, \tag{25}$$

$$\Omega_\Lambda = \frac{\left(\frac{a}{a_\Lambda}\right)^2}{\left(\frac{a_i}{a}\right)^{\beta_i - 2} + \left(\frac{a}{a_\Lambda}\right)^2 - k}. \tag{26}$$

In Figure 1, we present the dependence of the deceleration parameter q on the scale factor a (in dimensionless units) for Friedmann models containing different types of matter ($\beta = 1, 2, 3, 4$) together with the cosmological constant ($\beta = 0$), taking different types of curvature ($k = \pm 1, 0$) into account.

The behavior of the deceleration parameter under $\beta = 1, k = 1$ (closed universe) means that the universe expands until a certain value of scale factor is attained, and then it stops.

3. Friedmann Universe with Radiation and Cosmological Constant

Radiation was the dominating type of matter in the universe at the early stages of its evolution. Models of the universe with radiation in the presence of the cosmological constant have been widely studied in [13,

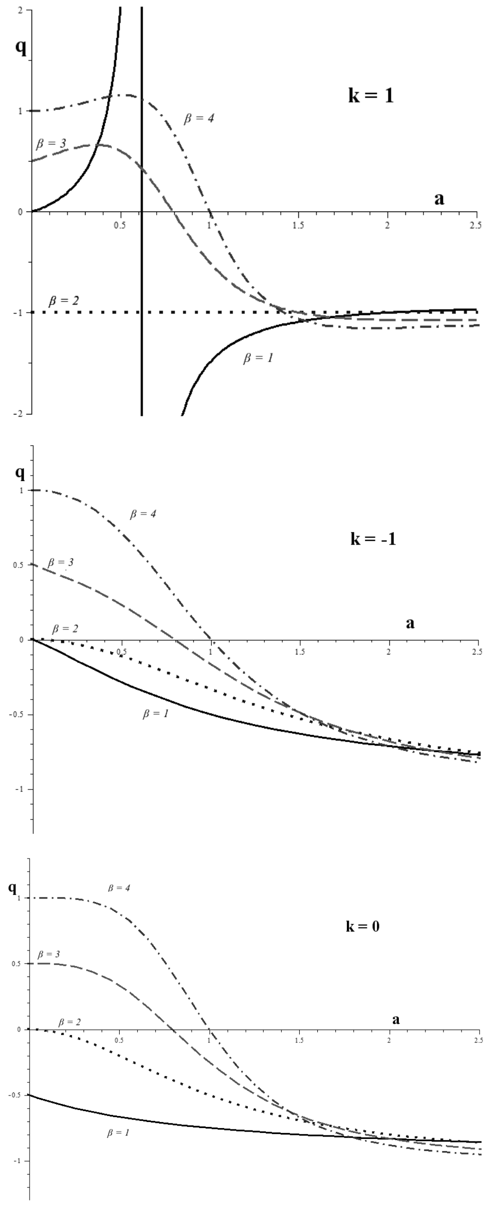

Fig. 1. Deceleration parameter q as a function of the scale factor a for different types of matter ($\beta = 1, 2, 3, 4$) with cosmological constant ($\beta = 0$) and different types of curvature ($k = \pm 1$, 0)

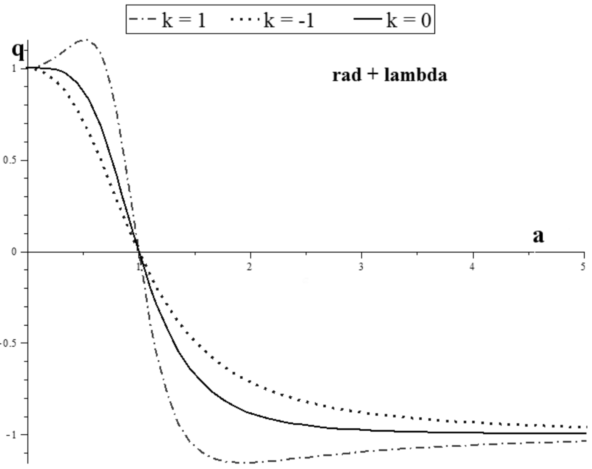

Fig. 2. Deceleration parameter q as a function of the scale factor a for the universe with cosmological constant and radiation, assuming different types of curvature ($k = \pm 1, 0$)

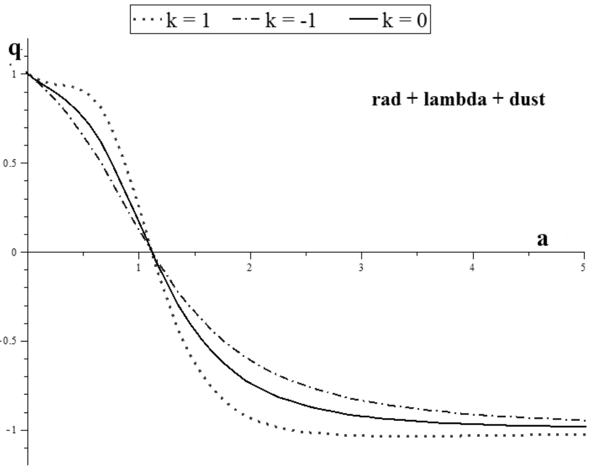

Fig. 3. Deceleration parameter q as a function of the scale factor a for the universe with cosmological constant, radiation and dust, assuming different types of curvature ($k = \pm 1, 0$)

14]. It turns out that it is possible to find the exact solution of system (8)–(9) in a unique form for all three types of space curvature [15]. The metric for this solution reads:

$$ds^2 = \frac{da^2}{\frac{a_{\rm rad}^2}{a^2} + \frac{a^2}{a_\Lambda^2} - k} - a^2 \left[d\chi^2 + F^2(\chi)\, d\sigma^2\right], \qquad (27)$$

where

$$a^2(t) = \sqrt{a_{\rm rad}^2\, a_\Lambda^2 - \frac{k^2\, a_\Lambda^4}{4}}\, \sinh\left[\frac{2}{a_\Lambda}\, c(t - t_0)\right] + \frac{k\, a_\Lambda^2}{2}. \qquad (28)$$

With the initial condition $a(t = 0) = 0$, it is possible to determine the constant of integration t_0:

$$t_0 = \frac{a_\Lambda}{2c}\, \text{arcsinh}\left(\frac{k\, a_\Lambda^2}{2}\, \frac{1}{\sqrt{a_{\rm rad}^2\, a_\Lambda^2 - \frac{k^2\, a_\Lambda^2}{4}}}\right). \qquad (29)$$

Finally, the scale factor reads:

$$a^2(t) = a_{\rm rad}\, a_\Lambda\, \sinh\left(\frac{2ct}{a_\Lambda}\right) - \frac{k\, a_\Lambda^2}{2}\left[\cosh\left(\frac{2ct}{a_\Lambda}\right) - 1\right]. \qquad (30)$$

In the special case with no cosmological constant $\Lambda \to 0$ ($a_\Lambda \to \infty$), the above solution becomes the well-known solution for the universe with radiation only:

$$a^2(t) = 2\, a_{\rm rad}\, ct - k\, (ct)^2. \qquad (31)$$

For small times, when the latter term is negligible as compared with the first term, all three types of the solution ($k = \pm 1$, 0) behave themselves identically.

In the model with cosmological constant and radiation, as well as with cosmological constant and dust [16], a case of special interest is the solution for $k = 1$:

$$a^2(t) = a_\Lambda \sinh\left(\frac{ct}{a_\Lambda}\right) \cosh\left(\frac{ct}{a_\Lambda}\right) \times \times \left[2\, a_{\rm rad} - a_\Lambda\, \tanh\left(\frac{ct}{a_\Lambda}\right)\right]. \qquad (32)$$

There are three possible scenarios for the universe evolution, depending on the ratio $a_{\rm rad}/a_\Lambda$:

- If $a_{\rm rad}/a_\Lambda < 1/2$, the universe will collapse at the time $t = (a_\Lambda/c)\, \text{arctanh}(2a_{\rm rad}/a_\Lambda)$.
- If $a_{\rm rad}/a_\Lambda = 1/2$, the model will turn into the static Einstein universe.
- If $a_{\rm rad}/a_\Lambda > 1/2$, we end up with the open universe which becomes close to the de Sitter one at late stages.

In Figures 2–4, we show the deceleration parameter q and the energy density parameters $\Omega_{\rm rad}$ and Ω_Λ as functions of the time (parametrized by the scale factor $a(t)$) in various models of the universe, using dimensionless units. In Fig. 2, we present the dependence of the deceleration parameter q on the scale factor a for the universe with cosmological constant and radiation, taking all three possible types of curvature ($k = \pm 1$, 0) into account. Analogously, in Fig, 3, the model with an additional contribution from dust is illustrated. Finally, in Fig. 4, the energy density parameters $\Omega_{\rm rad}$ and Ω_Λ are plotted as functions of the scale factor a for the universe containing the cosmological constant and radiation, for all types of curvature ($k = \pm 1$, 0). We conclude that, in the model of the universe with cosmological constant and radiation, the transition from deceleration to acceleration takes place at present times (when $a = 1$) (Fig. 2). Adding the dust into the model transfers this tome moment to later stages of the evolution (Fig. 3).

4. Friedmann Universe with Domain Walls and Cosmic Strings

Besides the cosmological constant, there are some other suitable types of matter with negative pressure, which could possibly cause the observed accelerated expansion of the universe, most notably domain walls and cosmic strings.

Let us consider a homogeneous universe filled with cosmic strings, which represent relic topological defects with the effective equation of state $p_{\rm str} = -\frac{1}{3}\,\varepsilon_{\rm str}$ and the energy density

$$\varepsilon_{\rm str} = \frac{a_{\rm str}}{a^2}, \qquad (33)$$

where $a_{\rm str}$ is a dimensionless constant. From (2), we find

$$a(t) = \pm\sqrt{a_{\rm str} - k}\,c(t - t_0). \qquad (34)$$

Thus, we arrive at the condition $a_{\rm str} > k$. However, the above solution implies that $a(t) \propto t$, which corresponds to the flat spacetime (Robertson–Milne solution [17]) and is not in agreement with cosmological observations.

If the universe is filled with both dust and domain walls, the total energy density is

$$\varepsilon = \frac{a_{\rm dust}}{a^3} + \frac{1}{a_{\rm dw}\,a}. \qquad (35)$$

The Friedmann equation and the deceleration parameter can be written in the form:

$$\frac{1}{c}\frac{{\rm d}a}{{\rm d}t} = \sqrt{\frac{a_{\rm dust}}{a} + \frac{a}{a_{\rm dw}} - k}, \qquad (36)$$

$$q = -\frac{1}{2}\,\frac{\frac{a_{\rm dust}}{a_{\rm dw}} - \frac{a^2_{\rm dust}}{a^2}}{\frac{a^2_{\rm dust}}{a^2} + \frac{a_{\rm dust}}{a_{\rm dw}} - k\,\frac{a_{\rm dust}}{a}}. \qquad (37)$$

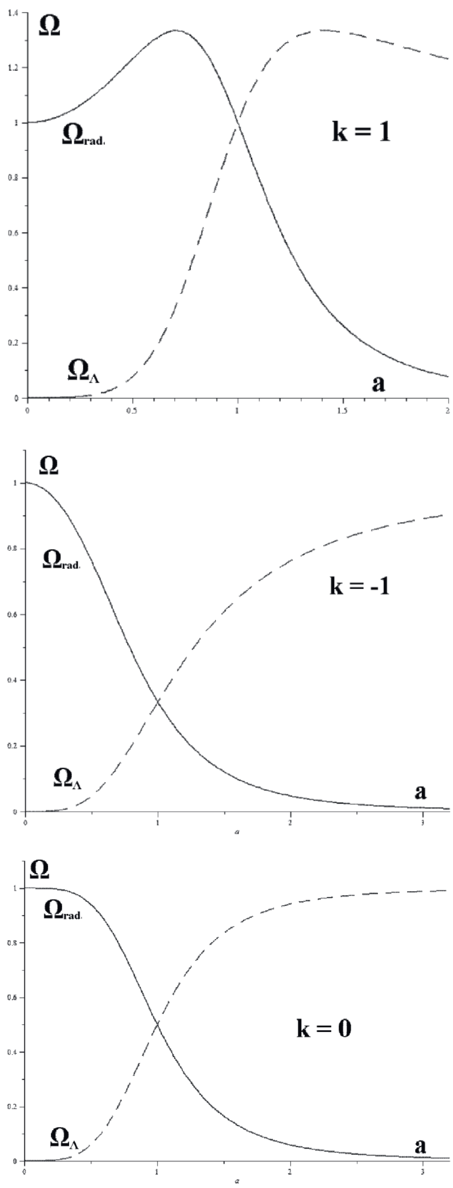

Fig. 4. Energy density parameters $\Omega_{\rm rad}$ and Ω_Λ as functions of the scale factor a for the universe with cosmological constant and radiation, assuming different types of curvature ($k = \pm 1, 0$)

Denoting $x \equiv a_{\rm dust}/a$ and $y \equiv a_{\rm dust}/a_{\rm dw}$, we can write:

$$q = -\frac{1}{2}\,\frac{y - x^2}{x^2 + y - kx}. \qquad (38)$$

In Fig. 5, we plot the dependences of the deceleration parameter q on the time (parametrized by the

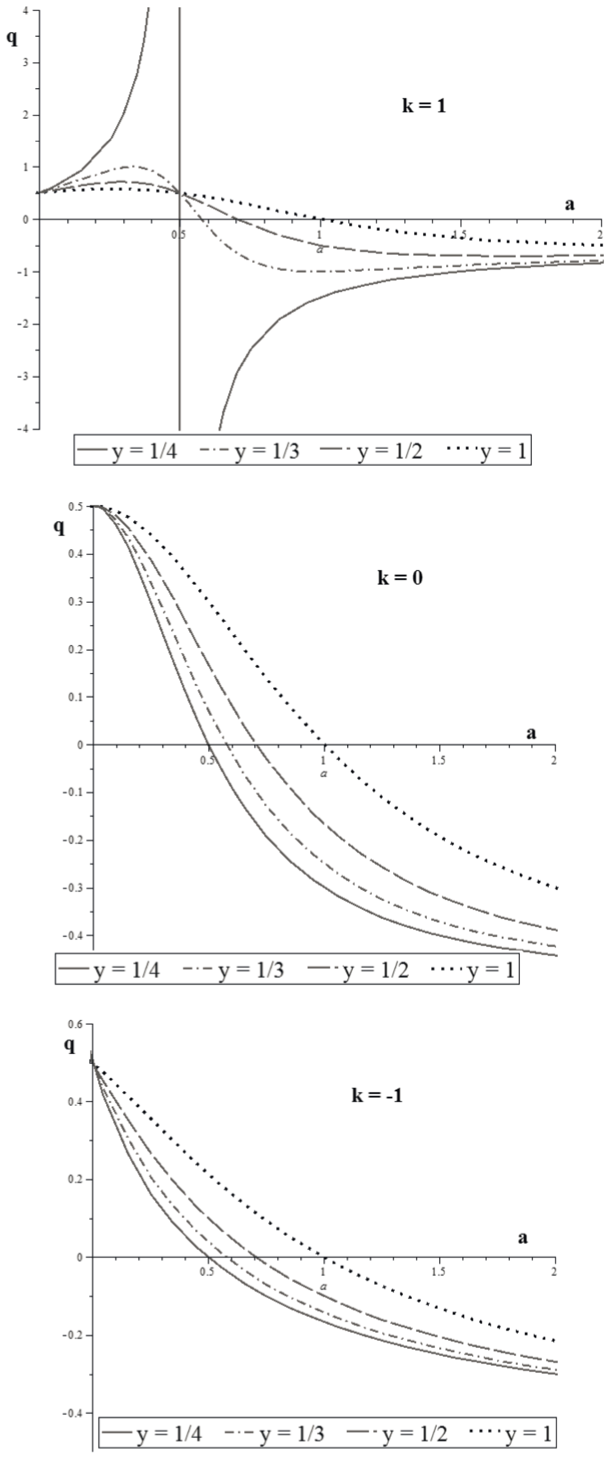

Fig. 5. Deceleration parameter q as a function of the scale factor a for the universe with dust and domain walls, assuming different types of curvature ($k = \pm 1, 0$)

scale factor a) for the universe containing dust and domain walls with regard for all three types of curvature ($k = \pm 1$, 0). We observe that, for $y \leq 1/4$, the deceleration parameter q exhibits a discontinuity, which means that the closed universe also stops its expansion. On the other hand, for $y > 1/4$, the deceleration turns into the acceleration. We should mention that, although domain walls look to be appropriate candidate for dark energy, their existence is in contradiction to the theory of scalar perturbations at late stages of the evolution of the universe [18].

5. Conclusion

In this work, the Friedmann models of the universe filled with various types of matter are considered. For each model, the corresponding evolution of the deceleration parameter with respect to the time parametrized by the scale factor is constructed. The obtained plots are in accordance with the well known fact that, for the matter with $\beta_i > 2$, the expansion of the universe is decelerated, while, for $\beta_i < 2$, the accelerated expansion takes place. For the model of the universe with cosmological constant and radiation, assuming the curvature parameter $k = 1$, we have shown that three qualitatively different scenarios are possible according to the ratio $a_{\rm rad}/a_\Lambda$. In addition to this, some alternative types of matter with negative pressure are considered instead of the cosmological constant. The model with dust and domain walls leads to different types of the universe evolution depending on the ratio $a_{\rm dust}/a_{\rm dw}$. It turned out that, for $a_{\rm dust}/a_{\rm dw} \leq 1/4$, the universe stops its expansion. For $a_{\rm dust} \approx a_{\rm dw}$, the resulting model is consistent with the cosmological observations, though there are arguments against topological defects as the nature of the dark energy.

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КОСМОЛОГІЧНІ МОДЕЛІ ФРІДМАНА
З РІЗНИМИ РІВНЯННЯМИ СТАНУ МАТЕРІЇ

Р е з ю м е

Розглянуто фрідмановські моделі з різними типами матерії. Для цих моделей досліджено поведінку космологічних параметрів. Розглянуто також космологічні моделі, в яких замість космологічної сталої враховані інші види матерії з негативним тиском. Показано, що в моделі з пилом і доменними стінками можливі різні типи еволюції всесвіту залежно від співвідношення між константами моделі. Запропоновано реалізацію такої моделі, що не суперечить спостережуваним даним.